General Compute and Forward for Virtual Full Duplex Relaying

A Project Report

submitted by

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THESIS CERTIFICATE

This is to certify that the thesis titled **General Compute and Forward for Virtual Full-Duplex Relaying**, submitted by **Roshan S.Sam (EE16M080)**, to the Indian Institute of Technology, Madras, for the award of the degree of **Master of Technology**, is a bona fide record of the research work done by her under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

KEYWORDS: Virtual full-duplex relaying; Successive Relaying; General Compute and Forward (GCoF); Non-integer penalty; Cutset upper bound

Motivated by the wireless backhaul application, multihop virtual full duplex relaying using a successive relaying protocol based on compute-and-forward (CoF) was proposed recently by Hong and Caire. The channel gain in each hop was assumed to be equal. In this work, we consider multihop virtual full duplex relaying where the gain in the different hops can be unequal. We use the recently proposed general compute-and-forward (GCoF) scheme along with successive relaying. GCoF eliminates the non-integer penalty present in CoF or the CoF with simple power allocation used earlier. We determine the achievable rate of virtual full duplex relaying using GCoF for the multihop case and show that this rate is within a constant gap (also independent of the number of hops) of the cutset upper bound under some mild assumptions.

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ABBREVIATIONS

CoF	Compute and Forward
DPC	Dirty Paper Coding
QMF	Quantize Map and Forward
DF	Decode and Forward
AF	Amplify and Forward
IC	Interference Channel

NOTATION

$\mathcal{N}(\mu, \boldsymbol{\sigma^2})$	Normal distribution with mean μ and variance σ^2
$i.i.d$	Independent and Identically distributed
$\mathbb{R}$	Set of real numbers

CHAPTER 1

INTRODUCTION

In communication networks, relays are used to improve the network coverage and throughput when the source and destination are far apart and cannot communicate with each other efficiently. In a full duplex relay, the relay transmits and receives in the same time slot. The implementation of a full duplex relay is difficult because of the huge amount of self-interference .

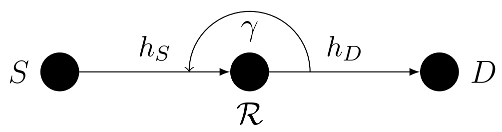
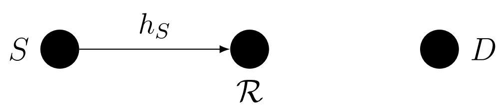
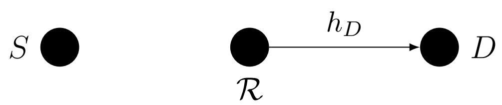


Figure 1.1: Two-hop relay network with full-duplex relay

In case of a half-duplex relay, the relay can either transmit or receive at a time. Since a half-duplex relay can forward a message from source to destination over two time slots, it makes an inefficient use of the radio channel resource.



(a) Time Slot 1



(b) Time Slot 2

Figure 1.2: Two-hop relay network with half-duplex relay

Hence, a full duplex relay is implemented by using 2 half-duplex relays. This is called a virtual full duplex relay. At each time slot, one of the relays receives a new data from the source while the other relay forwards the processed data to the destination. The role of relays is swapped at each time interval. This relaying operation is known as successive relaying. The main issue is the inter-relay interference, corresponding to the self interference in full-duplex relays.

(a) Even Time Slot (b) Odd Time Slot

Figure 1.3: Virtual full-duplex relay

The relay does some form of processing on the received data before transmitting to destination. Depending on the processing that is done, we have different relaying schemes. The capacity and the corresponding optimal relaying schemes of general relay channels are not known. Among the various relaying schemes, compute-and-forward (CoF) relaying Nazer and Gastpar (2011) is an important relaying scheme. CoF and the related physical layer network coding strategies were initally studied extensively in the context of two-way relay channels Wilson *et al.* (2010); Nam *et al.* (2010).

1.1 Previous work

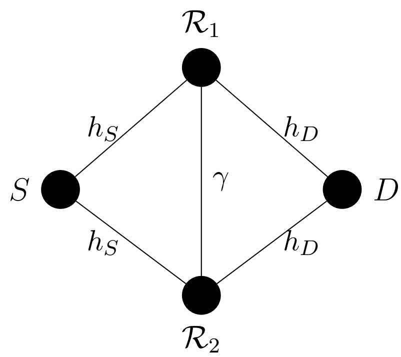


Figure 1.4: 2 hop network

Consider the 2-hop relay network in Fig. 1.4. In Hong and Caire (2015), they consider the case when the channel coefficients h_S and h_D are equal to unity, and each node has the same

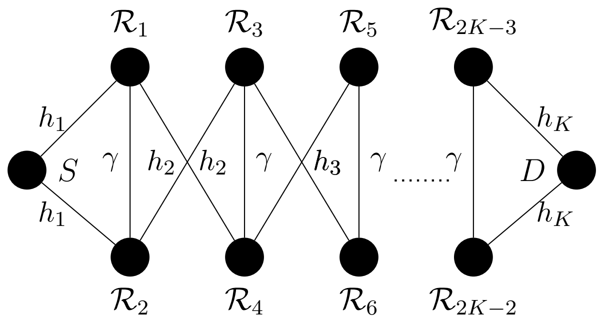


Figure 1.5: K-hop network

average transmit power constraint. Some relevant results in Hong and Caire (2015) are that:
(1) successive relaying with dirty paper coding (DPC) achieves the cutset bound when SNR ≥ 1,
(2) CoF with a simple power allocation can achieve rates within a constant gap of the cutset bound for a K-hop relay network with equal hop gains (Fig. 1.5 with $h_1 = h_2 = \ldots = h_K = 1$), and
(3) the gap between CoF with power allocation and the cutset bound grows linearly in the number of hops. The cutset bound and gap results in this paper are valid only for the network with equal hop gains. The equal hop gains are assumed to be achieved in the backhaul application by appropriate placement of nodes and power adjustment.

1.2 Proposed Algorithm

In this work, we consider the more general K-hop network in Fig. 1.5. We, therefore, do not require each hop to have the same gain. (Note that this unequal gain network cannot be reduced to a network with equal gains and equal power constraints at all nodes.) We do assume as in Hong and Caire (2015) that the distance between two relays in the same hop is much smaller than the hop distance, so that the gain for the two relays in each hop is the same. The CoF scheme used in Hong and Caire (2015) is based on Theorem 1 in Nazer and Gastpar (2011). The rates are found using two methods: with Power Allocation (PA) and without Power Allocation. In this work, we use the General Compute and Forward (GCoF) formula given in Zhu and Gastpar (2017) for our protocol. We will denote our scheme as the GCoF scheme. We obtain the following results:

(1) For the 2-hop network, we show that successive relaying with DPC is within 1 bit of the cutset bound under all conditions,

(2) We derive an expression for the rate achieved using the GCoF scheme over a K-hop network,

(3) We show that the gap between GCoF and the cutset bound is finite as long as the first or last hop is the bottleneck, i.e., $\min\{h_1^2, h_2^2, h_3^2, ..., h_K^2, \gamma^2\} = h_1^2$ or h_K^2, and h_k^2SNR $\geq 1, \forall k$. Furthermore, this gap does not grow linearly with the number of hops as in Hong and Caire (2015).

(4) Also, in Zhu and Gastpar (2015) they consider a symmetric many-to-one Interference Channel (IC) and show the conditions under which each user can achieve capacity. In this work, we extend this result to a non-symmetric many-to-one IC. The coding scheme used for this channel is based on the GCoF scheme.

1.3 Organisation of the thesis

Chapter 2 describes the system model that is used throughout the paper.

In chapter 3, the expression for the cut-set bound for a 2-hop network is found.

In chapter 4, we discuss the Dirty Paper Coding and the rate that can be achieved by it. We also find the gap between the cutset bound and the rate of DPC. We then state and prove the conditions under which DPC achieves the cutset bound.

Chapter 5 discusses the Compute and Forward protocol. The GCoF rate is found for a 2-hop and 3-hop network and then extended to a general K-hop network. We also derive an upper bound on the cut-set bound for a K-hop network. The gap between the cut-set bound and the rate achieved by GCoF is also studied for special cases.

The non-symmetric many-to-one IC results are given in Chapter 6.

The simulation results are given in Chapter 7.

CHAPTER 2

SYSTEM MODEL

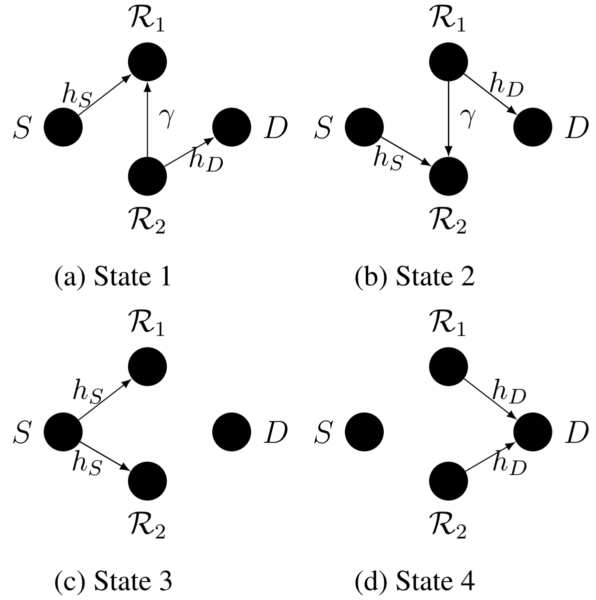


Figure 2.1: States of a 2-hop network

In successive relaying, only the state 1 and state 2 of the 4 states given in Fig. 2.1 (i.e, Fig. 2.1(a) and Fig. 2.1(b)) are taken by the 2-hop network. In even time slots, the network is in state 1 and in odd time slots, it is in state 2. In the even time slots, the source sends a message to relay 1 and the relay 2 sends to destination. Because of the inter-relay interference level γ, the relay 1 also receives what is sent by relay 2. The role of relays 1 and 2 is reversed in odd time slots. The encoding and decoding is done over n channel uses of a discrete time Gaussian channel.

For odd time slot ,

$$\underline{\mathbf{y}}_{\mathcal{R}_2}[t] = h_S \underline{\mathbf{x}}_S[t] + \gamma \underline{\mathbf{x}}_{\mathcal{R}_1}[t] + \underline{\mathbf{z}}_{\mathcal{R}_2}[t]$$
$$\underline{\mathbf{y}}_D[t] = h_D \underline{\mathbf{x}}_{\mathcal{R}_1}[t] + \underline{\mathbf{z}}_D[t]$$

For even time slot ,

$$\underline{\mathbf{y}}_{\mathcal{R}_1}[t] = h_S\underline{\mathbf{x}}_S[t] + \gamma\underline{\mathbf{x}}_{\mathcal{R}_2}[t] + \underline{\mathbf{z}}_{\mathcal{R}_1}[t]$$
$$\underline{\mathbf{y}}_D[t] = h_D\underline{\mathbf{x}}_{\mathcal{R}_2}[t] + \underline{\mathbf{z}}_D[t]$$

where $\gamma \in \mathbb{R}$ is the inter-relay interference level and $h_S \in \mathbb{R}$ and $h_D \in \mathbb{R}$ are the channel gains from the source to relay and relay to destination respectively. Here $\underline{\mathbf{x}}_S[t] \in \mathbb{R}^{1\text{x}n}$ and $\underline{\mathbf{x}}_{\mathcal{R}_k}[t] \in \mathbb{R}^{1\text{x}n}$ are the signals transmitted by source and relay k . Also, $\underline{\mathbf{y}}_D[t] \in \mathbb{R}^{1\text{x}n}$ and $\underline{\mathbf{y}}_{\mathcal{R}_k}[t] \in \mathbb{R}^{1\text{x}n}$ denote the received signals at destination and relay k respectively. Noise is assumed to be i.i.d. Gaussian with zero mean and unit variance (denoted by $\mathcal{N}(0,1)$). Power constraint at each transmitter is denoted by SNR.

CHAPTER 3

CUTSET BOUND

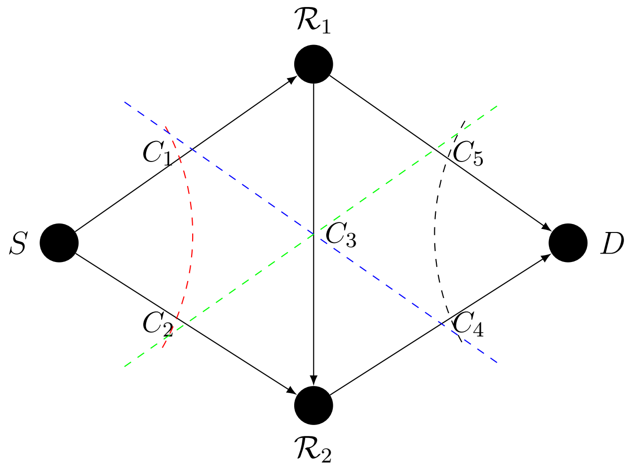
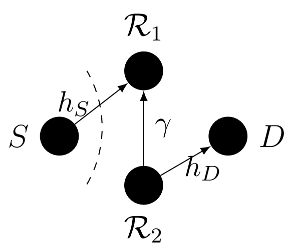
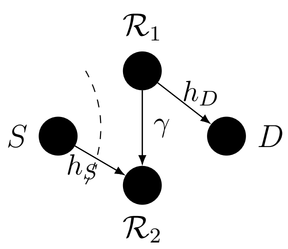


Figure 3.1: $C = \min\{C_1 + C_2, C_2 + C_3 + C_4, C_4 + C_5, C_1 + C_5\}$

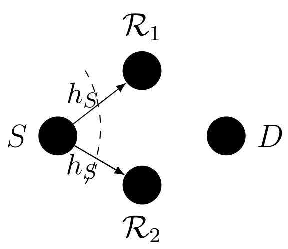
Cut-set bound serves as an upper bound for capacity. Suppose we want to find the maximum information flow from S to D and each of the edges have capacities C_i (see Fig. 3.1). The maximum information flow across any cut-set cannot be greater than the sum of the capacities across cut edges. Thus minimizing the maximum flow across cut sets yields an upper bound on the capacity of the network. The cut-set bound for the 2-hop network is given by considering 4 cuts as follows. The rate corresponding to each state for a given cut is shown below.



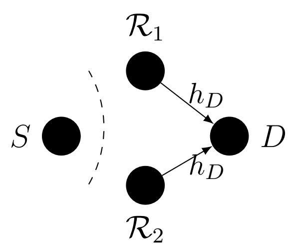
(a) State 1-C_1



(b) State 2-C_1

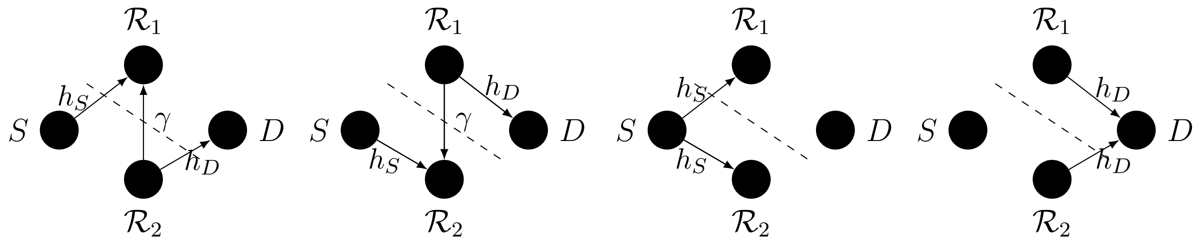


(c) State 3-C_2



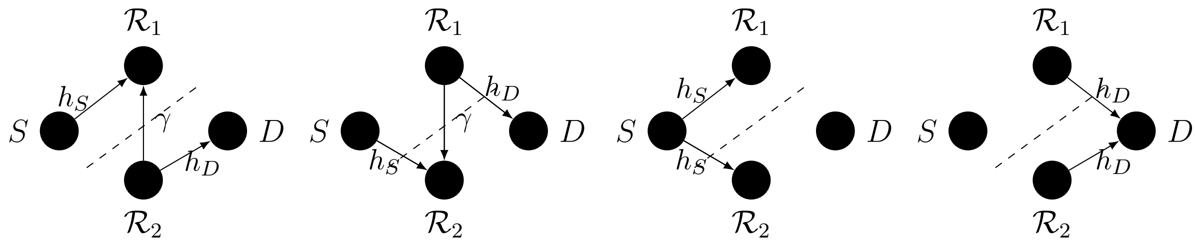
(d) State 4-0

Figure 3.2: Cut 1



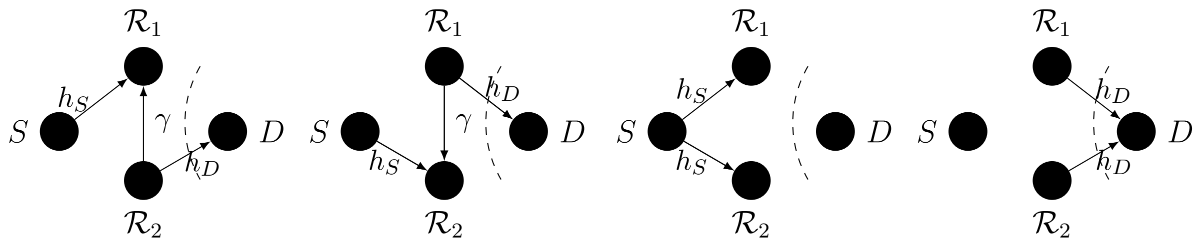
(a) State 1-C_3 (b) State 2-0 (c) State 3-C_1 (d) State 4-C_4

Figure 3.3: Cut 2



(a) State 1-0 (b) State 2-C_3 (c) State 3-C_1 (d) State 4-C_4

Figure 3.4: Cut 3



(a) State 1-C_4 (b) State 2-C_4 (c) State 3-0 (d) State 4-C_5

Figure 3.5: Cut 4

Here, $C_1 = C(h_S^2\text{SNR})$,

$C_2 = C(2h_S^2\text{SNR})$,

$C_3 = C\left((h_S^2 + h_D^2 + \gamma^2)\text{SNR} + h_D^2 h_S^2 \text{SNR}^2 + \frac{\gamma^2}{h_D^2}\right)$,

$C_4 = C(h_D^2\text{SNR})$,

$C_5 = C(4h_D^2\text{SNR})$

The cut-set bound is obtained as follows.

$$R_{cut-set} = \max_{\substack{t_1,t_2,t_3,t_4 \\ \text{s.t } t_1+t_2+t_3+t_4=1 \\ t_1,t_2,t_3,t_4 \geq 0}} \min\{I_1,\ I_2,\ I_3,\ I_4\}$$

where

$$\begin{aligned}
I_1 &\triangleq t_1 C(h_S^2 \text{SNR}) + t_2 C(h_S^2 \text{SNR}) + t_3 C(2h_S^2 \text{SNR}) \\
I_2 &\triangleq t_2 C\left((h_S^2 + h_D^2 + \gamma^2) SNR + h_D^2 h_S^2 \text{SNR}^2 + \frac{\gamma^2}{h_D^2}\right) + t_3 C(h_S^2 \text{SNR}) + t_4 C(h_D^2 \text{SNR}) \\
I_3 &\triangleq t_1 C\left((h_S^2 + h_D^2 + \gamma^2) \text{SNR} + h_D^2 h_S^2 \text{SNR}^2 + \frac{\gamma^2}{h_D^2}\right) + t_3 C(h_S^2 \text{SNR}) + t_4 C(h_D^2 \text{SNR}) \\
I_4 &\triangleq t_1 C(h_D^2 \text{SNR}) + t_2 C(h_D^2 \text{SNR}) + t_4 C(4h_D^2 \text{SNR})
\end{aligned}$$

Here, t_1, t_2, t_3, t_4 is the fraction of time the 2-hop network is in states 1,2,3 and 4. Also, I_1, I_2, I_3, I_4 is the maximum information flow corresponding to the 4 possible cuts of a 2-hop network.

$C(\text{SNR}) = 0.5 \log_2(1 + \text{SNR})$

CHAPTER 4

SUCCESSIVE RELAYING WITH DPC

Dirty Paper Coding (DPC) is a technique for efficient transmission of data through a channel subjected to some interference known to the transmitter. The technique consists of precoding the data in order to cancel the effect caused by interference. For a two-hop network with single relay stage and inter-relay interference level γ, where $h_S = h_D = 1$, Dirty Paper Coding (DPC) is optimal, i.e., it can achieve the performance of ideal full-duplex relay.[Changiz Rezaei *et al.* (2008) Chang *et al.* (2007)]

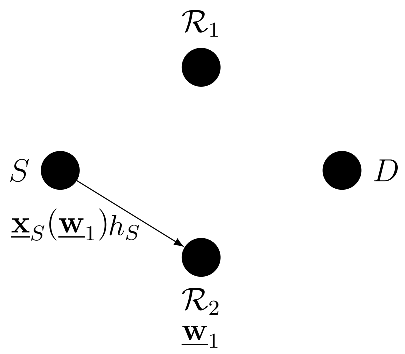


Figure 4.1: Time slot 1

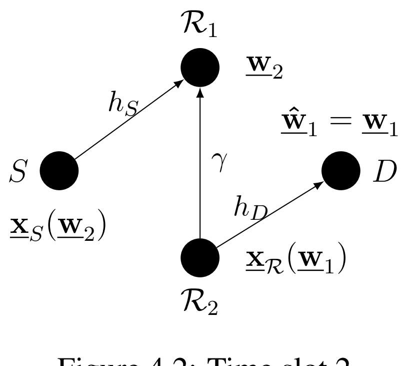


Figure 4.2: Time slot 2

For a 2-hop network, we can achieve the rate of ideal full-duplex relay by using half-duplex relays with successive relaying and DPC. We analyse the action of DPC in each of the time

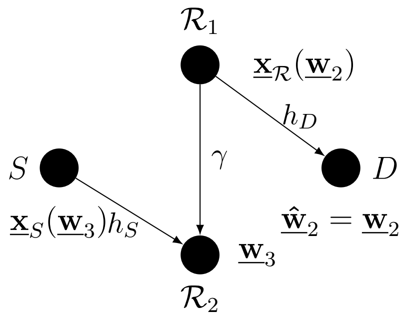


Figure 4.3: Time slot 3

slots. In the first time slot, (See Fig. 4.1) the source encodes the first message as $\underline{\mathbf{x}}_S(\underline{\mathbf{w}}_1)$ and transmits it to relay 2. The relay 2 decodes $\underline{\mathbf{w}}_1$. In the second time slot, (See Fig. 4.2) relay 2 re-encodes $\underline{\mathbf{w}}_1$ as $\underline{\mathbf{x}}_{\mathcal{R}}(\underline{\mathbf{w}}_1)$ and transmits it to destination and the destination decodes $\underline{\mathbf{w}}_1$. Also, the source encodes the second message as $\underline{\mathbf{x}}_S(\underline{\mathbf{w}}_2)$ and transmits it to relay 1. The encoding is done in such a way that the relay 1 can decode $\underline{\mathbf{w}}_2$. In the third time slot, (See Fig. 4.3) the source transmits $\underline{\mathbf{x}}_S(\underline{\mathbf{w}}_3)$ and the destination decodes $\underline{\mathbf{w}}_2$ and the cycle continues. The role of relays 1 and 2 can be reversed in the above time slots. So, in the even time slot, the 2-hop network is in the state given in Fig. 4.2 and in the odd time slot, the state given in Fig. 4.3.

In the even time slot, $\mathcal{R}_1$ can successfully decode the message if the source transmits at a rate $R_{S1} \leq C(h_S^2\text{SNR})$ and destination can decode reliably if $\mathcal{R}_2$ transmits at a rate $R_{2D} \leq C(h_D^2\text{SNR})$.

In the odd time slot, $\mathcal{R}_2$ can successfully decode the message if the source transmits at a rate $R_{S2} \leq C(h_S^2\text{SNR})$ and destination can decode reliably if $\mathcal{R}_1$ transmits at a rate $R_{1D} \leq C(h_D^2\text{SNR})$.

Hence, the rate of DPC can be expressed as follows.

$$R_{DPC} = \max_{R_{S1},R_{S2},R_{1D},R_{2D}} \frac{1}{2}[min(R_{S1}, R_{1D}) + min(R_{S2}, R_{2D})]$$

Hence, we obtain the following result.

$$\text{If } h_S \geq h_D,\ R_{DPC} = C(h_D^2\text{SNR})$$
$$\text{If } h_D > h_S,\ R_{DPC} = C(h_S^2\text{SNR})$$

4.1 Gap between cut-set bound and the rate of DPC

4.1.1 Upper Bound

Lemma 1.

$$\max_{x\in X} \min\{f_1(x), f_2(x)\} \leq \min\{\max_{x\in X} f_1(x), \max_{x\in X} f_2(x)\}$$

We use the above lemma to compute the gap between cut set bound and rate of DPC. The upper bound for the 2-hop network is obtained by considering cuts I_1 and I_4 alone.

$$R_{cut-set} \leq R_{upper} = \max_{\substack{t_1,t_2,t_3,t_4 \\ \text{s.t}\, t_1+t_2+t_3+t_4=1 \\ t_1,t_2,t_3,t_4\geq 0}} \min\{I_1,\ I_4\}$$

where

$$I_1 \triangleq t_1 C(h_S^2\text{SNR}) + t_2 C(h_S^2\text{SNR}) + t_3 C(2h_S^2\text{SNR})$$
$$I_4 \triangleq t_1 C(h_D^2\text{SNR}) + t_2 C(h_D^2\text{SNR}) + t_4 C(4h_D^2\text{SNR})$$

Proof.

$$\begin{aligned} R_{upper} = \text{maximize } & R \\ \text{subject to } & R \leq C_S\tilde{t} + C_{BC}t_3 \\ & R \leq C_D\tilde{t} + C_{MAC}t_4 \\ & \tilde{t} + t_3 + t_4 = 1 \end{aligned}$$

where

$$C_S = C(h_S^2\text{SNR}), C_{BC} = C(2h_S^2\text{SNR})$$
$$C_D = C(h_D^2\text{SNR}), C_{MAC} = C(4h_D^2\text{SNR})$$
$$\tilde{t} = t_1 + t_2$$

which can be written as

$$
\begin{aligned}
R_{upper} = \text{maximize } R& \\
\text{subject to } R &\leq C_S\tilde{t} + C_{BC}t_3 \\
R &\leq C_D\tilde{t} + C_{MAC}(1 - \tilde{t} - t_3)
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \max\min\left\{C_S\tilde{t} + C_{BC}t_3, (C_D - C_{MAC})\tilde{t} + C_{MAC} - C_{MAC}t_3\right\} \\
&\therefore \max_{\substack{\tilde{t},t_3 \\ \tilde{t}+t_3\leq 1}} \min\left\{C_S\tilde{t} + C_{BC}t_3, (C_D - C_{MAC})\tilde{t} + C_{MAC} - C_{MAC}t_3\right\} \\
&\leq \min\Big\{ \max_{\substack{\tilde{t},t_3 \\ \tilde{t}+t_3\leq 1}} (C_S\tilde{t} + C_{BC}t_3), \ \max_{\substack{\tilde{t},t_3 \\ \tilde{t}+t_3\leq 1}} \{(C_D - C_{MAC})\tilde{t} \\
&\qquad + C_{MAC} - C_{MAC}t_3\}\Big\} \text{(using Lemma 1)} \\
&\leq \min(C_{BC}, C_{MAC})
\end{aligned}
$$

Hence $R_{upper} \leq \min(C_{BC}, C_{MAC})$

We try to find the gap between cut-set bound and the rate of DPC using the above result.

If $|h_S| > |h_D|$

$$
\begin{aligned}
R_{cut-set} - R_{DPC} &\leq R_{upper} - R_{DPC} \\
&\leq C_{MAC} - 0.5\log(1 + h_D^2\text{SNR}) \\
&= 0.5\log(1 + 4h_D^2\text{SNR}) \\
&\qquad - 0.5\log(1 + h_D^2\text{SNR}) \\
&= 0.5\log\left(\frac{1 + 4h_D^2\text{SNR}}{1 + h_D^2\text{SNR}}\right) \\
&\leq 1
\end{aligned}
$$

If $|h_S| \leq |h_D|$

$$
\begin{aligned}
R_{cut-set} - R_{DPC} &\leq R_{upper} - R_{DPC} \\
&\leq C_{BC} - 0.5\log(1 + h_S^2\text{SNR})
\end{aligned}
$$

$$
\begin{aligned}
&= 0.5\log(1 + 2h_S^2\text{SNR}) \\
&\qquad - 0.5\log(1 + h_S^2\text{SNR}) \\
&= 0.5\log\left(\frac{1 + 2h_S^2\text{SNR}}{1 + h_S^2\text{SNR}}\right) \\
&\leq 0.5
\end{aligned}
$$

□

4.2 Condition under which DPC achieves the cutset bound

Theorem 1. *$R_{DPC} = R_{cut-set}$ if and only if $|h_S| = |h_D| = h$, when $|h|^2 SNR \geq 1$*

Proof. Throughout this section we use the following results.

There is a dual linear program associated with every linear program.

Consider the linear program

$$
\begin{aligned}
&\text{maximize } b^T\lambda \\
&\text{subject to } A^T\lambda \leq c \\
&\lambda \geq 0
\end{aligned} \tag{4.1}
$$

The dual of the above program is given by

$$
\begin{aligned}
&\text{minimize } c^T x \\
&\text{subject to } Ax \geq b \\
&x \geq 0
\end{aligned} \tag{4.2}
$$

The strong duality theorem is stated below.

□

Theorem 2. *If either of the problems (4.1) or (4.2) has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal i.e, $b^T\lambda^* = c^T x^*$. If* *either problem has an unbounded objective, the other problem has no feasible solution.*

The complementary slackness theorem is stated below.

Theorem 3. *Let x and λ be feasible solutions for the primal and dual programs, respectively, in the pair (4.1) or (4.2). A necessary and sufficient condition that they both be optimal solutions is that for all i and j*

$x_i > 0 \Rightarrow a_i^T \lambda = c_i$

$x_i = 0 \Leftarrow a_i^T \lambda < c_i$

$\lambda_j > 0 \Rightarrow a^j x = b_j$

$\lambda_j = 0 \Leftarrow a^j x > b_j$

Here, a_i denotes the ith column of A and a^j denotes the jth row of A.
The optimization problem for the cut-set bound in Chapter 3 can be rewritten in matrix form as follows.

$$\text{maximize} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ R \end{bmatrix}$$

$$\text{subject to} \begin{bmatrix} a_1 & a_1 & a_2 & 0 & 1 \\ 0 & a_3 & a_1 & a_4 & 1 \\ a_3 & 0 & a_1 & a_4 & 1 \\ a_4 & a_4 & 0 & a_5 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ R \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \tag{4.3}$$

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ R \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where

$$
\begin{aligned}
a_1 &= -C(h_S^2 \text{SNR}) \\
a_2 &= -C(2h_S^2 \text{SNR}) \\
a_3 &= -C\left((h_S^2 + h_D^2 + \gamma^2)\text{SNR} + h_D^2 h_S^2 \text{SNR}^2 + \frac{\gamma^2}{h_D^2}\right) \\
a_4 &= -C(h_D^2 \text{SNR}) \\
a_5 &= -C(4h_D^2 \text{SNR})
\end{aligned}
$$

The above problem is of the form

$$
\begin{aligned}
\text{maximize } & b^T \lambda \\
\text{subject to } & A^T \lambda \leq c \\
& \lambda \geq 0
\end{aligned}
$$

Note: The last 2 rows of (4.3) is obtained as follows.

$$
\begin{aligned}
& t_1 + t_2 + t_3 + t_4 = 1 \\
\Rightarrow\ & t_1 + t_2 + t_3 + t_4 \geq 1 \ \&\ t_1 + t_2 + t_3 + t_4 \leq 1 \\
\Rightarrow\ & -t_1 - t_2 - t_3 - t_4 \leq -1 \ \&\ t_1 + t_2 + t_3 + t_4 \leq 1
\end{aligned}
$$

The dual problem of the above optimization problem is given by

$$
\text{minimize} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}
$$

$$\text{subject to} \begin{bmatrix} a_1 & 0 & a_3 & a_4 & 1 & -1 \\ a_1 & a_3 & 0 & a_4 & 1 & -1 \\ a_2 & a_1 & a_1 & 0 & 1 & -1 \\ 0 & a_4 & a_4 & a_5 & 1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.4}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is of the form

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax \geq b \\ &\qquad\qquad x \geq 0 \end{aligned}$$

We prove the necessary condition for DPC to achieve cutset bound. We assume $t_3^* = t_4^* = 0$ & $t_1^* = t_2^* = 0.5$. The optimization problem reduces to

$$\begin{aligned} &\text{maximize } R \\ &\text{subject to } R \leq C(h_S^2 \text{SNR}) \\ &\qquad R \leq 0.5C\left((h_S^2 + h_D^2 + \gamma^2)\text{SNR} + h_D^2 h_S^2 \text{SNR}^2 + \frac{\gamma^2}{h_D^2}\right) \\ &\qquad R \leq C(h_D^2 \text{SNR}) \end{aligned}$$

Below we consider the case when $h_S > h_D$

$$
\begin{aligned}
&0.5C\left((h_S^2 + h_D^2 + \gamma^2)\text{SNR} + h_D^2 h_S^2 \text{SNR}^2 + \frac{\gamma^2}{h_D^2}\right) \\
&= 0.25 \log\left(1 + (h_S^2 + h_D^2 + \gamma^2)\text{SNR} + h_D^2 h_S^2 SNR^2 + \frac{\gamma^2}{h_D^2}\right) \\
&> 0.25 \log(1 + 2h_D^2 \text{SNR} + h_D^4 \text{SNR}^2) \\
&= 0.25 \log((1 + h_D^2 \text{SNR})^2) \\
&= 0.5 \log(1 + h_D^2 \text{SNR}) \\
&= C(h_D^2 \text{SNR})
\end{aligned}
$$

Hence optimal R is $C(h_D^2\text{SNR})$

Next we consider the case when $h_D > h_S$

$$
\begin{aligned}
&0.5C\left((h_S^2 + h_D^2 + \gamma^2)\text{SNR} + h_D^2 h_S^2 \text{SNR}^2 + \frac{\gamma^2}{h_D^2}\right) \\
&= 0.25 \log\left(1 + (h_S^2 + h_D^2 + \gamma^2)\text{SNR} + h_D^2 h_S^2 \text{SNR}^2 + \frac{\gamma^2}{h_D^2}\right) \\
&> 0.25 \log(1 + 2h_S^2 \text{SNR} + h_S^4 \text{SNR}^2) \\
&= 0.25 \log((1 + h_S^2 \text{SNR})^2) \\
&= 0.5 \log(1 + h_S^2 \text{SNR}) \\
&= C(h_S^2 \text{SNR})
\end{aligned}
$$

Hence optimal R is $C(h_S^2\text{SNR})$

$$
\therefore t^* = \begin{bmatrix} t_1^* \\ t_2^* \\ t_3^* \\ t_4^* \\ R^* \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ \min\{\bar{a}_1, \bar{a}_4\} \end{bmatrix}
$$

By the Strong Duality Theorem, we have

$$c^T x^* = b^T \lambda^*$$
$$= b^T t^*$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \\ x_6^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ \min\{\bar{a}_1, \bar{a}_4\} \end{bmatrix}$$

$$x_5^* - x_6^* = \min\{\bar{a}_1, \bar{a}_4\}$$

The second and third inequalities of (4.3) are strictly less than zero since $R < 0.5\bar{a}_3$

By Complementary Slackness theorem,

$x_2^* = x_3^* = 0$

We first consider the case when $h_S > h_D$

$x_5^* - x_6^* = \bar{a}_4$

Since $t_1^* > 0, t_2^* > 0, R > 0$, by Complementary Slackness Theorem, the 1st, 2nd and 5th inequalities of (4.4) become equalities. Hence, we have

$a_1 x_1 + a_3 x_3 + a_4 x_4 + x_5 - x_6 = 0$

$a_1 x_1 + a_3 x_2 + a_4 x_4 + x_5 - x_6 = 0$

$x_1 + x_2 + x_3 + x_4 = 1$

Since, $x_2^* = 0, x_3^* = 0$ & $x_5^* - x_6^* = -a_4$

$a_1 x_1 + a_4 x_4 - a_4 = 0$

$x_1 + x_4 = 1$

Since $x_2^* = 0, x_3^* = 0$ and $x_5^* - x_6^* = -a_4$

$a_1x_1 + a_4x_4 - a_4 = 0$

$x_1 + x_4 = 1$

Solving, we get $x_1^* = 0, x_4^* = 1$

We check if $x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*$ satisfies the 3rd and 4th inequalities of (4.4)

We first check the third inequality.

$a_2x_1 + a_1x_2 + a_1x_3 + x_5 - x_6 = -a_4 > 0$

$(\because\ -a_4 = C(h_D^2\text{SNR}))$

Next, we check the fourth inequality.

$a_4x_2 + a_4x_3 + a_5x_4 + x_5 - x_6 = a_5 - a_4 < 0$

$(\because\ -a_4 = C(h_D^2\text{SNR})\ \&\ -a_5 = C(4h_D^2\text{SNR}))$

Hence, we find that the $4th$ inequality is not satisfied.

Below we consider the case when $h_D > h_S$

$x_5^* - x_6^* = \bar{a}_1$

$a_1x_1 + a_4x_4 = a_4$

$x_1 + x_4 = 1$

Solving, we get $x_1^* = 1, x_4^* = 0$

Again, we check the third inequality of (4.4).

$a_2x_1 + a_1x_2 + a_1x_3 + x_5 - x_6 = a_2 - a_1 < 0$

$(\because\ -a_1 = C(h_S^2\text{SNR})\ \&\ -a_2 = C(2h_S^2\text{SNR}))$

Next, we check the fourth inequality.

$a_4x_2 + a_4x_3 + a_5x_4 + x_5 - x_6 = -a_1 > 0$

$(\because\ -a_1 = C(h_S^2\text{SNR}))$

Hence, we find that the third inequality is not satisfied.

To satisfy the 3rd and 4th inequalities in both the cases, $a_1$should be equal to a_4,i.e., $|h_S| = |h_D| = h$

Hence, the necessary condition for

$t_1^* = t_2^* = 0.5; t_3^* = t_4^* = 0$ is $|h_S| = |h_D| = h$

The sufficient condition for successive relaying can be proved by following similar steps to the proof of Lemma 1 given in Hong and Caire (2015)

CHAPTER 5

COMPUTE AND FORWARD

Upon receiving the data from the source, the relay does some form of processing on it and then forwards it to destination. Depending on the processing that is done, there are different encoding schemes for the relay.

- Amplify and Forward (AF): Here, the relay acts as a repeater and forwards a scaled version of the received signal to the destination.
- Quantize-Map and Forward (QMF) : The relay quantizes the received signal, maps it to a codeword, and then forwards it.
- Decode and Forward (DF) : The relay decodes the received message, re-encodes it and then forwards it to the destination.

Yet another scheme is the Compute and Forward (CoF) which we study in detail.
In case of CoF, the source maps the messages $\mathbf{w}_k$ to lattice codewords $\mathbf{t}_k$ which are then encoded and transmitted as $\mathbf{x}_k = \varepsilon_k(\mathbf{w}_k)$ to the relay. The relay receives a noisy linear combination of what is sent by the source and the other relay.

$$\mathbf{y} = h_S\mathbf{x}_1 + \gamma\mathbf{x}_2 + \mathbf{z}$$

Here, $\gamma\mathbf{x}_2$ is the interference signal. Instead of treating interference as noise, the relay decodes an integer linear combination of the message signal and interference signal $\mathbf{u}$ from the received $\mathbf{y}$.

$$\mathbf{u} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$$

This is further encoded and transmitted to destination where the messages are recovered using forward substitution.

5.1 Lattice codes

Below we give a short description of the lattice codewords used in CoF scheme.

- A lattice Λ is a discrete subgroup of $\mathbb{R}^n$ which satisfies the property that if $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{\Lambda}$, then $\mathbf{t}_1 + \mathbf{t}_2 \in \mathbf{\Lambda}$.
- The lattice quantizer is defined as

$$Q_\Lambda(\mathbf{x}) = \text{argmin}_{\mathbf{t}\in\Lambda} ||\mathbf{t} - \mathbf{x}||$$

- The fundamental Voronoi region is defined as

$$\mathcal{V} := \{\mathbf{x} \in \mathbb{R}^n : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$$

- In a general K-user Gaussian MAC, for every user, a lattice $\mathbf{\Lambda}_k$ which is good for AWGN channel coding is chosen. We denote the coarsest lattice among them by $\mathbf{\Lambda}_c$. Also, we construct K lattices $\mathbf{\Lambda}_k^s$ which are simultaneously good and which satisfies the condition $\mathbf{\Lambda}_k^s \subset \mathbf{\Lambda}_c$. The lattice $\mathbf{\Lambda}_k^s$ is used as the shaping region of the codebook of user k. They have the following second moment.

$$\frac{1}{n\text{Vol}(\mathcal{V}_k^s)} \int_{\mathcal{V}_k^s} ||\mathbf{x}||^2 \, d\mathbf{x} = \beta_k^2 P$$

  Here, $\underline{\beta} = (\beta_1, \beta_2, ...\beta_K)$ where $\beta_k, k = 1, ...K$ are K non-zero real numbers.

In this section, we compare the performance of CoF that is computed using General Compute and Forward Formula in Zhu and Gastpar (2017) (We denote it by R_{GCoF}) with DPC for a 2-hop network at high SNR.

Theorem 4. *A General Compute-and-Forward Formula : Consider a K-user Gaussian MAC with channel coefficients $h = (h_1, ..., h_K)$ and equal power constraint P. Let $\beta_1, ..., \beta_K$ be K nonzero real numbers. The computation rate tuple $(R_1^a, ..., R_K^a)$ with respect to the sum $\mathbf{u} := \left[\sum_{k=1}^K a_k \mathbf{t}_k\right] mod\, \mathbf{\Lambda}_f^s$ is achievable with*

$$R_K^a = \left[0.5 \log \left(\|\tilde{a}\|^2 - \frac{P(h^T \tilde{a})^2}{1 + P\|h\|^2} \right)^{-1} + 0.5 \log \beta_k^2 \right]^+$$

where $\tilde{a} := [\beta_1 a_1, ..., \beta_K a_K]$ and $a_k \in Z$ for all $k \in [1 : K]$. Here, $\mathbf{\Lambda}_f^s$ is the finest lattice among $\mathbf{\Lambda}_k^s$.

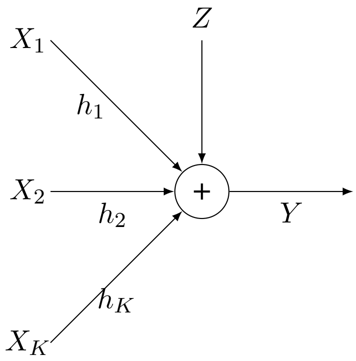


Figure 5.1: K-user Gaussian MAC

5.2 CoF for a 2-hop network

Below we discuss the action of Compute and Forward (CoF) protocol in each of the time slots.

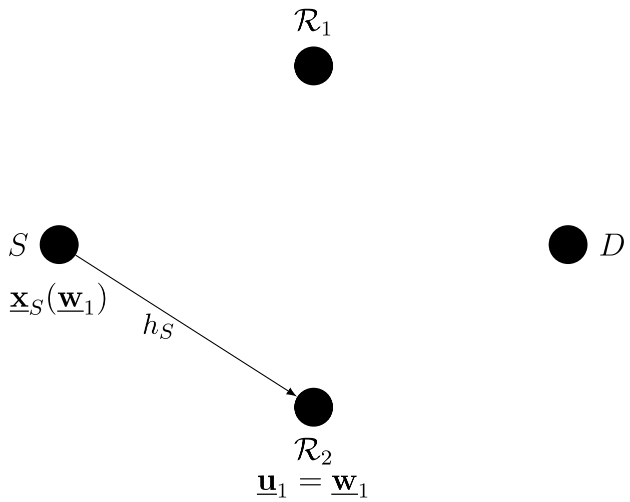


Figure 5.2: Time slot 1

In the first time slot, (See Fig. 5.2) the source encodes the first message $\underline{\mathbf{w}}_1$ as $\underline{\mathbf{x}}_S(\underline{\mathbf{w}}_1)$ and transmits it to relay 2. The relay 2 decodes $\underline{\mathbf{w}}_1$.

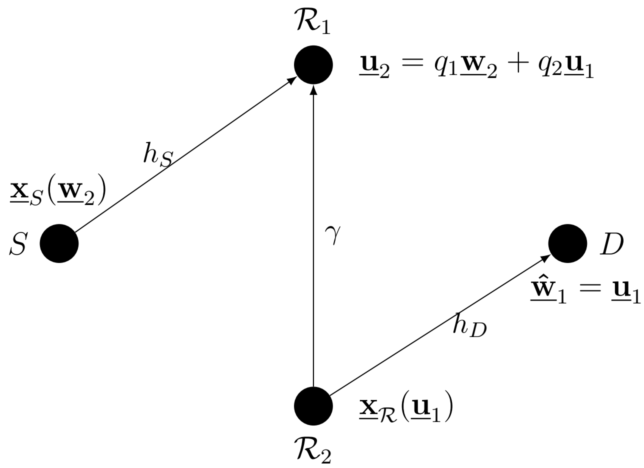


Figure 5.3: Time slot 2

In the second time slot, (See Fig. 5.3) relay 2 re-encodes $\underline{\mathbf{w}}_1$ as $\underline{\mathbf{x}}_{\mathcal{R}}(\underline{\mathbf{u}}_1)$ and transmits it to destination and the destination decodes $\underline{\mathbf{w}}_1$. Also, the source encodes the second message as $\underline{\mathbf{x}}_S(\underline{\mathbf{w}}_2)$ and transmits it to relay 1. The relay 1 receives $\underline{\mathbf{x}}_S(\underline{\mathbf{w}}_2) + \gamma\underline{\mathbf{x}}_{\mathcal{R}}(\underline{\mathbf{u}}_1)$. The relay decodes an integer linear combination of what is sent by the source and the other relay $\underline{\mathbf{u}}_2 = a_2\underline{\mathbf{w}}_2 + a_1\underline{\mathbf{u}}_1$.

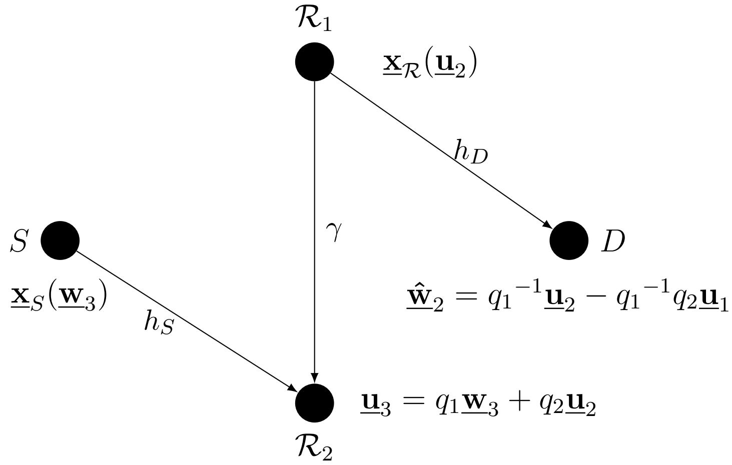


Figure 5.4: Time slot 3

In the third time slot, (See Fig. 5.4) the relay 1 re-encodes $\underline{\mathbf{u}}_2$ and transmits it as $\underline{\mathbf{x}}_{\mathcal{R}}(\underline{\mathbf{u}}_2)$ to destination. The destination decodes $\underline{\mathbf{w}}_2$ using forward substitution ($\hat{\underline{\mathbf{w}}}_2 = {a_2}^{-1}\underline{\mathbf{u}}_2 - {a_2}^{-1}a_1\underline{\mathbf{u}}_1$). This cycle is continued in successive time slots.

Below, we find the rate that can be achieved by the CoF protocol. This is found by using the

General Compute and Forward formula (Theorem 4). We denote the rate by R_{GCoF}.
Compared to the CoF scheme in Hong and Caire (2015), the GCoF coding scheme has additional lattice scaling coefficients that allow us to choose the scaling depending on the channel parameters. This reduces the rate loss due to mismatch between the channel coefficients and the integer combination computed at the relays. Our choice of these scaling coefficients and the integer combination will help us overcome the non-integer penalty.

From Fig. 5.3, the source and the relay 2 can be considered as 2 users and the other relay 1 as destination. This forms a 2 user Gaussian MAC with channel coefficients h_S and γ respectively. Using the General Compute and Forward formula, the relay can reliably decode the linear combination $a_1\underline{\mathbf{w}}_1 + a_2\underline{\mathbf{w}}_2$ if

$$R_S \leq 0.5\log\left(\frac{1+\text{SNR}(h_S^2+\gamma^2)}{\beta_1^2a_1^2+\beta_2^2a_2^2+\text{SNR}(h_S\beta_2a_2-\gamma\beta_1a_1)^2}\right)+0.5\log(\beta_1^2)$$
$$R_{\mathcal{R}} \leq 0.5\log\left(\frac{1+\text{SNR}(h_S^2+\gamma^2)}{\beta_1^2a_1^2+\beta_2^2a_2^2+\text{SNR}(h_S\beta_2a_2-\gamma\beta_1a_1)^2}\right)+0.5\log(\beta_2^2)$$

where, R_S denotes the rate at which the source can transmit and $R_{\mathcal{R}}$ denotes the rate at which the relay can transmit. β_1, β_2 are non zero real numbers which control the rates and $a_1, a_2 \in Z$. β_1, β_2 and a_1, a_2 can be chosen to optimize performance. We would like the term $\text{SNR}(h_S\beta_2a_2-\gamma\beta_1a_1)^2$ in the denominator to be zero, so that there is no penalty at high SNR. In order to completely eliminate this loss, we consider $h_S\beta_2a_2-\gamma\beta_1a_1 = 0$. We take $\beta_1^2a_1^2+\beta_2^2a_2^2 = 1$. Solving these two equations, gives: Hence, we get the following values.
$\beta_1 = \frac{h_S}{\sqrt{\gamma^2+h_S^2}}, a_1 = 1$
$\beta_2 = \frac{\gamma}{\sqrt{\gamma^2+h_S^2}}, a_2 = 1$
Substituting the above, we take the achievable rate of the source as $R_1 = \min\{R_S, R_{\mathcal{R}}\}$

$$\begin{aligned}R_1 =& \min\Big\{0.5\log\Big(1+\text{SNR}(h_S^2+\gamma^2)\Big)+0.5\log\Big(\frac{h_S^2}{h_S^2+\gamma^2}\Big),\\ &0.5\log\Big(1+\text{SNR}(h_S^2+\gamma^2)\Big)+0.5\log\Big(\frac{\gamma^2}{h_S^2+\gamma^2}\Big)\Big\}\\ =& \min\Big\{0.5\log\Big(\frac{h_S^2}{h_S^2+\gamma^2}+h_S^2\text{SNR}\Big), 0.5\log\Big(\frac{\gamma^2}{h_S^2+\gamma^2}+\gamma^2\text{SNR}\Big)\Big\}\end{aligned}$$

The decoded linear combination at the relay is reliably transmitted to destination if the relay transmits at a rate $R \leq R_2$

where $R_2 = 0.5\log(1 + h_D^2\text{SNR})$

Hence, the achievable rate of GCoF is given by

$$R_{GCoF} = \min\{R_1, R_2\}$$

Clearly, R_{GCoF} satisfies the following condition.

$$R_{GCoF} > \min\left\{0.5\log(h_S^2\text{SNR}), 0.5\log(h_D^2\text{SNR}), 0.5\log(\gamma^2\text{SNR})\right\}$$

Also, when $\min\{h_S, h_D, \gamma\} \neq \gamma$

$$R_{GCoF} > \min\left\{0.5\log(h_S^2\text{SNR}), 0.5\log(h_D^2\text{SNR})\right\}$$

The rate of DPC is given by

$$R_{DPC} = \min\left\{0.5\log(1 + h_S^2\text{SNR}), 0.5\log(1 + h_D^2\text{SNR})\right\}$$

When $h_S^2\text{SNR}, h_D^2\text{SNR} \geq 1$ and $\min\{h_S, h_D, \gamma\} \neq \gamma$, we get $|R_{DPC} - R_{GCoF}| < 0.5$. Thus, R_{GCoF} achieves the rate of DPC within 0.5 bits.

The above results are also found for the 3-hop case.

- In time slot 2 (Fig. 5.5(b)), relay 2 can reliably decode the linear combination if the source transmits at a rate

$$\begin{aligned} R &\leq 0.5\log\left(1 + \text{SNR}(h_1^2 + \gamma^2)\right) + 0.5\log\left(\frac{h_1^2}{h_1^2 + \gamma^2}\right) \\ &= 0.5\log\left(\frac{h_1^2}{h_1^2 + \gamma^2} + h_1^2\text{SNR}\right) \end{aligned}$$

  Also, the relay 1 has to transmit at a rate

$$\begin{aligned} R &\leq 0.5\log\left(1 + \text{SNR}(h_1^2 + \gamma^2)\right) + 0.5\log\left(\frac{\gamma^2}{h_1^2 + \gamma^2}\right) \\ &= 0.5\log\left(\frac{\gamma^2}{h_1^2 + \gamma^2} + \gamma^2\text{SNR}\right) \end{aligned}$$

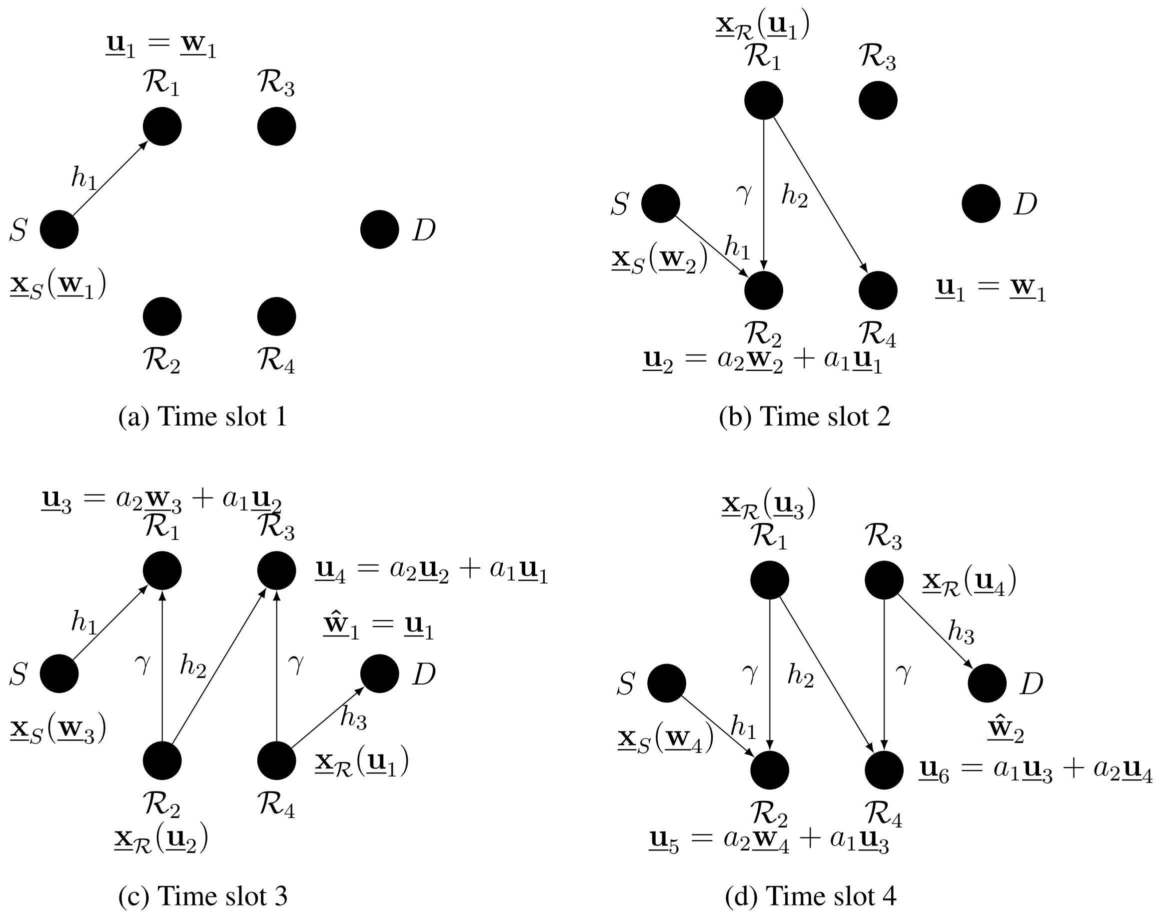


Figure 5.5: GCoF for a 3-hop network

- In the next time slot (Fig. 5.5(c)), relay 3 can reliably decode the linear combination if the relay 2 transmits at a rate

$$
\begin{aligned}
R &\leq 0.5\log\left(1 + \text{SNR}(h_2^2 + \gamma^2)\right) + 0.5\log\left(\frac{h_2^2}{h_2^2 + \gamma^2}\right) \\
&= 0.5\log\left(\frac{h_2^2}{h_2^2 + \gamma^2} + h_2^2\text{SNR}\right)
\end{aligned}
$$

  Also, the relay 4 has to transmit at a rate

$$
\begin{aligned}
R &\leq 0.5\log\left(1 + \text{SNR}(h_2^2 + \gamma^2)\right) + 0.5\log\left(\frac{\gamma^2}{h_2^2 + \gamma^2}\right) \\
&= 0.5\log\left(\frac{\gamma^2}{h_2^2 + \gamma^2} + \gamma^2\text{SNR}\right)
\end{aligned}
$$

- The decoded linear combination can be reliably transmitted to destination if relay 4 transmits at a rate

$$R \leq 0.5\log(1 + h_3^2\text{SNR})$$

  This is seen in Fig. 5.5(c).

Hence

$$R_{GCoF} = \min\Big\{0.5\log\Big(\frac{h_1^2}{h_1^2+\gamma^2} + h_1^2\text{SNR}\Big), 0.5\log\Big(\frac{\gamma^2}{h_1^2+\gamma^2} + \gamma^2\text{SNR}\Big),$$
$$0.5\log\Big(\frac{h_2^2}{h_2^2+\gamma^2} + h_2^2\text{SNR}\Big), 0.5\log\Big(\frac{\gamma^2}{h_2^2+\gamma^2} + \gamma^2\text{SNR}\Big), 0.5\log(1+h_3^2\text{SNR})\Big\}$$

5.3 CoF for a K-hop case

Choosing the scaling coefficients at each stage in a similar manner, we get the rate for the general K-hop case to be the following expression. The last term is for the destination to decode the message. Each of the first $K-1$ hops contributes two terms, one for each relay to reliably decode the linear combination.

$R_{GCoF} =$

$$\min\Big\{\min_{i=1,2,..,K-1}\Big\{0.5\log\Big(\frac{h_i^2}{h_i^2+\gamma^2} + h_i^2\text{SNR}\Big)\Big\}, \min_{i=1,2,..,K-1}\Big\{0.5\log\Big(\frac{\gamma^2}{h_i^2+\gamma^2} + \gamma^2\text{SNR}\Big)\Big\},$$
$$0.5\log(1+h_K^2\text{SNR})\Big\}$$

From the above, we get

$$R_{GCoF} > \min\Big\{\min_{i=1,2,..,K}\Big\{0.5\log(h_i^2\text{SNR}),\Big\}, 0.5\log(\gamma^2\text{SNR})\Big\} \tag{5.1}$$

Corollary 1. *For the case when SNR* ≥ 1, $h_1 = h_2 = \ldots = h_K = 1$ *and* $\gamma > 1, R_{GCoF}$ *achieves the cut-set upper bound within 0.5 bits.*

Proof. When $h_1 = h_2 = \ldots = h_K = 1, \gamma > 1, R_{GCoF} > 0.5\log(\text{SNR})$. From (Hong and Caire, 2015, Lemma 3), the cut-set upper bound is $0.5\log(1+\text{SNR})$. Therefore, $|0.5\log(1+\text{SNR}) - 0.5\log(\text{SNR})| \leq 0.5$ for SNR ≥ 1. □

5.4 Gap between cut-set bound and $\mathrm{R_{GCoF}}$ for a K-hop network

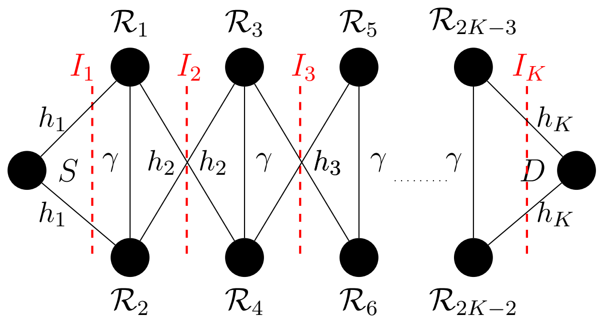


Figure 5.6: Cuts for K-hop network

Theorem 5. *For a K-hop network, we have the following results.*

- *If* $\min\{h_1^2, h_2^2, h_3^2, \dots, h_K^2, \gamma^2\} = h_1^2$, *and* $h_1^2 SNR \geq 1$, *then* $R_{cut-set} - R_{GCoF} \leq 0.5 \log 3$.
- *If* $\min\{h_1^2, h_2^2, h_3^2, ..., h_K^2, \gamma^2\} = h_K^2$, *and* $h_K^2 SNR \geq 1$, *then* $R_{cut-set} - R_{GCoF} \leq 0.5 \log 5$.

Proof. An upper bound using cutsets, for the K-hop case, can be obtained by considering the cuts as shown in Fig. 5.6.

$$\begin{aligned} R_{cut-set} &= \max \min\{I_1, I_2, I_3, .., I_K\} \\ &\leq \min\{\max I_1, \max I_2, \max I_3, .., \max I_K\}. \end{aligned}$$

The maximum of I_1 for such a channel is bounded by $C(2h_1^2\text{SNR})$. Similarly, the maximum of I_k is $2C(h_k^2\text{SNR})$ for $k = 2, 3, \dots, K-1$. Maximum of I_K is bounded by $C(4h_K^2\text{SNR})$. Hence, we get

$$R_{cut-set} \leq \min\{C(2h_1^2\text{SNR}), 2C(h_2^2\text{SNR}), \dots, 2C(h_{K-1}^2\text{SNR}), C(4h_K^2\text{SNR})\} \quad (5.2)$$

Using the lower bound for R_{GCoF} in (5.1) and the upper bound for $R_{cut-set}$ in (5.2), we bound the gap between the cut set bound and the rate achieved by GCoF for the following cases.

(1) If $\min\{h_1^2, h_2^2, h_3^2, ..., h_K^2, \gamma^2\} = h_1^2$, then

$$R_{cut-set} \leq C(2h_1^2\text{SNR}),$$
$$R_{GCoF} > 0.5\log(h_1^2\text{SNR}).$$

Therefore

$$R_{cut-set} - R_{GCoF} < 0.5\log\Big(\frac{1+2h_1^2\text{SNR}}{h_1^2\text{SNR}}\Big).$$

If $h_1^2\text{SNR} \geq 1$, then

$$R_{cut-set} - R_{GCoF} \leq 0.5\log\ 3.$$

(2) If $\min\{h_1^2, h_2^2, h_3^2, ..., h_K^2, \gamma^2\} = h_K^2$, then

$$R_{cut-set} \leq C(4h_K^2\text{SNR}),$$
$$R_{GCoF} > 0.5\log(h_K^2\text{SNR}).$$

Therefore

$$R_{cut-set} - R_{GCoF} < 0.5\log\Big(\frac{1+4h_K^2\text{SNR}}{h_K^2\text{SNR}}\Big).$$

If $h_K^2\text{SNR} \geq 1$, then

$$R_{cut-set} - R_{GCoF} \leq 0.5\log\ 5.$$

□

We illustrate the results with three numerical examples here.
(1) We consider a 3-hop network where $h_1 \sim \mathcal{N}(2,1)$, $h_2 \sim \mathcal{N}(3,1)$, $h_3 \sim \mathcal{N}(3.5,1)$, and $\gamma \sim \mathcal{N}(5,1)$. Using the realizations where the condition $\min\{h_1^2, h_2^2, h_3^2, \gamma^2\} = h_1^2$ is satisfied, we plot the average achieved rate of the GCoF scheme versus SNR in Fig. 5.7. The averaged cut-set bound is also shown. The gap result is shown by plotting the sum of the achieved rate

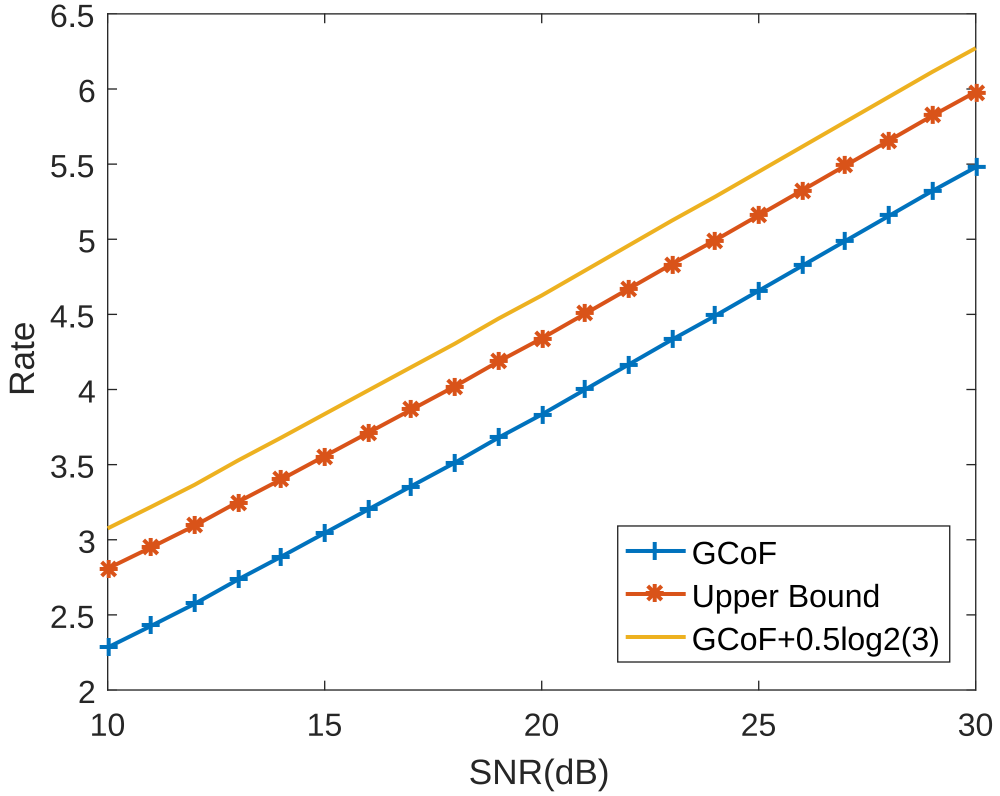


Figure 5.7: Average R_{GCoF} vs SNR when $\min\{h_1^2, h_2^2, h_3^2, \gamma^2\} = h_1^2$

and the derived gap.

(2) We consider a 3-hop network where $h_1 \sim \mathcal{N}(2.5,1)$, $h_2 \sim \mathcal{N}(2,1)$, $h_3 \sim \mathcal{N}(1,1)$, and $\gamma \sim \mathcal{N}(5,1)$. Using the realizations where the condition $\min\{h_1^2, h_2^2, h_3^2, \gamma^2\} = h_3^2$ is satisfied, we plot the average achieved rate of the GCoF scheme versus SNR in Fig. 5.8. The averaged cut-set bound is also shown. The gap result is shown by plotting the sum of the achieved rate and the derived gap.

(3) We consider a 3-hop network where $h_1 \sim \mathcal{N}(2,1)$, $h_2 \sim \mathcal{N}(1,1)$, $h_3 \sim \mathcal{N}(4.5,1)$, and $\gamma \sim \mathcal{N}(5,1)$. Using the realizations where the condition $\min\{h_1^2, h_2^2, h_3^2, \gamma^2\} = h_2^2$ is satisfied, we plot the average achieved rate of the GCoF scheme versus SNR in Fig. 5.9. The averaged cut-set bound is also shown.

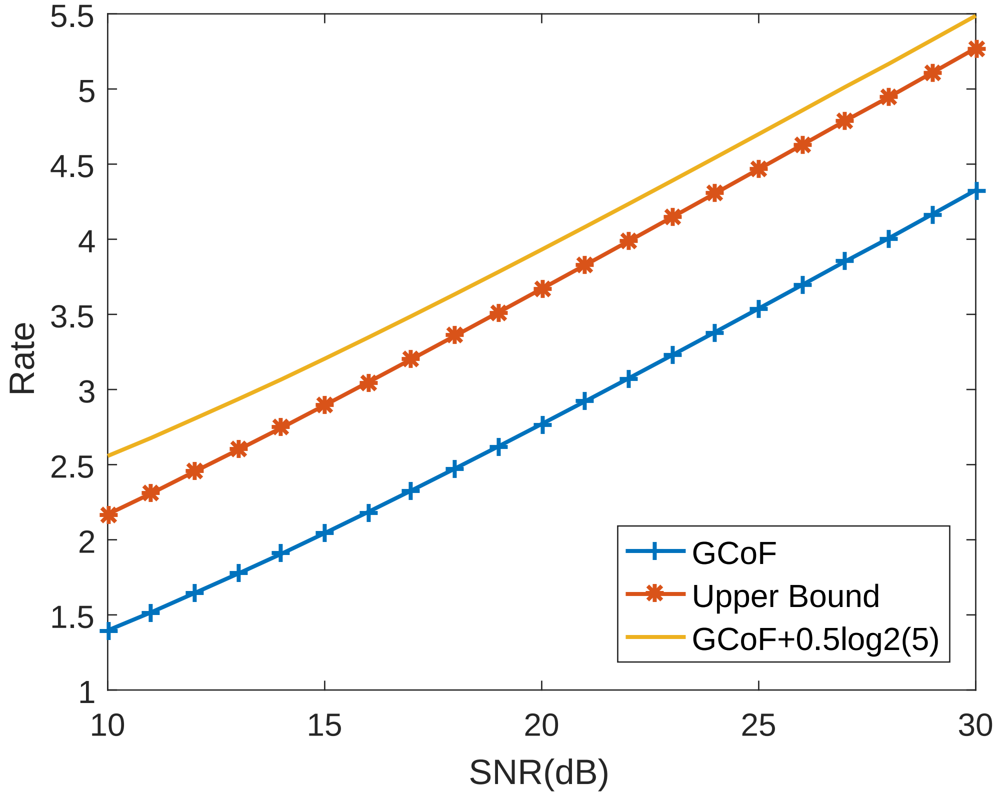


Figure 5.8: Average R_{GCoF} vs SNR when $\min\{h_1^2, h_2^2, h_3^2, \gamma^2\} = h_3^2$

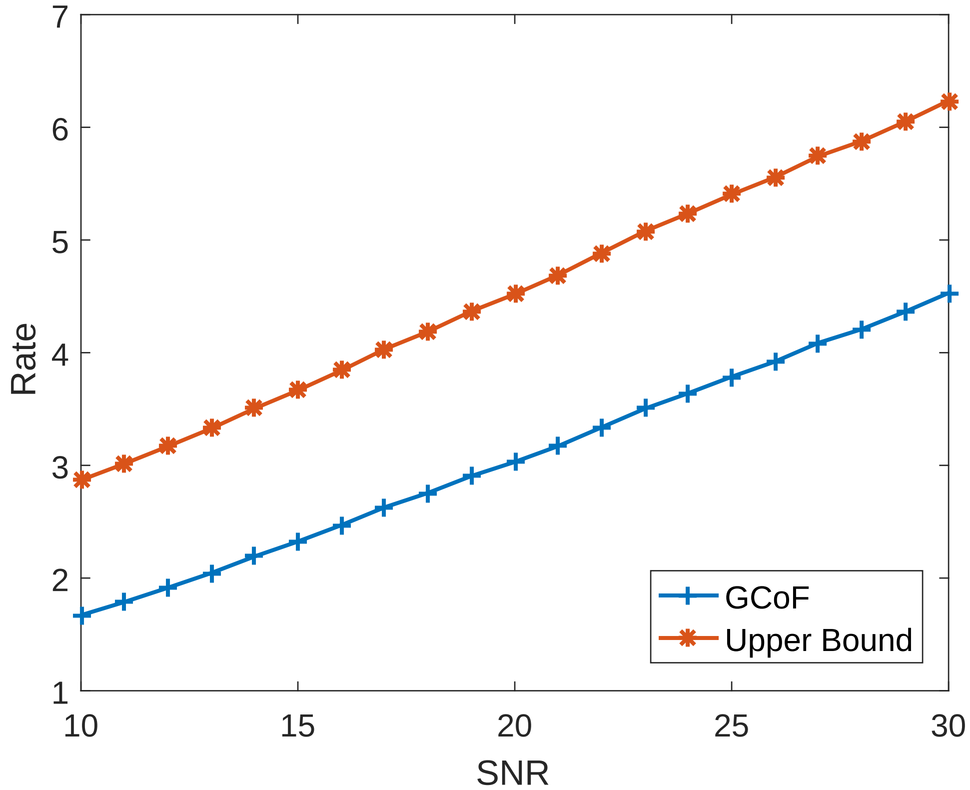


Figure 5.9: Average R_{GCoF} vs SNR when $\min\{h_1^2, h_2^2, h_3^2, \gamma^2\} = h_2^2$

CHAPTER 6

MANY TO ONE INTERFERENCE CHANNEL

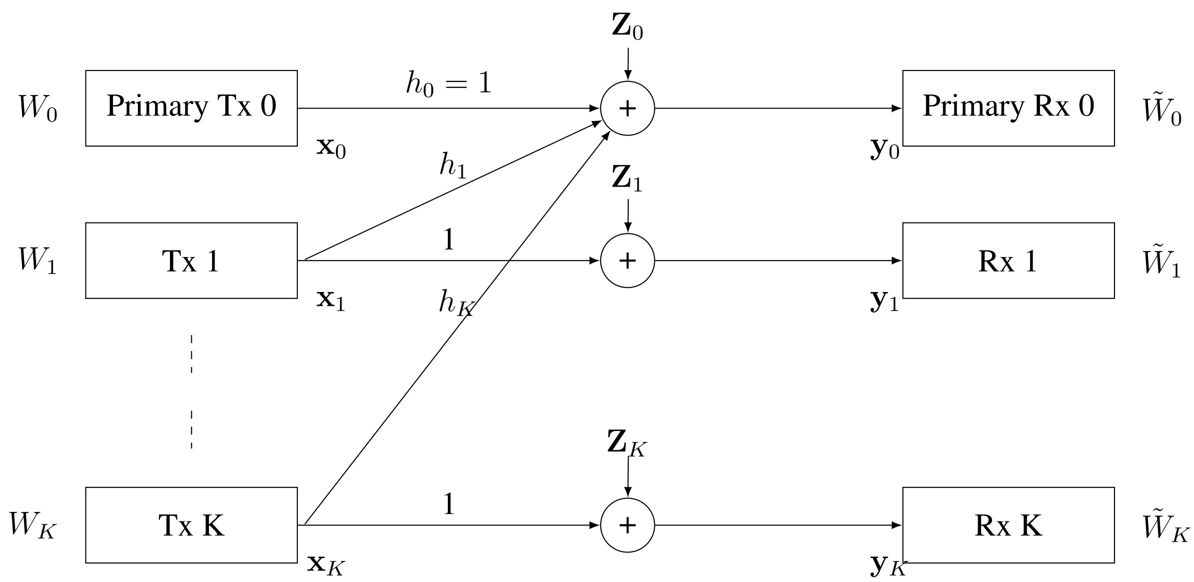


Figure 6.1: Many to one Interference channel

Consider a many to one interference channel as shown in Fig. 6.1. Each user $k, k \in [0 : K]$ has a message W_k to send to its corresponding receiver. Each transmitter has an encoder $\mathcal{E}_k : W_k \to \mathbb{R}^n$ which maps the message to its channel input as $\mathbf{x}_k = \mathcal{E}_k(W_k), k \in [1 : K]$.

$$
\begin{aligned}
\mathbf{y}_0 =& \mathbf{x}_0 + \sum_{k=1}^{K} h_k \mathbf{x}_k + \mathbf{z}_0, \\
\mathbf{y}_k =& \mathbf{x}_k + \mathbf{z}_k, \quad k \in [1 : K],
\end{aligned}
$$

The noise $\mathbf{z}_k \in \mathbb{R}^n$ is assumed to be i.i.d. Gaussian with zero mean and unit variance. This system is referred to as the many-to-one interference channel, since only Receiver 0 experiences interference from other transmitters.

In Zhu and Gastpar (2015), they show that for a symmetric channel, (i.e., $h_1 = h_2 = \ldots = h_K = h$), every user can achieve the capacity under some channel conditions.
Here, we extend the theorem to a non-symmetric channel. The result is stated below.

Theorem 6. *Consider a non-symmetric (non-cognitive) many to one interference channel with K+1 users. If* $h_k^2 > \frac{(1+P_k)(1+P_0)}{P_k}$*,*$(k = 1, 2, ...K)$ *then each and every user can achieve capacity . Here,* P_0 *is the power of the primary transmitter and* P_k*, the power of the* k^{th} *transmitter.*

$$R_0 = \frac{1}{2} log(1 + P_0)$$

$$R_k = \frac{1}{2} log(1 + P_k)$$

Proof. We use theorem 2 in Zhu and Gastpar (2015) which is stated below to prove the above result.

For any given positive numbers $\underline{\beta}$ and coefficient matrix $\mathbf{A}$, define $\mathcal{L}_k := \{\ell \in [1 : L] | a_k(\ell) \neq 0\}$ with $L \in [1 : K + 1]$. If $r_k(\mathbf{a}_{\ell|1:\ell-1}, \underline{\beta}) > 0$ for all $\ell \in \mathcal{L}_k, k \in [0 : K]$, then the following rate is achievable for the many-to-one interference channel

$$\begin{aligned} R_0 &\leq \min_{\ell \in \mathcal{L}_0} \tilde{r}_0(\mathbf{a}_{\ell|1:\ell-1}, \underline{\beta}) \\ R_k &\leq \min \left\{ \frac{1}{2} \log \left(1 + h_k^2 P_k\right), \min_{\ell \in \mathcal{L}_k} \tilde{r}_k(\mathbf{a}_{\ell|1:\ell-1}, \underline{\beta}) \right\} \end{aligned}$$

for $k \in [1 : K]$

$$\tilde{r}_k(\mathbf{a}_{\ell|1:\ell-1}, \underline{\beta}) := \max_{\alpha_1,...,\alpha_\ell \in \mathbb{R}} \frac{1}{2} \log^+ \left(\frac{\beta_k^2 P_k}{\tilde{N}_0(\ell)} \right)$$

$$\begin{aligned} \tilde{N}_0(\ell) :=& \sum_{k \geq 1} \left(\alpha_\ell h_k - a_k(\ell)\beta_k - \sum_{j=1}^{\ell-1} \alpha_j a_k(j) \beta_k \right)^2 P_k \\ &+ \left(\alpha_\ell - a_0(\ell)\beta_0 - \sum_{j=1}^{\ell-1} \alpha_j a_0(j) \beta_0 \right)^2 P_0 + \alpha_\ell^2. \end{aligned}$$

Like in the GCoF protocol, the primary receiver Rx0 decodes integer linear combinations of the messages sent by the transmitters. Using these, the primary receiver Rx0 decodes the message sent by the primary transmitter Tx0. Here, $\mathbf{A}$ is the coefficient matrix whose rows indicate the linear combination of the messages that is decoded by the primary receiver Rx0.

$$\mathbf{A} = \begin{pmatrix} a_0(1) & a_1(1) & a_2(1) & \dots & a_K(1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0(L) & a_1(L) & a_2(L) & \dots & a_K(L) \end{pmatrix},$$

$\tilde{r}_k(\mathbf{a}_{\ell|1:\ell-1}, \underline{\beta})$ refers to the rate at which transmitter k can transmit so that the $l-$th sum can be decoded, given that the previous $l-1$ sums are already decoded. The term $\frac{1}{2}\log(1+h_k^2 P_k)$ is for the receiver Rxk,$k \in [1:K]$ to decode the message sent by transmitter Txk reliably.
This result is proved by taking the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

in Gnanasambandam (2017)
Here, we take the coefficient matrix given below.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} a_0(1) & a_1(1) & \dots & a_K(1) \\ a_0(2) & a_1(2) & \dots & a_K(2) \end{pmatrix}$$

When the primary user decodes the first equation, the variance of the equivalent noise is given by

$$\tilde{N}_0(1) := \alpha_1^2 + \sum_{k\geq 1}(\alpha_1 h_k - a_k(1)\beta_k)^2 P_k + (\alpha_1 - a_0(1)\beta_0)^2 P_0.$$

Like in the GCoF protocol, the encoder maps the messages to lattice codewords which are scaled and then transmitted. β_k denotes the scaling coefficient for each user. By choosing different β_k for each user, we can adjust the rate of individual user and achieve a better rate region. We choose $\beta_k = h_k$ and $\beta_0 = 1$
The parameters $\alpha_1, \alpha_2, \dots, \alpha_l$ is used to balance the effect of the noise that comes due to the non-integer channel gain and also the additive white Gaussian noise at the receiver. We find the

optimal $\alpha_1, \alpha_2, \ldots, \alpha_l$ so as to minimize the effective noise and hence maximize the rate. This is found by differentiating $\tilde{N}_0(l)$ with $\alpha_1, \alpha_2, \ldots, \alpha_l$.

$$\begin{aligned}\tilde{N}_0(1) :&= \alpha_1^2 + \sum_{k\geq 1}(\alpha_1 h_k - h_k)^2 P_k + (\alpha_1 - 0)^2 P_0 \\ &= \alpha_1^2 + (\alpha_1 - 1)^2 \sum_{k\geq 1} h_k^2 P_k + \alpha_1^2 P_0.\end{aligned}$$

We have to find α_1 that minimizes $\tilde{N}_0(1)$ so as to get maximum rate.

$$\frac{d\tilde{N}_0(1)}{d\alpha_1} = 2\alpha_1 + 2(\alpha_1 - 1)\sum_{k\geq 1} h_k^2 P_k + 2\alpha_1 P_0 = 0$$

$$\alpha_1 = \frac{\sum\limits_{k\geq 1} h_k^2 P_k}{1 + P_0 + \sum\limits_{k\geq 1} h_k^2 P_k}$$

Substituting for α_1 in $\tilde{N}_0(1)$ we get

$$\tilde{N}_0(1) = \frac{(\sum\limits_{k\geq 1} h_k^2 P_k)(1 + P_0)}{1 + P_0 + \sum\limits_{k\geq 1} h_k^2 P_k}$$

$$\begin{aligned}\tilde{r}_k(\mathbf{a}_1, \underline{\beta}) &= \frac{1}{2} log\left(\frac{\beta_k^2 P_k}{\tilde{N}_0(1)}\right) \\ &= \frac{1}{2} log\left(\frac{(h_k^2 P_k)(1 + P_0 + \sum\limits_{k\geq 1} h_k^2 P_k)}{(1 + P_0)(\sum\limits_{k\geq 1} h_k^2 P_k)}\right) \\ &> \frac{1}{2} log\left(\frac{h_k^2 P_k}{1 + P_0}\right)\end{aligned}$$

Hence $\tilde{r}_k(\mathbf{a}_1, \underline{\beta}) > \frac{1}{2} log\left(\frac{h_k^2 P_k}{1+P_0}\right)$

Decoding this sum does not impose any constraint on R_0 since the linear combination does not contain W_0.

When the primary user decodes the second equation, the variance of the equivalent noise is given by

$$
\begin{aligned}
\tilde{N}_0(2) := {} & \alpha_2^2 + \sum_{k \geq 1} (\alpha_2 h_k - a_k(2) h_k - \alpha_1 a_k(1) \beta_k)^2 P_k + \\
& (\alpha_2 - a_0(2)\beta_0 - \alpha_1 a_0(1)\beta_0)^2 P_0 \\
= {} & \alpha_2^2 + \sum_{k \geq 1} (\alpha_2 h_k - h_k - \alpha_1 h_k)^2 P_k + (\alpha_2 - 1)^2 P_0
\end{aligned}
$$

We have to find α_1 and α_2 to minimize $\tilde{N}_0(2)$ and hence maximize the rate.

$$
\begin{aligned}
\frac{d\tilde{N}_0(2)}{d\alpha_1} &= 2(\alpha_2 - \alpha_1 - 1)(-1) \sum_{k \geq 1} h_k^2 P_k = 0 \\
&\Rightarrow \alpha_2 = \alpha_1 + 1 \qquad (6.1)
\end{aligned}
$$

$$
\frac{d\tilde{N}_0(2)}{d\alpha_2} = 2\alpha_2 + 2(\alpha_2 - \alpha_1 - 1) \sum_{k \geq 1} h_k^2 P_k + 2(\alpha_2 - 1) P_0 = 0
$$

Substituting for (6.1) in the above expression, we get

$$
\begin{aligned}
\alpha_2 &= \frac{P_0}{1 + P_0} \\
\alpha_1 &= \frac{-1}{1 + P_0}
\end{aligned}
$$

Substituting for α_1 and α_2 in $\tilde{N}_0(2)$

$$
\tilde{N}_0(2) = \frac{P_0}{1 + P_0}
$$

$$
\begin{aligned}
r_k(\mathbf{a}_{2|1}, \underline{\beta}) &= \frac{1}{2} log \left(\frac{\beta_k^2 P_k}{\tilde{N}_0(2)} \right) \\
&= \frac{1}{2} log \left(\frac{(h_k^2 P_k)(1 + P_0)}{P_0} \right)
\end{aligned}
$$

$$R_0 = \frac{1}{2}log\left(\frac{(h_0^2P_0)(1+P_0)}{P_0}\right)$$
$$= \frac{1}{2}log\big((h_0^2)(1+P_0)\big)$$

Since $h_0 = 1$, we get $R_0 = \frac{1}{2}log(1+P_0)$, i.e, the primary transmitter 0 can achieve capacity.

We know $R_k = \min\{r_k(\mathbf{a}_1, \underline{\beta}), r_k(\mathbf{a}_{2|1}, \underline{\beta}), \frac{1}{2}log(1+P_k)\}$
Also, we know $r_k(a_1, \underline{\beta}) > \frac{1}{2}log\left(\frac{h_k^2 P_k}{1+P_0}\right)$
If $h_k^2 > \frac{(1+P_k)(1+P_0)}{P_k}$, then $r_k(\mathbf{a}_1, \underline{\beta}) > \frac{1}{2}log(1+P_k)$
Also, $r_k(\mathbf{a}_{2|1}, \underline{\beta}) > \frac{1}{2}log\frac{(1+P_k)(1+P_0)^2}{P_0}$

Hence, we have $\min\{r_k(\mathbf{a}_1, \underline{\beta}), r_k(\mathbf{a}_{2|1}, \underline{\beta}), \frac{1}{2}log(1+P_k)\} = \frac{1}{2}log(1+P_k)$

Hence, each and every user can achieve capacity if $h_k^2 > \frac{(1+P_k)(1+P_0)}{P_k}$

□

CHAPTER 7

SIMULATION RESULTS

We plot the rate vs interference for the 3 hop network for various encoding schemes. This is considered for the case when $h_1 = h_2 = h_D = 1$ and SNR=20 dB

- CoF without Power Allocation given in Hong and Caire (2015)
  $R_{CoF}^{(K)} = R_{CoF}^{(1)}$

- CoF with Power Allocation given in Hong and Caire (2015)
  $R_{CoF-P}^{(K)} = 0.5\log(\text{SNR}) + 0.5K\log(\gamma_{max}^2)$
  where $\gamma_{max} = \max\left\{\frac{\gamma}{\lceil\gamma\rceil}, \frac{\lfloor\gamma\rfloor}{\gamma}\right\}$

- With General Compute and Formula -R_{GCoF}
- Decode and Forward scheme given in Hong and Caire (2015)
  $R_{DF} = \min\left\{0.5\log(1+\text{SNR}), \max\left\{0.5\log\left(1+\frac{\text{SNR}}{1+\gamma^2\text{SNR}}\right), 0.25\log\left(1+(1+\gamma^2)\text{SNR}\right)\right\}\right\}$
- Quantize Map and Forward scheme given in Hong and Caire (2015)
  $R_{QMF} = 0.5\log\left(1 + \frac{\text{SNR}^{K+1}}{(1+\text{SNR})^{K+1}-\text{SNR}^{K+1}}\right)$
- Amplify and Forward Scheme given in Hong and Caire (2015)
  $R_{AF} = 0.5\log\left(1 + \left(\frac{1+\text{SNR}}{1+(1+\gamma^2)\text{SNR}}\right)^K . \frac{\text{SNR}^{K+1}}{(1+\text{SNR})^{K+1}-\text{SNR}^{K+1}}\right)$

We compare it with the upper bound. This is shown in Fig. 7.1.

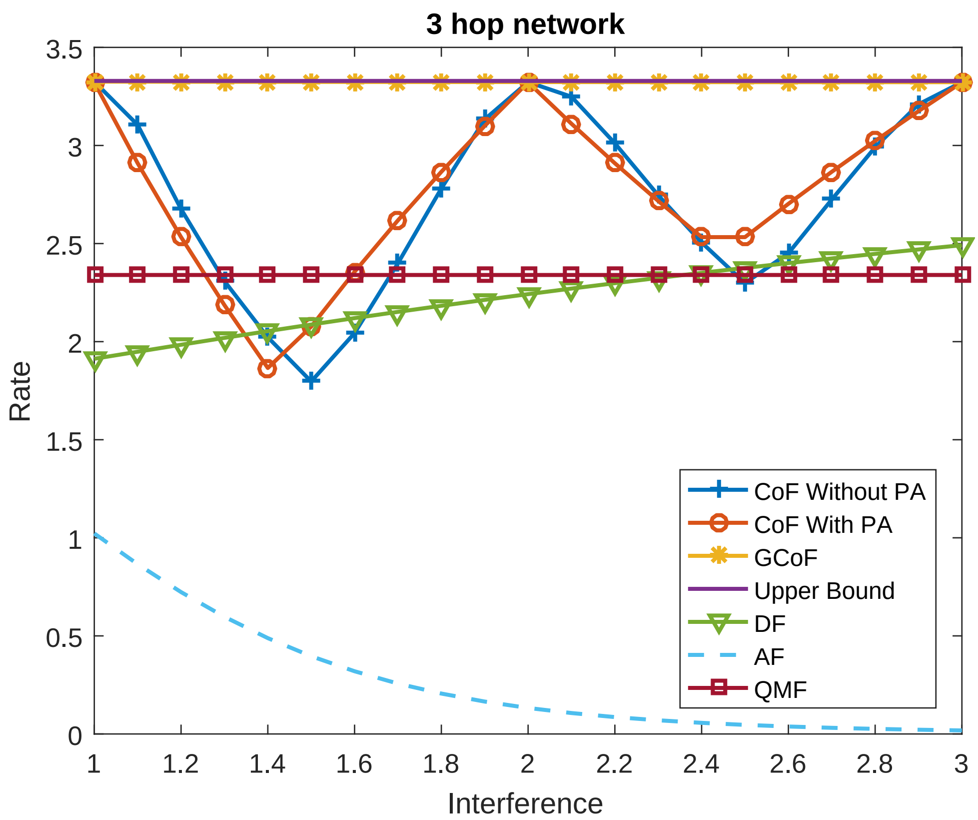


Figure 7.1: Plot of rate vs interference

We observe that R_{GCoF} almost achieves the cut-set bound. The rate of the CoF scheme with and without power allocation fluctuates as the interference levels vary. The Quantize Map and Forward (QMF) scheme has a constant gap with interference. The rate of Amplify and Forward (AF) scheme decreases as the interference levels increase. Also, the rate of Decode and Forward (DF) scheme increases with interference.

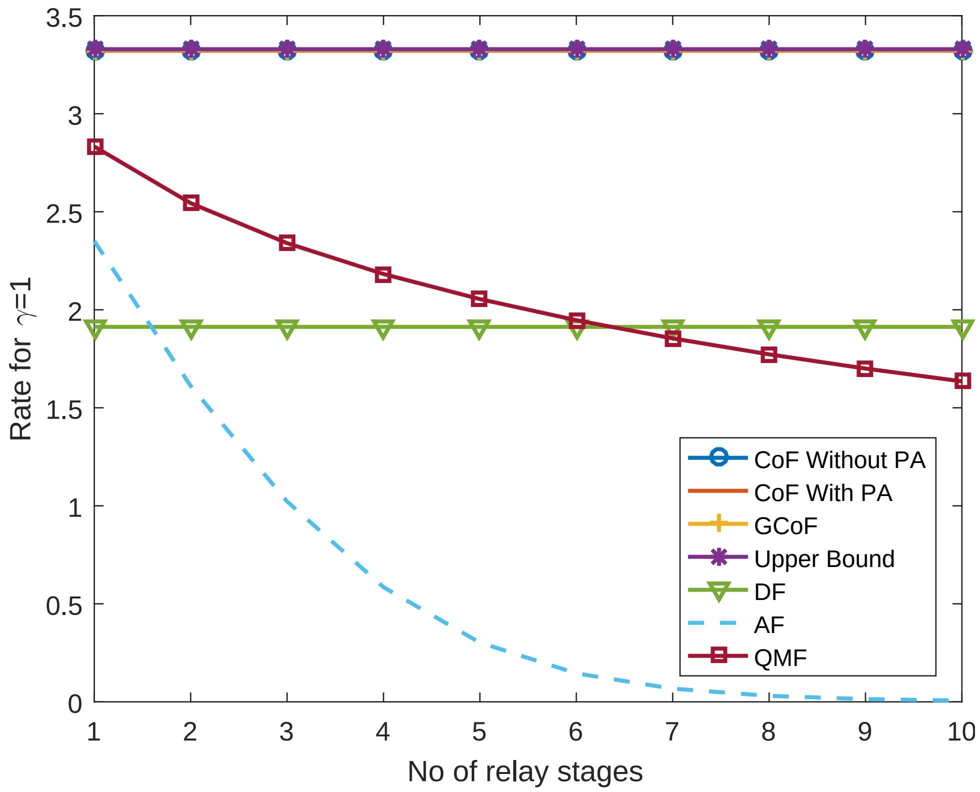


Figure 7.2: Plot of rate vs number of relay stages

We also compare the rate vs the no of relay stages for different encoding schemes. This is shown by simulation in Fig. 7.2. This is done when the channel coefficients h_S and h_D are equal to 1 and inter-relay interference level γ is also equal to 1. We observe that CoF without PA, CoF with PA, GCoF almost achieves the upper bound. The DF scheme has a constant gap with the upper bound. For QMF scheme, the gap increases logarithmically with the number of relay stages. For AF scheme, the gap increases much faster.

CHAPTER 8

CONCLUSION

In this work, we extend the virtual full-duplex relaying scheme in Hong and Caire (2015) by considering unequal gain across hops for the channel coefficients. We found the gap between the cut-set bound and DPC for a 2-hop network. We also showed the conditions under which DPC achieves the cut set bound.
We use the General Compute-and-forward (GCoF) scheme to derive an expression for the rate achieved using the GCoF scheme over a K-hop network. We show that the gap between GCoF and the cutset bound is finite as long as the first or last hop is the bottleneck. Furthermore, this gap does not grow linearly with the number of hops as in Hong and Caire (2015). Also, for a 2 hop network, R_{GCoF} can achieve the rate of DPC within 0.5 bits.
We also considered the case when the channel coefficients are equal to unity. We showed that R_{GCoF} achieves the cut-set bound within 0.5 bits for a K-hop network.
We also studied the non-symmetric many-to-one interference channel and showed that all the users achieve the capacity under certain channel conditions.

REFERENCES

1. **Chang, W.**, **S.-Y. Chung**, and **Y. H. Lee** (2007). Capacity bounds for alternating two-path relay channels.

2. **Changiz Rezaei, S. S.**, **S. Oveis Gharan**, and **A. K. Khandani** (2008). Cooperative strategies for the half-duplex gaussian parallel relay channel: Simultaneous relaying versus successive relaying.

3. **Gnanasambandam, A.** (2017). On the many-to-one and one-to-many gaussian interference channels.

4. **Hong, S. N.** and **G. Caire** (2015). Virtual full-duplex relaying with half-duplex relays. *IEEE Transactions on Information Theory*, **61**(9), 4700–4720. ISSN 0018-9448.

5. **Nam, W.**, **S. Y. Chung**, and **Y. H. Lee** (2010). Capacity of the gaussian two-way relay channel to within $1/2$ bit. *IEEE Transactions on Information Theory*, **56**(11), 5488–5494. ISSN 0018-9448.

6. **Nazer, B.** and **M. Gastpar** (2011). Compute-and-forward: Harnessing interference through structured codes. *IEEE Transactions on Information Theory*, **57**(10), 6463–6486. ISSN 0018-9448.

7. **Wilson, M. P.**, **K. Narayanan**, **H. D. Pfister**, and **A. Sprintson** (2010). Joint physical layer coding and network coding for bidirectional relaying. *IEEE Transactions on Information Theory*, **56**(11), 5641–5654. ISSN 0018-9448.

8. **Zhu, J.** and **M. Gastpar** (2015). Lattice codes for many-to-one interference channels with and without cognitive messages. *IEEE Transactions on Information Theory*, **61**(3), 1309–1324.

9. **Zhu, J.** and **M. Gastpar** (2017). Gaussian multiple access via compute-and-forward. *IEEE Transactions on Information Theory*, **63**(5), 2678–2695. ISSN 0018-9448.