Jitter Minimization Techniques in Time Sensitive Communication Networks: A Statistical Approach

A Project Report

Submitted by

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Thesis Certificate

This is to certify that the thesis titled **Jitter Minimization Techniques in Time Sensitive Communication Networks, A Statistical Approach**, submitted by **Rohit Dnyaneshwar Chimate**, to the Indian Institute of Technology, Madras, for the award of the degree of **Master of Technology**, is a bona fide record of the research work carried out by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Abstract

Voice communication being the traditional and most widely used form of communication by sheer number, has to be improved upon in terms of its quality and cost. The cost optimization could be done by supporting the voice service by transitioning to packet infrastructure. The problem with that approach is that the jitter introduced by a node in packet infrastructure would be much higher than that of typical jitter in legacy TDM networks(like PDH/SDH networks). This being the genesis of the TDM over PSN problem, which predominantly handles this through manipulation of service times in the jitter buffer.

In this thesis work, We also look into the service times in the intermediate routers in a TDM over PSN problem to choose a statistical distribution for them such that the output jitter is minimal. The key to such service distribution is to identify a positive distribution such that the mean and variance are analytically independent,i.e., should allow the user to choose mean and variance independently. When the utilization of the corresponding queue is near to 1,then the IDTs variance could be controlled by service variance.

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List of symbol

λ	Poisson arrival rate
μ	Exponential Service rate
ρ	Utilization Factor
μ_G	Service rate for any General Distribution
$f(.)$	Probabbility Density Function
$F(.)$	Cumulative Density Function
σ^2	Variance
σ^2_{IDI}	IDI Variance
$\Gamma(.)$	Gamma Function
erf(.)	Error function
E[X]	Mean of Distribution
E[X^2]	Second Moment Of Distribution
Q(Z)	Z-Transform Of Queue length
B*(S)	Laplace transform of Distribution
Q_bar	Mean Queue length
P_N	Probability that there are N number of customer there in queue.
N_q	Number of customer in queue
W_q	Mean waiting time in queue

Abbreviations

TDM	Time Division Multiplexing
PSN	Packet Switch Network
PDH	Physical Digital Hierarchy
SDH	Synchronous Digital Hierarchy
IP	Internet Protocol
CES	Circuit Emulation Service
MEN	Metro Ethernet Network
CPRI	Common Public Radio Interface
RRH	Remote Radio Head
TSN	Time Sensitive Switch
CoE	CPRI over Ethernet
IDI	Inter Departure Interval
TDMoPSN	TDM Over PSN
SONET	Synchronous Optical Network
MPLS	Multi-Protocol Lable Switching
ATM	Asynchronous transfer mode
BBU	Base Band Unit
PDF	Probability Distribution Function
CDF	Cumulative Distribution Function
M/M/1	Poisson arrival , Exponential Service rate with single server (Kendall's Notation)
M/G/1	Poisson arrival , General distribution Service rate with single server

Chapter 1

Introduction

1.1 TDM over PSN (TDMoPSN)

Traffic smoothing in communication /computer networks is an old-age problem in which the deterministic approaches like 'leaky bucket' algorithm is very common. In that approach there might be packet losses if a burst occurs. Pseudowire, PDH over IP or SDH over IP, CES in MEN, and the latest problem of CPRI over Ethernet are all have one common aspect that is they all involve TDM over PSN. In CPRI fronthaul network[1] ,stringent bandwidth, latency, and jitter requirements play important role. Ethernet can be a cost-effective solution to carry the CPRI traffic. The jitter management plays significant role for CPRI over Ethernet. So we can consider there is Poisson arrival process for incoming packets from different Remote Radio Heads(RRHs) to CPRI network. In general, we have different data rates for different RRHs, so random packet arrivals and different data rate produces variable delay at the transmitting side of CPRI fronthaul network. The latencies at switches are unavoidable due to introduced delays but manageable once the values of non-deterministic variable delays are constant, known or predictable. For that we have to do scheduling of arrival packets. In CPRI fronthaul networks there are Time Sensitive Network (TSN) switches where COE mapping and scheduling of arriving packets takes place. We proposed a statistical model in which, we are able minimize output jitter by queue modeling. This Statistical model can be apply at TSN switches of CPRI fronhaul network.

1.2 Queuing model for jitter minimization

In case of scheduling/ modeling queues or in our case random arriving packets, we generally prefer M/M/1 queue which is consider as the most elementary of queuing models. Although M/M/1 queuing model gives better modeling results for mean queue length, mean waiting time and for output variance but it can be possible to find new queuing model such that it will give better performance for all parameters in comparison with M/M/1 queue. Output jitter can be seen as variance of Inter Departure Intervals (IDIs). So for minimizing jitter we have to focus on minimizing IDI variance. So, while modeling any queue we can consider two possibilities one is queue is non-empty and other is queue is empty or arrival process is taking place continuously. So,for non empty queue departure will follow service distribution,because arrival process is not affecting to departure if server is serving non-empty queue. which says that if we could able to minimize the variance of service distribution it will automatically reduce jitter (IDI variance). So, we try to find few service distribution so that we can control mean and variances independently such that by keeping mean constant we must able to reduce variance. For second case where queue is not full the IDI variance will not depend on service distribution, here in this case arrival process will also have to consider in account. In this case we have to find a model such that its overall output IDI variance is less than that of M/M/1 queue. We have tried to model M/G/1 queue with different types of service distribution so that we will able to brings out better output jitter in comparison with M/M/1.

1.3 Thesis Overview

This thesis gives a new perspective on the above problems, by exploiting the effect of correlations in the service intervals on the IDI process. In this thesis we are going to discuss about how we can achieve output jitter minimum than that of M/M/1 queuing model by introducing M/G/1 queuing model with distribution such as Beta prime and Inverse Gaussian as a service distribution of M/G/1 queue. In Chapter 1, we have discussed about background and introduction of time sensitive networks and queuing models. In Chapter 2, we mentioned some applications related to TDMoPSN which are most widely used in real world. Chapter 3 Background materials, we briefly discussed about different type of distribution which we are going to use for modeling queues. While in search of required distribution we almost went through 10 different types of positive random variable distribution. We also mentioned all parameters like mean, variance, probability distribution function and analytical independence of their parameters briefly for each distribution in this section. In Chapter 4, we have found out two distributions which are fitted perfectly as per our requirement. So, we analyze queue parameters e.g. Mean queue length, mean waiting time, IDI variance etc. in detailed manner by considering these distributions as a service of queuing models. We also plot graphs for both distributions to compare parameters with M/M/1 queue.

Chapter 2

Real Time Application of TDMoPSN

2.1 SDH Over IP Synchronous Digital Hierarchy

Synchronous Digital Hierarchy (SDH) and Synchronous Optical Network (SONET)[3] are refer to group of fiber optic transmission rates that can transport digital signal with different capacity. The need to reduce network operating cost and increase revenue were the drivers behind the introduction of SDH. The charecteristics of SDH , however, make it more suitable for this application, because it offers better transmission quality , enormous routing flexibility and support the facility such as path self healing.
In the multiplexing process, payloads are layered into lower order and higher order virtual containers, each including a range of overhead functions for management and error monitoring. Transmission is then supported by the attachment of further layers of overheads. This layering of functions in SDH, both for traffic and management, suits the layered concept of a service based network better than the transmission oriented PDH standards.To support range of operations, SDH includes a management layer whose communications are transported within dedicated data communication channel time slots inside the interface rate. Managing capacity in the network involves operations such as protection for circuit recovery, provisioning for all the allocation of capacity to preferred routes, consolidation of traffic from unfilled bearers onto fewer bearers in order to reduce waste of traffic capacity and the sorting of different traffic types from mixed payloads into separate destinations for each type traffic.

2.2 Circuit Emulation Service(CES) over Metro Ethernet Network

CES allows the transport of synchronous circuit such as TI/E1 over asynchronous network so that it is received error-free with constant delay. The object of this is to allow MEN[2] service providers to offer TDM services to customers.

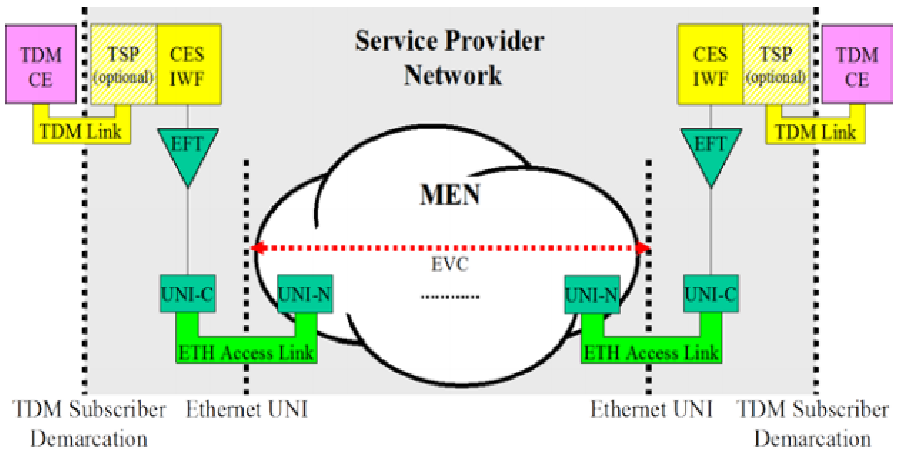

Fig.2.1: Circuit Emulation Service (TDM) over MEN

Hence it allows MEN service providers to extend their reach and addressable customer base. The MEN must maintain the bit integrity, timing information and other client-payload specific characteristics of the transported traffic without causing any degradation that would exceed the requirements for the given service.

2.3 Pseudo Wire

Pseudowire is a mechanism for emulating various networking or telecommunications services across packet-switched networks that use Ethernet, IP, or MPLS. Services emulated can include T1 leased line, frame relay, Ethernet, ATM, TDM, or SONET/SDH[4].

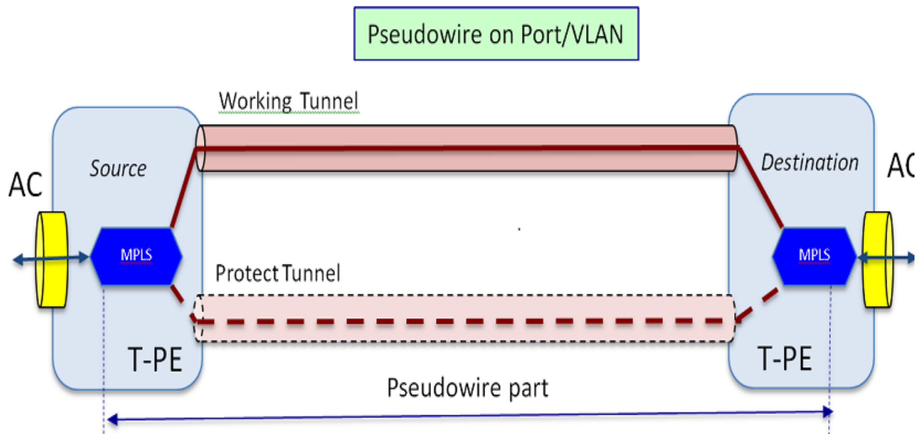

Fig.2.2: Psuedowire on VLAN

Psuedowire provide the functions in order to emulate the behavior and characteristics of the native service like Managing the signaling, timing, order, or other aspects of the service at the boundaries of the Psuedowire. The applicability of Psuedowire to a particular service depends on the sensitivity of that service or the CE implementation , and on the ability of the adaptation layer to mask them.The PSN carrying a Psuedowire will subject payload packets to loss, delay, delay variation, and re-ordering. During a network transient there may be a sustained period of impaired service.

2.4 CPRI Over Ethernet (5G)

CPRI (Common Public Radio Interface) is a specification for wireless communication networks that defines the key criteria for interfacing transport, connectivity and control communications between base-band units (BBUs) and remote radio units (RRUs)[1]. An important feature of CPRI is its support for separation between the base frequency band and the radio frequency band. Benefits of CPRI include, Base station manufactures can use one common protocol, The specifications are freely made available to the public and The public can contribute ideas and proposals about the CPRI specifications.

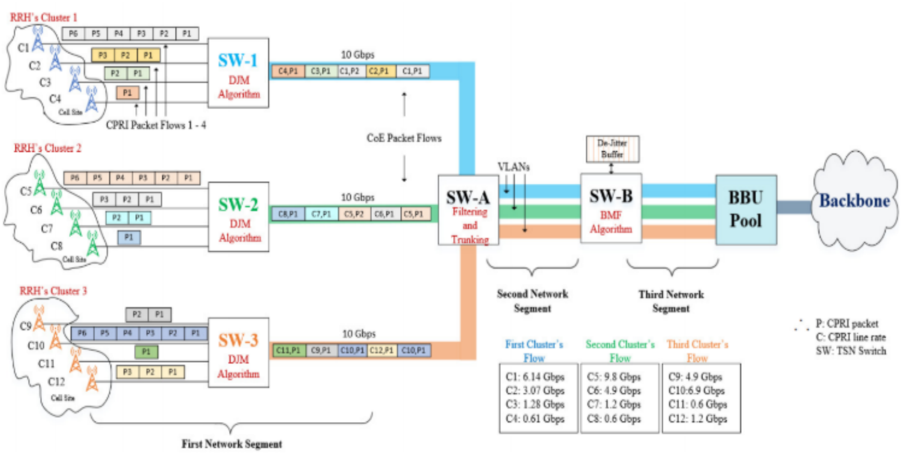

Fig.2.3: CPRI fronthaul network structure

So, in fig. 2.3 we can see SW-1, SW-2 ,SW-3 which are TSN switches. These switches used for managing packet arrivals from different RRHs and CoE mapping. For jitter minimization we either have to remove introduced variable delay or try to maintain constant delay in arriving packets. We can apply our proposed statistical model at TSN switches such that it will reduce overall output variance to acquire minimum jitter at the output.

2.5 Synchronous Ethernet (Sync E)

time division multiplexing (TDM) services such as T1/E1 and SONET/SDH require synchronized clocks at both the source and destination nodes. Similarly, wireless base stations require synchronization to a common clock to ensure a smooth call hand-off between adjacent cells. one gaining momentum is Synchronous Ethernet (SyncE)[5]. SyncE uses the physical layer interface to pass timing from node to node in the same way timing is passed in SONET/SDH or T1/E1. This gives telecom and wireless providers confidence that networks based on SyncE will be not only cost-effective, but also as highly reliable .

Chapter 3

Background Material

Mathematically we try to get Minimum Inter Departure Interval (IDI) Variance by two approaches Mentioned below:-

- Find such distributions whose variance (σ^2) can be controlled independently of their mean (μ) by keeping its mean constant.
- Find M/G/1 distribution whose IDI variance is less than that of M/M/1 queue.

3.1 Positive Distributions for Statistical modeling with first approach

The list of positive distributions, which have the property of variance could be reduced arbitrarily small while keeping the mean constant, are given below.

3.1.1 Beta-Prime distribution:-

Beta prime distribution is identified its density function as given below

$$f_{B2}\left(x;\alpha,\beta\right)=\frac{x^{\alpha-1}\left(1+x\right)^{-\alpha-\beta}}{B\left(\alpha,\beta\right)}$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters and the distribution is positive type $x > 0$.

3.1.1.1 Mean and variance

The mean is given by $\mu_{B2}=\frac{\alpha}{\beta-1}$ and variance by $\sigma^2_{B2}=\frac{\alpha(\alpha+\beta-1)}{(\beta-2)(\beta-1)^2}$ here α,β > 0. We are interested in looking into the form of mean & variance, which allows to see explicitly whether it is possible to change variance keeping mean constant.

3.1.2 Gamma Distribution and Erlang distribution

Gamma distribution is used to model in a number of real-world situations: the size of insurance claims (in economics), rainfall (weather prediction), multi-path fading of signal power (in wireless communication), inter-spike intervals (in neuroscience), bacterial gene expression (biology) etc. Here, we choose Gamma distribution for service, since it could model a real world (positive) quantity as a positive RV whose mean could independently be decided over its variance.

3.1.2.1 Gamma Distribution PDF and CDF

The cumulative distribution function (CDF) of the standard Gamma distribution is defined by

$$F(x;\alpha,\beta) = \int_0^x f(u;\alpha,\beta)\,du = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)} \tag{3.1}$$

where $\gamma(\alpha,\beta x)$ is the lower incomplete gamma function and $\alpha > 0, \beta > 0$.

Its probability density function is

$$f(x,\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \tag{3.2}$$

for $x > 0,\ \ \alpha,\beta > 0$.

3.1.2.2 The mean and variance is given by

mean of gamma distribution $=\mu_X = \frac{\alpha}{\beta}$ and variance $= \sigma_X^2 = \frac{\alpha}{\beta^2}$.

3.1.2.3 Moments

The mean of Gamma distribution is given by $\mu{=}\frac{\alpha}{\beta}$ and its variance $\sigma^2_{(x,\alpha,\beta)}{=}\frac{\alpha}{\beta^2}$.Now for analytical independence of mean and variance could be seen by rewriting the expression for variance as $\sigma^2_{(x,\alpha,\beta)}{=}\frac{(\frac{\alpha}{\beta})}{\beta}$ .But we know $\mu = \frac{\alpha}{\beta}$.Using this, rewriting the expression for variance, we get $\frac{\mu}{\beta}$.

3.1.3 Inverse Gamma distribution

The inverse gamma distribution is a two-parameter family of continuous probability distributions on the positive real line, which is the distribution of the reciprocal of a variable distributed according to the gamma distribution.

3.1.3.1 PDF and CDF of inverse gamma distribution

probability distribution function is f(x;α,β) $= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \mathrm{x}^{-\alpha-1} \exp(\frac{-\beta}{x})$,

cumulative distribution function is F(x;α,β)$= \frac{\Gamma(\alpha,\frac{\beta}{x})}{\Gamma(\alpha)}$ $\alpha,\ \beta > 0$.

3.1.3.2 Mean and Variance

mean = $\mu_{IG} = \frac{\beta}{\alpha-1}$ here $\alpha > 1$, Variance = $\sigma^2_{IG} = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ here $\alpha > 2$.

here analytical independence can be expressed as $\sigma^2_{IG} = \frac{\mu^2_{IG}}{\alpha-2}$
So, by controlling α we can control variance . for higher values of α variance will be minimum

3.1.4 Inverse Gaussian distribution

the inverse Gaussian distribution (also known as the Wald distribution) is a two-parameter family of continuous probability distributions with support on (0,∞). while the Gaussian describes a Brownian motion's level at a fixed time, the inverse Gaussian describes the distribution of the time a Brownian motion with positive drift takes to reach a fixed positive level.

3.1.4.1 PDF and CDF of inverse Gaussian distribution

PDF f(x, λ, μ) = $(\frac{\lambda}{2\pi x^3})^{1/2} \exp(\frac{-\lambda(x-\mu)^2}{2\mu^2 x})$,

CDF F(x, λ, μ) = $\Phi((\frac{\lambda}{x})^{1/2}(\frac{x}{\mu} - 1)) + \exp(\frac{2\lambda}{\mu})\Phi(-(\frac{\lambda}{x})^{1/2}(\frac{x}{\mu} - 1))$ here $\mu, \lambda > 0$

3.1.4.2 Mean and Variance

mean = $\mu_{I-gaussian} = \mu$ and variance = $\sigma^2_{I-gaussian} = \frac{\mu^3}{\lambda}$

analytical independence can be seen as $\sigma^2_{I-gaussian} = \frac{\mu^3_{I-gaussian}}{\lambda}$

3.1.5 Log-normal distribution

The log-normal distribution is important in the description of natural phenomena. This follows, because many natural growth processes are driven by the accumulation of many small percentage changes. These become additive on a log scale.The length of chess games tends to follow a log normal distribution.

3.1.5.1 PDF and CDF of log normal distribution

PDF f(x, μ, σ) = $\frac{1}{x\sigma\sqrt{2\pi}} \exp(-\frac{(\ln x-\mu)^2}{2\sigma^2})$,

CDF F(x, μ, σ) = $\frac{1}{2} + \frac{1}{2} erf(\frac{\ln x - \mu}{\sqrt{2}\sigma})$
Here, $\sigma > 0$, $\mu \ \epsilon$ (-∞, +∞)

3.1.5.2 Mean and Variance

mean = $\mu_{Log-N} = \exp(\mu + \frac{\sigma^2}{2})$
Variance = $\sigma^2_{Log-N} = [(\exp\sigma^2 - 1)\exp 2(\mu + \frac{\sigma^2}{2})]$

analytical independence $\sigma^2_{Log-N} = (\exp\sigma^2 - 1)(\mu_{Log-N})^2$

3.1.6 Scaled-Inverse Chi-Square distribution

the scaled distribution has an parameter τ^2, which scales the distribution horizontally and vertically, representing the inverse-variance of the original underlying process. Also, the scaled inverse chi-squared distribution is presented as the distribution for the inverse of the mean of ν squared deviates, rather than the inverse of their sum.

3.1.6.1 PDF and CDF of Scaled inverse chi squared distribution

PDF f(x, τ^2, ν) = $\frac{(\tau^2\frac{\nu}{2})^{\frac{\nu}{2}}\exp(-\frac{\tau^2\nu}{2x})}{\Gamma(\frac{\nu}{2})*x^{(\frac{\nu}{2}+1)}}$,

CDF F(x, τ^2, ν) = $\frac{\Gamma\left(\frac{\nu}{2},\frac{\tau^2\nu}{2x}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$

where, ν = Number of chi-squared degree of freedom , τ^2 =Scaling parameter.

3.1.6.2 Mean and Variance

mean = $\mu_{SCI} = \frac{\nu\tau^2}{\nu-2}$ for $\nu > 2$

Variance $\sigma^2_{SCI} = 2\ \frac{\nu^2\tau^4}{(\nu-2)^2(\nu-4)}$ for $\nu > 4$

analytical independence e $\sigma^2_{SCI} = 2\ \frac{\mu^2_{SCI}}{((\nu-4)}$

All above mentioned 7 distributions possess the property of controlling its variance by keeping its mean constant which satisfy our first approach.

3.2 Distributions for statistical modeling with second approach

In this approach we tried to find mean queue length by using Pollaczek-Khinchin formula which was difficult task because there are two forms of P-K formula but we mostly focused on first one which is

Q (z) = $\frac{B^*(\lambda-\lambda z)*(1-z)*(1-\rho)}{[B^*(\lambda-\lambda z)-z]}$ (Pollaczek-Khinchin formula)[6]

for mean queue length expression is Q_bar = $\frac{d}{dz}(Q(z))|_{z=1}$

here in this expression we have to find Laplace transform of given distribution.

There is one more form of P-K formula and whose calculation is easy to calculate mean queue length.

Q (z) = $\frac{B^*(\lambda-\lambda z)*(1-z)*(1-\rho)}{[B^*(\lambda-\lambda z)-z]}$,

Q'(1) = $\rho + \frac{\lambda^2 E[X^2]}{2(1-\rho)}$ [2 nd form of P-K Formula]

Using above formula we solve for below mentioned distribution queue length

3.2.1 Gamma and Erlang-k Distribution

Gamma distribution follows in many real life applications like in oncology , neuroscience , amount of rainfall etc. for example the amount of rainfall accumulated in reservoir is followed by gamma distribution. Other distribution like Erlang (when shape parameter can takes only positive integer value) , Exponential (shape parameter k=1) , Chi squared distribution are special cases of gamma distribution. Here we explained mean Queue length of Erlang-k distribution is less than M/M/1 mean queue length by using P-K formula.

$E[X] = \frac{1}{\mu} = \frac{k}{\beta}$ here k $\rightarrow$ Shape parameter (kϵN),β $\rightarrow$Scale parameter , $E[X^2] = \sigma_{Er}^2 + \mu_{Er}^2$; ($\mu_{Er}, \sigma_{Er}^2$ =Mean ,Variance of Erlang-2 distribution respectively)

$E[X^2] = \frac{k}{\beta^2} + \frac{k^2}{\beta^2} = \frac{k(k+1)}{\beta^2}$

By P-K Formula, for mean queue length Q_bar

Q_bar$_{Er} = \rho + \frac{(k+1)\rho^2}{2k(1-\rho)}$ (ρ =Utilization factor ,λ =Poisson arrival rate ,μ=Erlang-k Service rate)

For M/M/1 mean queue length Q_bar can be rewritten as $\rho + \frac{\rho^2}{(1-\rho)} < \rho + \frac{(k+1)\rho^2}{2k(1-\rho)}$ (Q_bar$_{Er}$,Erlang-k mean queue length)

Same expression will have for gamma also, but in case of gamma k can be any positive real value.

3.2.2 Inverse Gaussian Distribution

We will do the same analysis for mean queue length of Inverse Gaussian distribution as we did for Erlang-k distribution. It has two parameters μ(mean) and λ(shape) . $\mu, \lambda > 0$.

E[X] = $\frac{1}{\mu_{Inv-Gaussian}} = \mu$ here $\mu_{Inv-Gaussian}$is service rate of Inverse Gaussian Distribution ,

Variance $\sigma_{Inv-Gaissian}^2 = \frac{\mu^3}{\lambda}$, E[X^2] = $\sigma_{Inv-Gaissian}^2 + E^2[x] = \mu^2 + \frac{\mu^3}{\lambda}$

Q_bar$_{Inv-Gaussian} = \rho + \frac{\rho^2(1+\frac{\mu}{\lambda})}{2(1-\rho)}$

From above Expression we can say that if $\mu > \lambda$ then

Q_bar$_{Inv-Gaussian}$ > Q_bar$_{M/M/1}$

3.2.3 Chi-Squared Distribution

This distribution is used for hypothesis testing.It is special case of gamma distribution , which has one controlling parameter k (k ϵ N) called as degree of freedom. For k=2 Chi-squared behaves as a exponential distribution.

E[X] = $\frac{1}{\mu_{Chi-sqr}}$ = k , Var[X] = $\sigma^2_{Chi-sqr}$ = 2k , E[X^2] = k^2+2k

Q_bar$_{Chi-sqr}$ = ρ + $\frac{\rho^2 \frac{(k+2)}{k}}{2(1-\rho)}$

$\lambda, \mu_{Chi-sqr}$ are Poisson arrival rate, Chi-squared service rate respectively
So, for Chi-Squared distribution also we found that mean queue length is less than that of M/M/1.

3.2.4 Rayleigh Distribution

This distribution also have many real life application like it is used in MRI images,Human and animal nutrition linking , Wind direction analysis etc. It has only one scale parameter $\sigma(\sigma > 0)$.
Mean and variance is defined as E[X]=$\frac{1}{\mu_R} = \sigma\sqrt{\frac{\pi}{2}}$, Var[X] = $\frac{4-\pi}{2}\sigma^2$.
so, second moment E[X^2]= E^2[X] + Var[X] = $\frac{\sigma^2\pi}{2}$ + $\frac{4-\pi}{2}\sigma^2$=2σ^2
Mean queue length Q-bar$_R$
Q-bar$_R$= ρ + $\frac{2\rho^2}{\pi(1-\rho)}$
Hence, Q_bar$_R$<Q_bar$_{M/M/1}$

3.2.5 Pareto Distribution

Pareto distribution has widely application in real world . Sizes of sand particle,Human settlement rate in cities, wealth allocation in population this are the few applications where pareto distribution is used. In case of distribution of income of world population, 80% of wealth is controlled by 20% population this is also known as pareto principle.

Pareto distribution has two parameters α(Shape), x_m(Scale). Both α, x_mpossess positive real value. i.e. α, x_m>0.
Mean queue length analysis of Pareto,
E[X]= $\frac{1}{\mu_P} = \frac{\alpha x_m}{\alpha-1}$ $\alpha > 1$, μ_P $\rightarrow$Service rate with pareto distribution ,
Var[X]=$\frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}$ $\alpha > 2$, E[X^2]= E^2[X] + Var[X]= $\frac{\alpha^2 x_m^2}{(\alpha-1)^2}$ + $\frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}$
Q_bar$_P$= ρ + $\frac{\rho^2[1+\frac{1}{\alpha(\alpha-2)}]}{2(1-\rho)}$
So, for Pareto distribution also Q_bar$_P$< Q_bar$_{M/M/1}$.

3.2.6 Beta-Prime Distribution

Beta prime distribution has two controlling parameters α (Shape) ,β (Scale). Where $\alpha, \beta > 0$
E[X]= $\frac{1}{\mu_{B-P}} = \frac{\alpha}{\beta-1}$,Var[X] = $\frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1)$,

E[X^2]= E$^2[X]$+ Var[X]= $\frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1)$ + ($\frac{\alpha}{\beta-1})^2$

Mean queue length expression for Beta prime is, Here $\mu_{B-P}$is Beta prime service rate.
Q_bar$_{B-P}$=ρ + $\frac{\rho^2[\frac{1}{(\beta-2)}(\frac{\alpha+\beta-1}{\alpha})+1]}{2(1-\rho)}$

Now we know Expression for Q_bar$_{M/M/1}$,

Which is Q_bar$_{M/M/1}$ =$\rho + \frac{\rho^2}{1-\rho}$

So, by comparing Q_bar$_{B-p}$and Q_bar$_{M/M/1}$, Q_bar$_{B-p}$> Q_bar$_{M/M/1}$ only if $[\frac{1}{(\beta-2)}(\frac{\alpha+\beta-1}{\alpha}) + 1] > 2$ (In this case Coefficient of $\frac{\rho^2}{1-\rho}$ getting greater than 1(M/M/1)).

$\Longrightarrow \frac{1}{(\beta-2)}(\frac{\alpha+\beta-1}{\alpha}) + 1 > 2$

$\frac{1}{(\beta-2)}(\frac{\alpha+\beta-1}{\alpha})$ >1$\Longrightarrow \alpha + \beta - 1 > \alpha(\beta - 2)$. So, from this inequality this mean queue length expression Q_bar$_{B-p}$> Q_bar$_{M/M/1}$ holds.

3.3 Parameters for jitter minimization

- Positive and negative correlations.
- Queuing theory of M/M/1 and M/G/1 queues.
- Brief Discussion about Queue Modeling.

3.3.1 Positive and Negative Correlations:

Positively Correlated :- Positive correlation is a relationship between two variables such that if one variable increases or decreases then another one will also increase or decrease respectively.
Example 1

$X_{n+1} = \rho X_n$ for $I_n = 0$

$X_{n+1} = \rho X_n + E_n$ for $I_n = 1$ here I_n = Random binary sequence , E_n = Exponential random sequence , ρ = Probability of occurring zero .

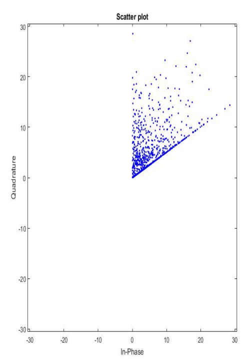

Figure 3.1 : Positive correlation example

Example 2

In day to day life also we observe many natural examples of positive correlations. one of them is relationship between rainfall and crop yield. We can easily say that as rainfall increase (up to certain limitation), crop yield also get increased.

we can write this relation in form of equations also,

Y= crop yield , R= rainfall amount , Y_0= Crop yield even when there is no rainfall , η =Rate of change of rainfall with respect to time.

$Y = Y_0 + \eta R$

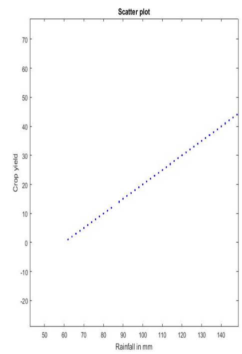

Figure 3.2 : Rainfall example of positive correlation

Negatively Correlated :- Negative correlation is a relationship between two variables in which one variable increases as the other decreases, and vice versa.
Example 1. We can considering example of smoking cigarettes versus life span of person. As someone smoking more number of cigarettes his/her life span will reduced by that proportion. So this two factors are negatively correlated.
If we wrote it in equation form then let
N = Number of cigarettes a person consumes , L= Life span of person ,
A= average life span of person , α= Rate with which life span reducing
L= A - αL

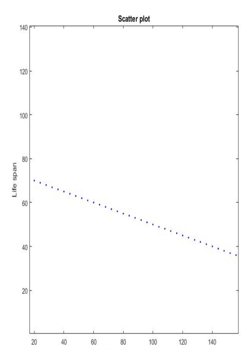

Figure 3.3 : Negative correlation example of cigarettes consumption

3.3.2 Queuing Theory basics of M/M/1 and M/G/1 queue:

3.3.2.1 M/M/1 Queue:-

The model name written in Kendall's Notation , M/M/1 means that the system has a Poisson arrival process, an exponential service time distribution, and one server. Arrivals occur at rate λ according to a Poisson process and move the process from state i to i + 1. Service times have an exponential distribution with rate parameter μ in the M/M/1 queue, where 1/μ is the mean service time. A single server serves customers one at a time from the front of the queue, according to a first-come, first-served discipline. When the service is complete the customer leaves the queue and the number of customers in the system reduces by one. The buffer is of infinite size, so there is no limit on the number of customers it can contain.

Some important Equations for M/M/1 Queue:-

λ =Arrival rate , μ =Service rate , $\rho = \frac{\lambda}{\mu}$ =Server Utilization ($\rho < 1$)

i. P_0 =1-ρ= Probability that there are Zero number of customer there in queue.

ii. $P_N = \rho^N(1-\rho)$= Probability that there are N number of customer there in queue.

iii.$Q = \frac{\rho}{1-\rho}$ =mean queue length (Mean number of customer in system)

iv. $N_q = \frac{\rho^2}{1-\rho}$= Mean number of customer in waiting in queue.

v. $W_q = \frac{\rho}{\mu(1-\rho)}$ Mean time spent by customer in queue.

3.3.2.2 M/G/1 Queue:-

An M/G/1 queue is a queue model where arrivals are Markovian (modulated by a Poisson process), service times have a General distribution and there is a single server. M (memory less) Poisson arrival process, intensity λ. G (General) general holding time distribution, mean S = 1/μ. This queue also has single server, load ρ = λS (in a stable queue one has ρ < 1). In spite of this, the mean queue length, waiting time, and sojourn time of the M/G/1 queue can be found. The results (the Pollaczek-Khinchin formulae) will be derived in the following.

Some important Equations for M/G/1 Queue:

i. Q (z) = $\frac{B^*(\lambda-\lambda z)*(1-z)*(1-\rho)}{[B^*(\lambda-\lambda z)-z]}$ (Pollaczek-Khinchin formula)
(λ =Arrival rate ,μ =Service rate , $\rho = \frac{\lambda}{\mu}$ =Server Utilization ($\rho < 1$).)

here $B^*(\lambda-\lambda z)=B^*(s)|_{s=(\lambda-\lambda z)}$= Laplace transform of pdf of general distribution, λ =Arrival Rate, ρ =Utilization Factor,
Q(z) = Z- transform of queue length.

ii. $Q = \rho + \frac{\lambda^2 E[X^2]}{2*(1-\rho)}$ (2nd representation of P-K formula)

here $E[X^2] = \sigma^2 + \mu^2$ = 2 nd moment generation of general distribution. , μ,σ^2 =mean and variance of general distribution respectively.

iii. $N_q = \frac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)}$

iv. $Wq = \frac{\lambda E[X^2]}{2(1-\rho)}$

By using 3rd equation mentioned in M/G/1 queue formulae we find mean queue length by considering different distributions. for most of distributions mean queue length is lesser than M/M/1 queue. For Beta-prime distribution mean queue length founds to be greater than of M/M/1. which is suitable for our model because as per 1981 paper IDI variance equation which is

IDI variance $= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - \left(\frac{1}{\lambda} - \frac{1}{\mu}\right) \frac{Q}{\lambda}$ [6]

Here we can see that IDI variance and mean queue length is negatively correlated. So we required larger queue length than M/M/1 model for lesser variance.

Chapter 4

Modeling the Jitter Buffer Queue

The dejitter buffer could be modeled as a queuing system in which service intervals are in our control. In this chapter we would consider those queues in which the service distributions (say, Y) are such that the following two requirements are satisfied (a) variance of IDI is less than that of variance of IATs (b) variance of IDI of the queue M/Y/1 is $<$ IDI variance of M/M/1 queue. We also study the cascading of such queues to explore the output jitter (equivalent to IDI variance) with property (a) resulting in further reduction of jitter.

4.1 M/B-P/1 Queue

M/B-P/1 queue is M/G/1 queue with poisson arrival process and Beta prime distribution as a service rate. So specification of beta prime queue is as follows
μ = mean = $\frac{\alpha}{\beta-1}$ (here, α and β are shape and scale parameter respectively.)
$\alpha, \beta > 0$
σ^2 =Variance = $\frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1)$
here just for derivation (to simplify mathematical analysis) we will assume α = 1,β = μ+1 , λ= μ-1
E[X] = $\frac{1}{\mu}$= $\frac{\alpha}{\beta-1}$ (satisfies as per our assumption)

$$Q = \rho + \frac{\lambda^2 E[X^2]}{2*(1-\rho)}$$

$$E[X^2] = \sigma^2 + \mu^2 = \frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1) + \left(\frac{\alpha}{\beta-1}\right)^2$$

$$\Longrightarrow Q = \rho + \lambda^2\left[\frac{\frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1)+(\frac{\alpha}{\beta-1})^2}{2(1-\rho)}\right]$$

By putting all values for α,β and λin terms of μ we get,

$$Q = \rho + \lambda^2\left[\frac{\frac{\beta}{(\beta-2)(\beta-1)^2}+(\frac{1}{\beta-1})^2}{2(1-\rho)}\right]$$

$Q = \rho + \lambda^2[\frac{\frac{2}{(\beta-2)(\beta-1)}}{2(1-\rho)}]$

$Q = \rho + \frac{\lambda}{\beta-2}\frac{\lambda}{\beta-1}\frac{1}{1-\rho}$ $(\beta - 1 = \mu, \beta - 2 = \lambda)$

$Q_{M/B-P/1} = \rho + \frac{\rho}{1-\rho}$

By comparing mean queue length results of M/M/1 and M/B-P/1 we can easily say that $Q_{M/B-P/1} > Q_{M/M/1}$ by amount of ρ .

4.1.1 Calculation of IDI variance for M/B-P/1

IDI variance $= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ \frac{Q}{\lambda}$
by putting value of Q , λwe get
$\sigma^2_{IDI,M/B-P/1} = \frac{(\mu-2)^2}{(\mu-1)^2\mu^2}$
similarly we can calculate IDI variance for M/M/1 then we get
$\sigma^2_{IDI,M/M/1} = \frac{(\mu-1)^2+1}{(\mu-1)^2\mu^2}$

4.1.2 we can plot IDI variance Vs mean for M/B-P/1 and M/M/1 queues.

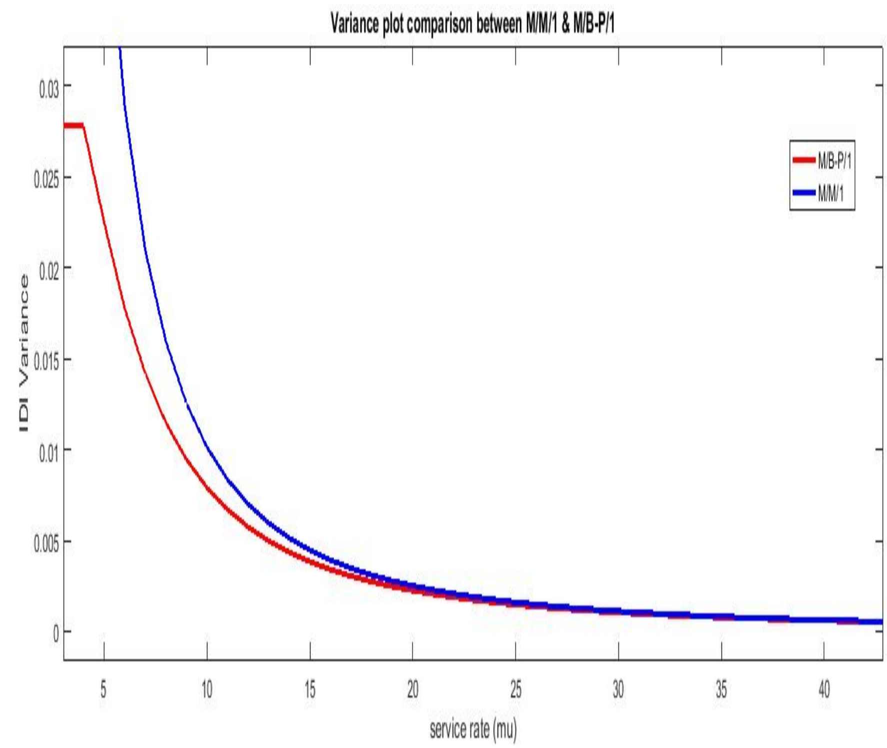

Figure 4.1 : M/M/1 and M/BP/1 IDI variance comparison

4.1.3 Mean queue length of Beta prime:-

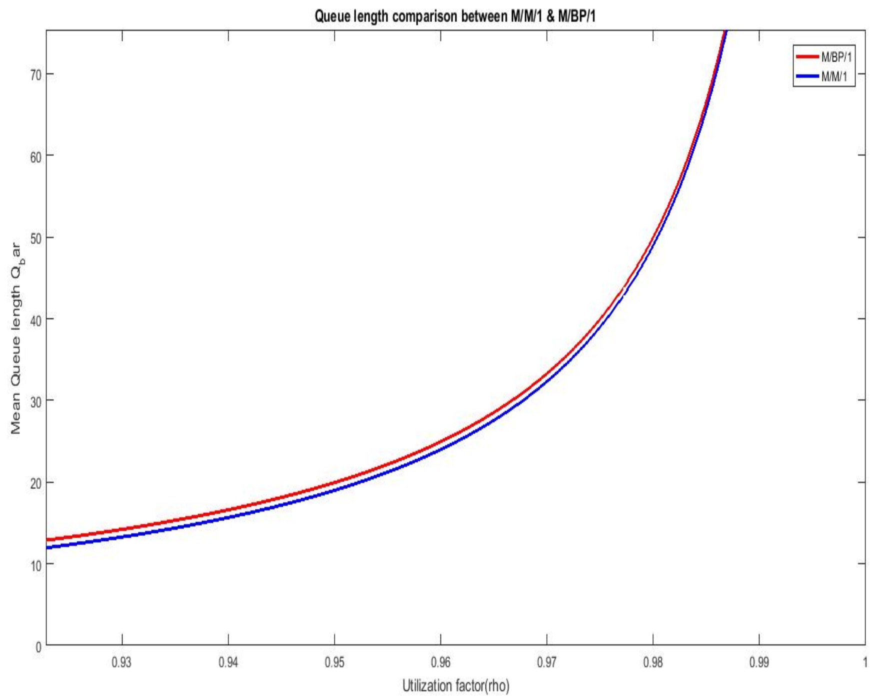

Figure 4.2 : Mean queue length comparison between M/M/1 and M/BP/1

4.1.4 Beta prime Mean waiting time comparison:

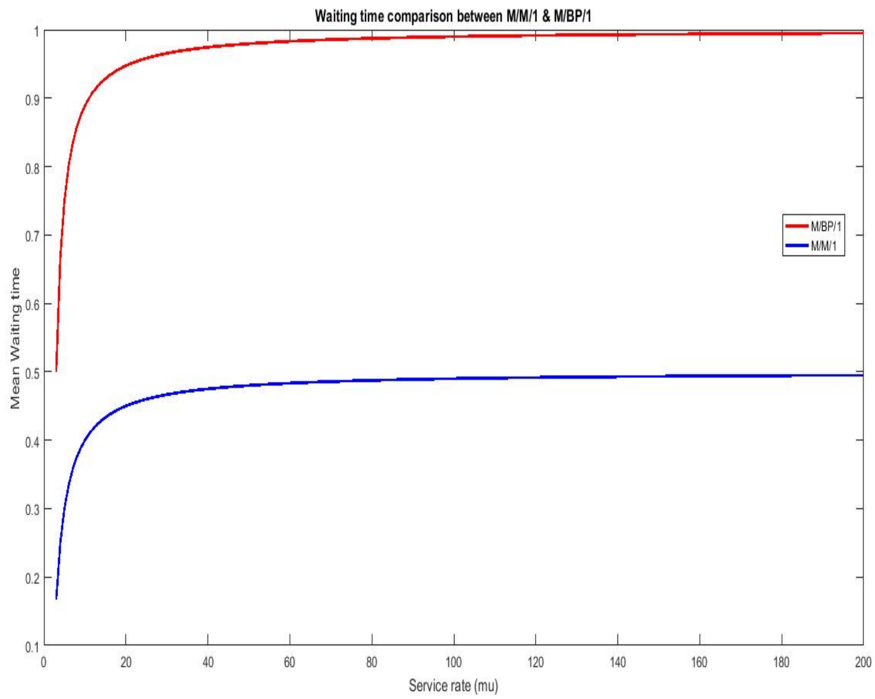

Figure 4.3 M/M/1 and M/BP/1 mean waiting time comparison

4.1.5 Correlation plot of Beta prime

we can explore Beta prime distribution little further and find out how its correlated.for that purpose we go through counts process and correlation plot.

- for 10000 points scatter plot looks like

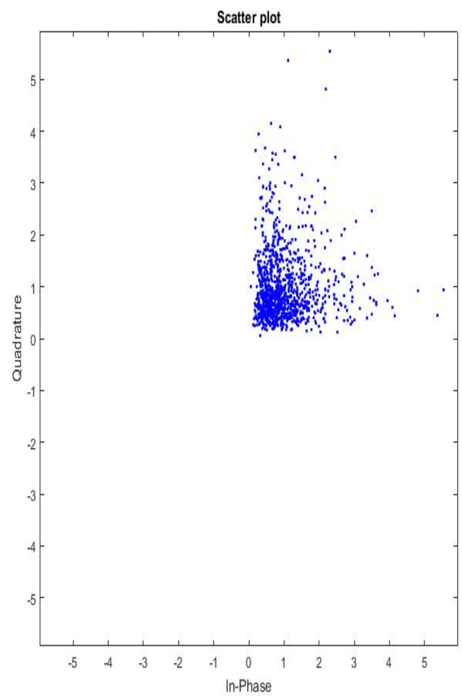

Figure 4.6 : Scatter plot for 1000 sample

4.2 M/I-G/1 Queue

We already discuss above in section 2.2.2 that M/I-G/1 queue has higher mean queue length than M/M/1 queue. So, by defining parameters condition lets analyses its IDI variance as compared to M/M/1

E[X] = $\frac{1}{\mu_{Inv-Gaussian}}$ = μ here $\mu_{Inv-Gaussian}$is service rate of Inverse Gaussian

Distribution , Variance $\sigma^2_{Inv-Gaissian}$ =$\frac{\mu^3}{\lambda}$ for this distribution λ, μthese are two shape and mean parameter respectively.
λ, μ>0 For calculation purpose lets assume $\mu = 2, \lambda = 1$,
$\mu_{Inv-Gaussian}$= Inverse Gaussian service rate,
λ(Poisson arrival rate) =($\mu_{Inv-Gaussian}$− r) ,Where r is any positive integer.

4.2.1 IDI variance of M/I-G/1 queue.

IDI variance $=\frac{r^2}{\mu^2\lambda^2}$
So, expression for IDI variance of M/M/1 queue is $\frac{(\mu-r)^2+r^2}{\mu^2\lambda^2} > \frac{r^2}{\mu^2\lambda^2}$(Expression of IDI variance of M/I-G/1)

4.2.2 IDI variance plot of M/IG/1

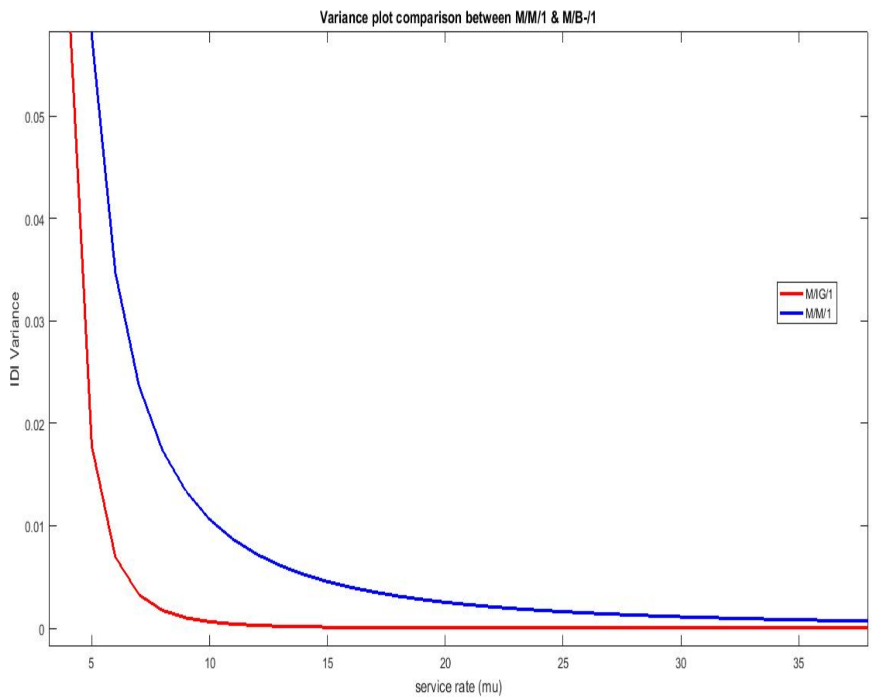

Figure 4.7 : Variance plot comparison between M/M/1 And M/IG/1

4.2.3 Mean queue length of M/IG/1

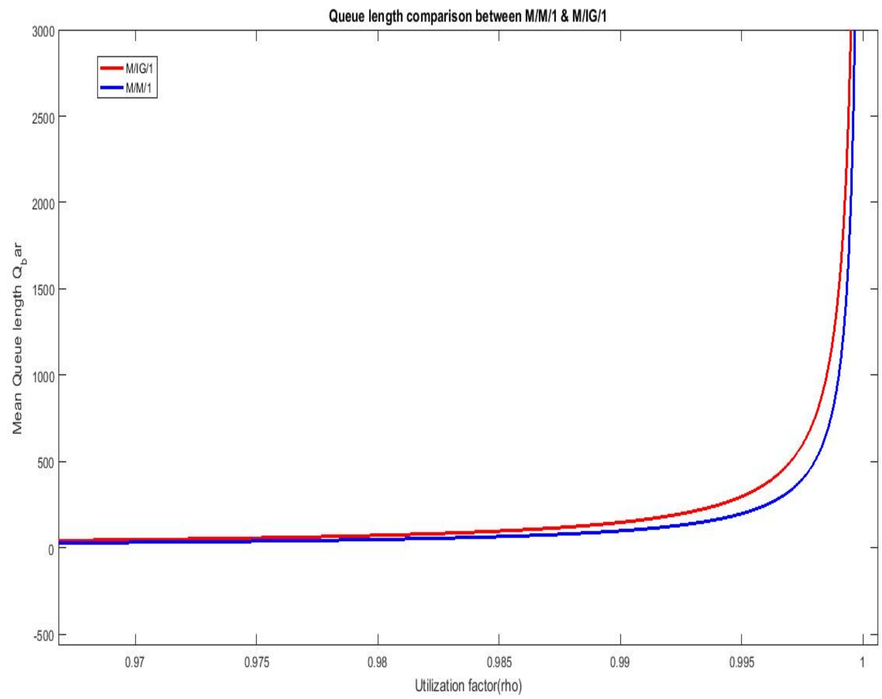

Figure 4.8 : Queue length comparison between M/M/1 and M/IG/1

4.2.4 Mean waiting time of M/IG/1

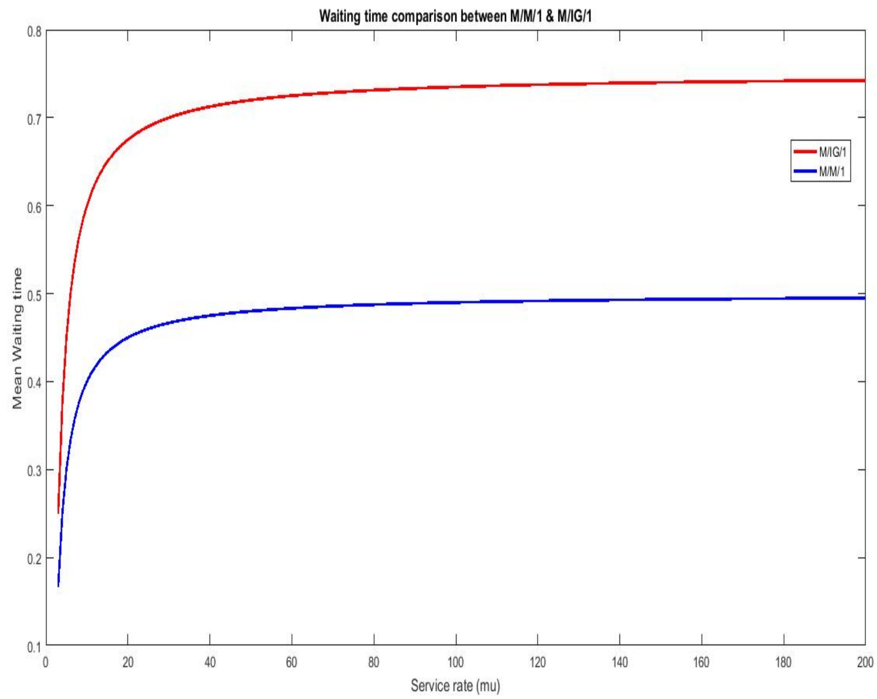

Figure 4.9 : Waiting time comparison between M/M/1 and M/IG/1

4.3 M/Wb/1

Weibull Distribution has wide area of application in engineering like In electrical engineering to represent over voltage occurring in an electrical system,In industrial engineering to represent manufacturing and delivery times,In radar systems to model the dispersion of the received signals level produced by some types of clutters etc. This distribution has two parameters k and λwhich are shape and scale parameter respectively.
Mean = $\lambda\Gamma(1+\frac{1}{k})$ and Variance = $\lambda^2[\Gamma(1+\frac{2}{k})-\Gamma^2(1+\frac{1}{k})]$
We will analyze M/Wb/1 queuing model for k=0.75

4.3.1 Mean Queue length of M/Wb/1

Mean queue length for M/Wb/1 $= \rho + \frac{\rho^2 \frac{\Gamma(1+\frac{2}{k})}{\Gamma^2(1+\frac{1}{k})}}{2(1-\rho)}$ for range of 0<k<1 , $Q_{IDI,M/Wb/1} > Q_{IDI.M/M/1}$.

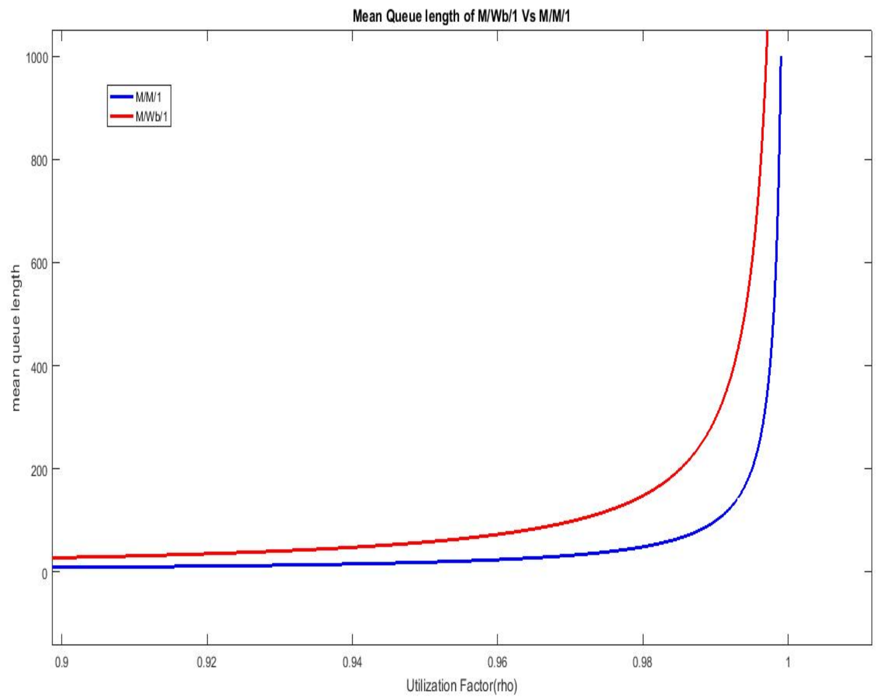

Fig. 4.10: Mean queue length comparison of M/Wb/1 and M/M/1

4.3.2 IDI variance plot of M/Wb/1 and M/M/1

IDI variance of M/Wb/1 can be expressed as $= \frac{0.1668(\mu-1)^2+1}{(\mu-1)^2\mu^2}$

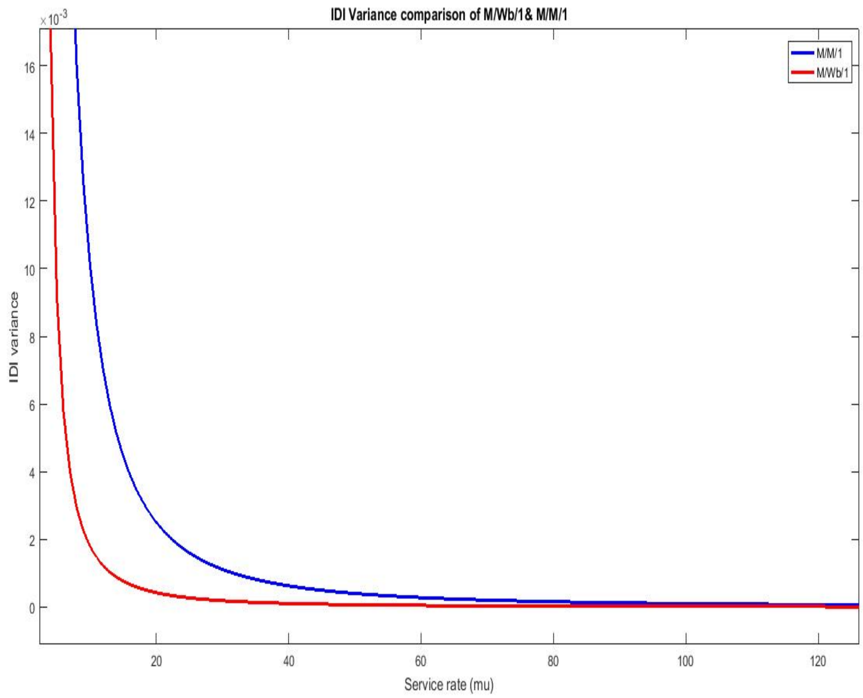

Fig. 4.11: IDI variance comparison of M/Wb/1 and M/M/1

4.4 Output Variance Reduction

Lets consider two Queues i.e. M/M/1 and M/BP/1 both has same arrival rate λ which is poisson, So we can say that Variance of inter arrival time for both distribution is same.
Which is, $\sigma^2_{M/M/1,IAT} = \sigma^2_{M/BP/1,IAT} = \frac{1}{\lambda^2}$ (1)

from burke's theorem we can say that, for M/M/1 $\sigma^2_{IAT} = \sigma^2_{IDI}$ (2)

We previously already showed that $\sigma^2_{IDI,M/BP/1} < \sigma^2_{IDI.M/M/1}$ (3)
now, for M/BP/1 queue variance ratio
R $= \frac{\sigma^2_{IAT,M/BP/1}}{\sigma^2_{IDI,M/BP/1}} = \frac{\sigma^2_{IDI.M/M/1}}{\sigma^2_{IDI,M/BP/1}} > 1$. From this result we can say that output variance is reduction takes place as compared to inter arrival intervals.

Chapter 5

Results and Conclusion

We have seen that the IDI variance for the M/G/1 Queue with Beta Prime and Inverse Gaussian Distribution as service Distributions found to be less than that of M/M/1 queue. In general for most of communication application we use M/M/1 as a queuing model for statistical modeling but where jitter plays major role in that case we can prefer this M/G/1 model as discussed to get better Quality of Service (QoS) in terms of delay variation and Jitter minimization.

5.1 Variance control by analytical independence of Mean(μ) & Variance(σ^2)

Distribution Name	Parameters	Mean (μ)	Variance($\sigma 2$)	Analytical independence[$\sigma^2 = f(\mu)$]
Beta Prime Distribution	$\alpha, \beta > 0$	$\frac{\alpha}{\beta-1}$	$\frac{\alpha(\alpha+\beta-1)}{(\beta-2)(\beta-1)^2}$	$\frac{\mu(\alpha+\beta-1)}{(\beta-1)^2}$, $\beta \uparrow$,$\sigma^2 \downarrow$
Gamma Distribution	$\alpha, \beta > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\frac{\mu}{\beta}$,$\beta \uparrow, \sigma^2 \downarrow$
Inverse Gamma Distribution	$\alpha, \beta > 0$	$\frac{\beta}{\alpha-1}$	$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$	$\frac{\mu^2}{\alpha-2}$, $\alpha \uparrow, \sigma^2 \downarrow$
Inverse Gaussian Distribution	$\mu, \lambda > 0$	μ	$\frac{\mu^3}{\lambda}$	$\frac{\mu^3}{\lambda}$,$\lambda \uparrow, \sigma^2 \downarrow$
Log Normal Distribution	$\mu > 0, \sigma\epsilon(-\infty, +\infty)$	$\exp(\mu + \frac{\sigma^2}{2})$	$[(\exp \sigma^2 - 1) \exp 2(\mu + \frac{\sigma^2}{2})]$	$(\exp \sigma^2 - 1)(\mu)^2$,$\sigma \downarrow$,$\sigma^2 \downarrow$
Scaled-Inverse Chi-Squared	ν ,$\tau^2 > 0$	$\frac{\nu\tau^2}{\nu-2}$	$\frac{2\nu^2\tau^4}{(\nu-2)^2(\nu-4)}$	$\frac{2\mu^2}{((\nu-4)}$, $\nu \uparrow$,$\sigma^2 \downarrow$

So, from above table we can observe that all mentioned distributions have analytical independence between their respective mean and variance. There is one controlling paraeter for each distribution which will leads to minimize required jitter by keeping mean constant, by varying that controlling parameter we can acquire our expected jitter.

As per designing perspective, we can design parameters of modeling distribution. Lets analyze Invesre Gamma distribution for designing CPRI standards.

mean of CPRI network is μ =1.2μs and allowed maximum jitter is σ =65ns.
So, mean of Inverse Gamma Distribution,
$\frac{\beta}{\alpha-1} = \frac{1}{1.2} \Rightarrow \beta = \frac{\alpha-1}{1.2}$
Variance of Inverse Gamma Distribution
$\sigma^2 = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \frac{\mu^2}{\alpha-2} = \frac{1}{1.2^2(\alpha-2)} \Longrightarrow \alpha - 2 = \frac{1}{1.2^2\sigma^2}$
$\Longrightarrow \alpha - 2 = \frac{1}{1.2^2(0.065)^2}$=164.365,$\alpha = 166.365$,$\beta$ =137.80

5.2 Queuing Models for Jitter minimization

In section 4, we have seen that we are able to find 3 such distribution as a servuce of M/G/1 which will give better performance than M/M/1 queue in terms of jitter minimization.

5.2.1 Mean Queue length

So, all analyzed distribution have larger mean queue length than M/M/1 queue. We compare Mean queue length for all distribution, plot for queue length will look like Fig.5.1

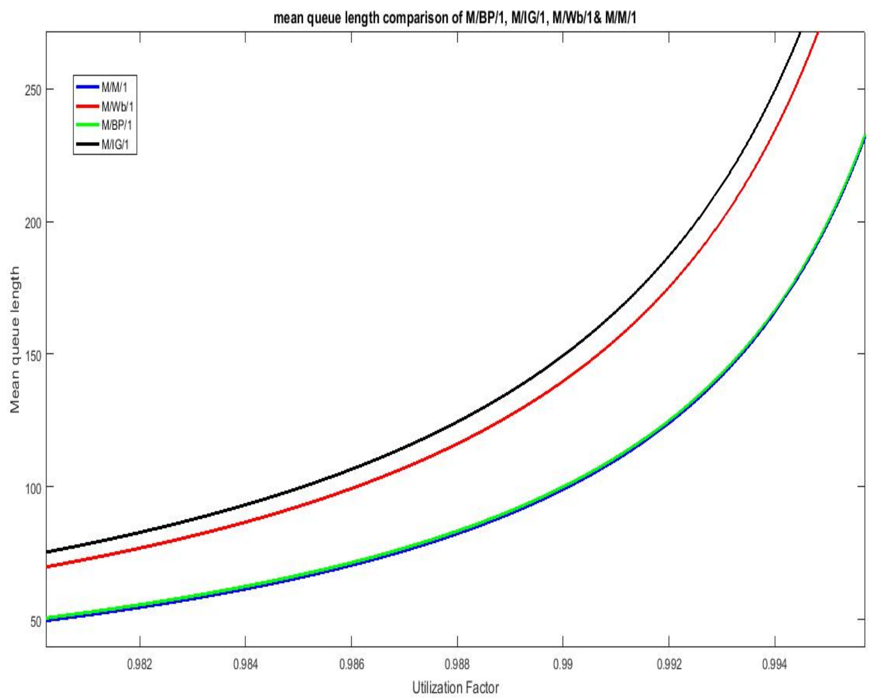

Fig.5.1: Mean queue length comparison of M/BP/1,M/IG/1,M/Wb/1 and M/M/1

We seen that mean queue length of queuing models follows order M/IG/1> M/Wb/1> M/BP/1> M/M/1

5.2.2 IDI Variance

For IDI variance of all distribution, we found that each distribution has less IDI variance as compared to M/M/1 model . There is restriction on Parameters of Beta prime and Weibull Distributions for obtaining less variance. While Inverse Gaussian Distribution will always give less variance for all parameters.Plot for IDI variance of mentioned distributions will looks like fig. 5.2

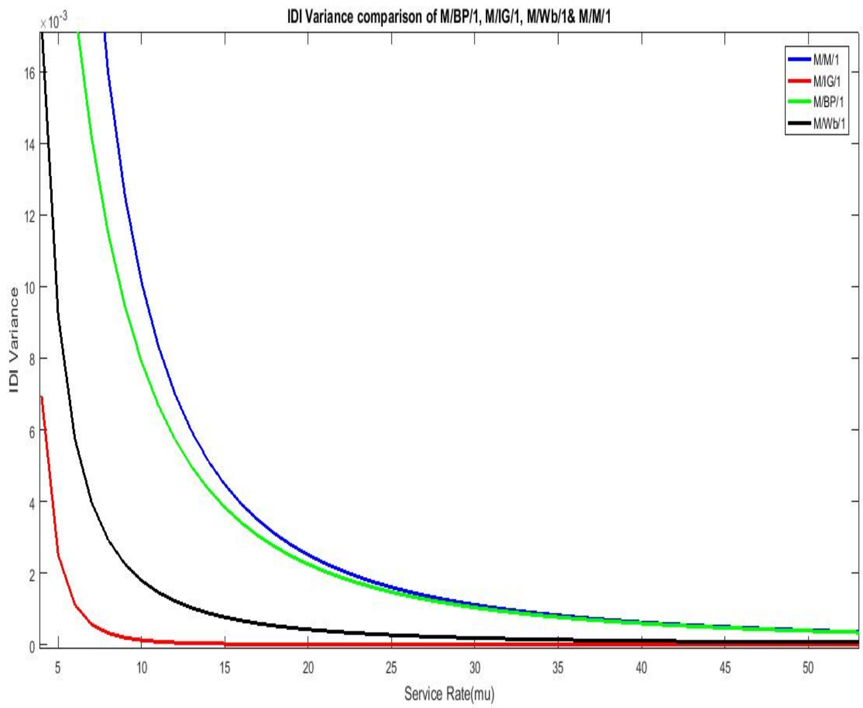

Fig.5.2: IDI Variance comparison of M/BP/1, M/IG/1, M/Wb/1 & M/M/1

IDI variance of this distribution follow this order M/M/1> M/BP/1> M/Wb/1> M/IG/1

5.3 Restrictions on parameters of M/BP/1, M/Wb/1 & M/IG/1

1. Beta prime Distribution has two parameters $\alpha, \beta > 0$.
For larger queue length and lesser Variance than M/M/1 queue α, β should satisfy below condition,
if,$\alpha + \beta - 1 > \alpha(\beta - 2)$.

2.Weibull Distribution has two parameters λ,k > 0
it will have its mean queue length greater than M/M/1 if 0<k<1, but variance will not be always positive for this range of k.
So, for positive variance, again k has range which is 0.72< k< 1.

3.Inverse Gaussian will give expected result for all values of its parameters.

References

[1] Muhammad Waqar, Ajung Kim, Peter K. CHO, "The Transport Scheme For Reducing Delays and Jitter In Ethernet Based 5G-Fronthoul Network", IEEE Access.2018.2864248, September 7,2018.

[2] " Introduction to Circuit Emulation Service Over Ethernet", The metro Ethernet Forum 2004, "http://www.mef.net/PDF_Documents/Introduction-to-CESoE.pdf".

[3] " Synchronous Digital Hierarchy (SDH)", The International Engineering Consortium, "http://www.iec.org/Web_proforum_tutorials/Chap04_part-2.pdf".

[4] S. Bryant, Ed. Cisco Systems, P. Pate, Ed. Overture network Inc. March 2005, " Pseudo Wire Emulation Edge-to-Edge (PWE3) Architecture", The Internet Society 2005.

[5] Slobodan Milijevic, Zarlink Semiconductor Inc., "The basics of synchronized Ethernet", EE Times-Asia, "https://www.microsemi.com/.../126475-the-basics-of-synchronized-ethernet-synce".

[6] Leonard Kleinrock, "Queueing Systems Volume 1: Theory Leonard Kleinrock".

Chapter 6

Appendix

Appendix A.1: Derivations for different distributions

A.1.1 Mean queue length derivations:

1. Gamma and Erlang distribution:

$E[X] = \frac{1}{\mu} = \frac{k}{\beta}$ here k $\rightarrow$ Shape parameter (kϵN),β $\rightarrow$Scale parameter , $E[X^2] = \sigma_{Er}^2 + \mu_{Er}^2$; ($\mu_{Er}, \sigma_{Er}^2$ =Mean ,Variance of Erlang-2 distribution respectively)

$E[X^2] = \frac{k}{\beta^2} + \frac{k^2}{\beta^2} = \frac{k(k+1)}{\beta^2}$

By P-K Formula, for mean queue length Q_bar

Q_bar$_{Er} = \rho + \frac{\lambda^2 E[X^2]}{2(1-\rho)}$ (ρ =Utilization factor ,λ =Poisson arrival rate ,μ=Erlang-k Service rate)

$$= \rho + \frac{\lambda^2 \frac{k(k+1)}{\beta^2}}{2(1-\rho)}$$

$$= \rho + \frac{\frac{(k+1)}{k}\rho^2}{2(1-\rho)} \; (\rho = \frac{\lambda}{\mu}, \frac{1}{\mu} = \frac{k}{\beta})$$

$$= \rho + \frac{(k+1)\rho^2}{2k(1-\rho)}$$

2. Inverse gamma Distribution:

E[X] = $\frac{1}{\mu_{Inv-Gaussian}} = \mu$ here $\mu_{Inv-Gaussian}$is service rate of Inverse Gaussian Distribution , Variance $\sigma_{Inv-Gaissian}^2 = \frac{\mu^3}{\lambda}$, E[X^2] = $\sigma_{Inv-Gaissian}^2 + E^2[x]$ $= \mu^2 + \frac{\mu^3}{\lambda}$

Q_bar$_{Inv-Gaussian} = \rho + \frac{\lambda_{Inv-Gaussian}^2 E[X^2]}{2(1-\rho)}$

$$= \rho + \frac{\lambda_{Inv-Gaussian}^2 (\mu^2 + \frac{\mu^3}{\lambda})}{2(1-\rho)}$$

$$= \rho + \frac{\lambda_{Inv-Gaussian}^2 \mu^2 (1 + \frac{\mu}{\lambda})}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2_{Inv-Gaussian} \frac{1}{\mu^2_{Inv-Gaussian}} (1+\frac{\mu}{\lambda})}{2(1-\rho)} \quad (\frac{1}{\mu_{Inv-Gaussian}} = \mu)$$

$$= \rho + \frac{\rho^2 (1+\frac{\mu}{\lambda})}{2(1-\rho)}$$

3. Chi-Squared Distribution:

E[X] = $\frac{1}{\mu_{Chi-sqr}}$ = k , Var[X] = $\sigma^2_{Chi-sqr}$ = 2k , E[X^2] = k^2+2k

Q_bar$_{Chi-sqr}$ = ρ + $\frac{\lambda^2 E[X^2]}{2(1-\rho)}$ ($\lambda, \mu_{Chi-sqr}$ are Poisson arrival rate, Chi-squared service rate respectively)

$$= \rho + \frac{\lambda^2 (k^2+2k)}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2 k(k+2)}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2 \frac{1}{\mu^2_{Chi-sqr}} \frac{(k+2)}{k}}{2(1-\rho)}$$

$$= \rho + \frac{\rho^2 \frac{(k+2)}{k}}{2(1-\rho)}$$

4.Rayleigh Distribution:

E[X]=$\frac{1}{\mu_R}$ = $\sigma\sqrt{\frac{\pi}{2}}$, Var[X] = $\frac{4-\pi}{2}\sigma^2$. so, second moment E[X^2]= E^2[X] + Var[X] = $\frac{\sigma^2\pi}{2}$ + $\frac{4-\pi}{2}\sigma^2$=2σ^2

Mean queue length Q-bar$_R$

Q-bar$_R$= ρ + $\frac{\lambda^2 E[X^2]}{2(1-\rho)}$

$$= \rho + \frac{\lambda^2 (2\sigma^2)}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2 (2\frac{2}{\pi\mu_R^2})}{2(1-\rho)} \quad (\sigma^2 = \frac{2}{\pi\mu_R^2})$$

$$= \rho + \frac{\frac{4}{\pi}\rho^2}{2(1-\rho)}$$

$$= \rho + \frac{2\rho^2}{\pi(1-\rho)}$$

5. Pareto Distribution:

E[X]= $\frac{1}{\mu_P}$ = $\frac{\alpha x_m}{\alpha-1}$ $\alpha > 1$, μ_P $\rightarrow$Service rate with pareto distribution , Var[X]= $\frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}$ $\alpha > 2$, E[X^2]= E^2[X] + Var[X]= $\frac{\alpha^2 x_m^2}{(\alpha-1)^2}$ + $\frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}$

Q_bar$_P$= ρ + $\frac{\lambda^2 E[X^2]}{2(1-\rho)}$

$$= \rho + \frac{\lambda^2 [\frac{\alpha^2 x_m^2}{(\alpha-1)^2} + \frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}]}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2 \frac{\alpha^2 x_m^2}{(\alpha-1)^2} [1 + \frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)\frac{\alpha^2 x_m^2}{(\alpha-1)^2}}]}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2 \frac{1}{\mu_P^2} [1+\frac{1}{\alpha(\alpha-2)}]}{2(1-\rho)}$$

$$= \rho + \frac{\rho^2 [1+\frac{1}{\alpha(\alpha-2)}]}{2(1-\rho)}$$

6. Beta prime Distribution:

E[X]= $\frac{1}{\mu_{B-P}}$ = $\frac{\alpha}{\beta-1}$,Var[X] = $\frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1)$, E[X^2] = E$^2[X]$+ Var[X]= $\frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1) + (\frac{\alpha}{\beta-1})^2$

Mean queue length expression for Beta prime is, Here $\mu_{B-P}$is Beta prime service rate.

Q_bar$_{B-P}$= $\rho + \frac{\lambda^2 E[X^2]}{2(1-\rho)}$

$$= \rho + \frac{\lambda^2[\frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1)+(\frac{\alpha}{\beta-1})^2]}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2(\frac{\alpha}{\beta-1})^2[\frac{\alpha}{(\beta-2)(\beta-1)^2}(\alpha+\beta-1)\frac{1}{(\frac{\alpha}{\beta-1})^2}+1]}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2(\frac{\alpha}{\beta-1})^2[\frac{1}{(\beta-2)}(1+\frac{\beta-1}{\alpha})+1]}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2\frac{1}{\mu^2_{B-P}}[\frac{1}{(\beta-2)}(\frac{\alpha+\beta-1}{\alpha})+1]}{2(1-\rho)}$$

$$=\rho + \frac{\rho^2[\frac{1}{(\beta-2)}(\frac{\alpha+\beta-1}{\alpha})+1]}{2(1-\rho)}$$

A.1.2 IDI variance Derivations:

1. Exponential Distribution(M/M/1):

IDI variance for M/M/1 then we get

$\sigma^2_{IDI,M/M/1} = \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ [\frac{\frac{\rho}{1-\rho}}{\lambda}\ ]$

$\sigma^2_{IDI,M/M/1} = \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{\mu-\lambda}{\lambda\mu})\ [\frac{\lambda}{\lambda}\ ]$ (put $\frac{\rho}{1-\rho} = \lambda$, $\lambda = \mu - 1$)

$\sigma^2_{IDI,M/M/1} = \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda\mu})$

$\sigma^2_{IDI,M/M/1} = \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\mu(\mu-1)})$

$\sigma^2_{IDI,M/M/1} = \frac{1}{\lambda^2} + \frac{2}{\mu}[\frac{1}{\mu} - \frac{1}{\mu-1}]$

$\sigma^2_{IDI,M/M/1} = \frac{1}{\lambda}[\frac{1}{\lambda} - \frac{2}{\mu^2}]$

$\sigma^2_{IDI,M/M/1} = \frac{(\mu^2-2(\mu-1))}{\lambda^2\mu^2}$

$\sigma^2_{IDI,M/M/1} = \frac{(\mu-1)^2+1}{(\mu-1)^2\mu^2}$

2. Beta-prime distribution:

IDI variance = $\frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ \frac{Q}{\lambda}$

by putting value of Q , λwe get

$\sigma^2_{IDI} = \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ [\frac{\rho+\frac{\rho}{1-\rho}}{\lambda}\ ]$

$\sigma^2_{IDI} = \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda\mu})[\frac{\rho+\lambda}{\lambda}\ ]$ ....(By simplification we get $\frac{\rho}{1-\rho} = \lambda$, just put $\lambda = \mu - 1, \rho = \frac{\lambda}{\mu}$)

$\sigma^2_{IDI} = \frac{1}{\lambda^2} + \frac{2}{\mu^2} - \frac{2}{\lambda\mu} - \frac{2}{\lambda\mu^2}$

$\sigma^2_{IDI} = \frac{1}{\lambda^2}$- $\frac{2}{\lambda\mu^2} + \frac{2}{\mu}[\frac{1}{\mu} - \frac{1}{\mu-1}]$........................ ($\lambda = \mu - 1$)

$\sigma^2_{IDI} = \frac{1}{\lambda^2}$- $\frac{4}{\lambda\mu^2}$

$\sigma^2_{IDI} = \frac{1}{\lambda}[\frac{1}{\lambda} - \frac{4}{\mu^2}]$

$\sigma^2_{IDI} = \frac{1}{\lambda}[\frac{\mu^2-4(\mu-1)}{\lambda\mu^2}]$

$\sigma^2_{IDI,M/B-P/1} = \frac{(\mu-2)^2}{(\mu-1)^2\mu^2}$

3. Inverse gaussian Distribution:

IDI variance $= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ \frac{Q-bar}{\lambda}$

$$= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ \frac{\rho + \frac{\rho^2(1+\frac{2}{1})}{2(1-\rho)}}{\lambda}$$

$$= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ \frac{\rho + \frac{3\rho^2}{2(1-\rho)}}{\lambda}$$

$$= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ \frac{(\frac{\rho}{1-\rho} + \frac{\rho^2}{2(1-\rho)})}{\lambda}$$

$$= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ \frac{(\frac{\rho}{1-\rho})(1+\frac{\rho}{2})}{\lambda}$$

$$= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\lambda} - \frac{1}{\mu})\ \frac{(\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}})(1+\frac{\rho}{2})}{\lambda}$$

$$= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{\mu-\lambda}{\mu\lambda})\ \frac{\lambda(1+\frac{\rho}{2})}{(\mu-\lambda)\lambda}$$

$$= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\mu\lambda})\ (1 + \frac{\rho}{2})$$

$$= \frac{1}{\lambda^2} + \frac{2}{\mu^2} - 2\ (\frac{1}{\mu\lambda})\ (1 + \frac{\lambda}{2\mu})$$

$$= \frac{1}{\lambda^2} + \frac{2\lambda}{\mu^2\lambda} - (\frac{2\mu+\lambda}{\mu^2\lambda})$$

$$= \frac{1}{\lambda^2} + \frac{2\lambda-2\mu-\lambda}{\mu^2\lambda}$$

$$= \frac{1}{\lambda^2} + \frac{\lambda-\mu-\mu}{\mu^2\lambda}$$

$$= \frac{1}{\lambda^2} - \frac{(\mu+r)}{\mu^2\lambda}$$

$$= \frac{\mu^2}{\mu^2\lambda^2} - \frac{(\mu+r)\lambda}{\mu^2\lambda^2}$$

$$= \frac{\mu^2-(\mu+r)(\mu-r)}{\mu^2\lambda^2}$$

$$= \frac{r^2}{\mu^2\lambda^2}$$

A.2 P-K 2nd Formula Derivation:

Q (z) = $\frac{B^*(\lambda-\lambda z)*(1-z)*(1-\rho)}{[B^*(\lambda-\lambda z)-z]}$,

A(z) = B*($\lambda - \lambda z$) , A(1) = 1 ,A'(1) = $\lambda E[x] = \rho$, A"(1) =$\lambda^2 E[X^2]$

Q (z) = $\frac{A(z)*(1-z)*(1-\rho)}{[A(z)-z]}$ = $\sum_{i=0}^{\infty} a_i z^i$ (Z- transform of queue length)

Q(1) = $\lim_{z\to1} \frac{(1-\rho)[(1-z)A'(z)-A(z)]}{A'(z)-1}$= -$\frac{(1-\rho)}{\rho-1}$ = 1 (for simplifying this we have used L'hospital's rule)

So, Q(1) =A(1) =1

Q(z) [A(z)-z] = (1-ρ) (1-z) A(z) (from main expression only)

Q'(z)[A(z)-z] + Q(z)[A'(z)-1] = (1-ρ) (1-z) A'(z) - (1-ρ) A(z) (By differentiating both sides)

Q"(z)[A(z)-z] + 2 Q'(z)[A'(z)-1] + Q(z) A"(z) = (1-ρ) (1-z) A"(z) -2 (1-ρ) A'(z) (By double differentiating both side)

Putting z = 1 and using earlier results we get

-2 Q'(1) (1-ρ) +$\lambda^2 E[X^2]$ = (1-ρ)(-2ρ)

Q'(1) = ρ + $\frac{\lambda^2 E[X^2]}{2(1-\rho)}$ [2 nd form of P-K Formula