Analyzing Scheduling Policies Based on the Age-of-Information Metric

A Project Report

submitted by

ARUNABH SRIVASTAVA

in partial fulfilment of the requirements
for the award of the degree of

DUAL DEGREE

DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY MADRAS.

June 2021

THESIS CERTIFICATE

This is to certify that the thesis titled **Analyzing Scheduling Policies Based on the Age-of-Information Metric**, submitted by **Arunabh Srivastava**, to the Indian Institute of Technology, Madras, for the award of the degree of **Dual Degree in Electrical Engineering**, is a bona fide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Prof. Abhishek Sinha
Research Guide
Assistant Professor
Dept. of Electrical Engineering
IIT-Madras, 600 036

Place: Chennai

Date: 15th June 2021

ACKNOWLEDGEMENTS

I would like to thank my guide Prof. Abhishek Sinha of the Electrical Engineering Department for introducing me to the wonderful field of Scheduling Policies in Wireless Communication and for helping me on my path towards becoming a competent researcher. His timely advice and support has been essential for the completion of this project. I would also like to thank Prof. Krishna Jagannathan for his guidance and mentoring in the early stages of the project.

ABSTRACT

KEYWORDS: Age-of-Information, Max-Age-Policy

Age-of-Information(AoI) is a recently proposed metric for quantifying the freshness of information from the UE's perspective in a communication network. AoI is an important factor in many real life applications like in environment sensing in IoT devices, intrusion detection and real time gaming, which are becoming increasingly important in the recent years.

In this thesis, we have analyzed various scheduling policies dependent on the Age-of-Information metric in the stochastic and online settings. First, we analyze the Max-Age Policy and the Max-Age with Throughput Constraints Policy using simulations and verify they are optimal and close to optimal in solving their respective stochastic problems. We have compared them with other popular policies and their own lower bound. We have then used simulations to show that all UEs have the same AoI frequency distribution under the action of the Max-Age Policy. Next, we have found a policy which minimizes the drift vector for the sum of square roots Lyapunov function for the long term age minimization problem. We then move to analyzing the Multi-Cell Max Weight(MMW) Policy as various parameters are varied and also verified the 2-Optimality of the MMW Policy. We then move to the online setting, where we show that the Max-Age Policy has competitive ratio of 2N and 3N for the average age and max age cost respectively. We also analyze the Receding Horizon Control Policy. Finally, we find an expression for the competitive ratio of the Best-Effort Uniform Updating Policy under the Energy Harvesting Online Setting.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABBREVIATIONS

AoI	Age of Information
UE	User Equipment
MA	Max Age
MW	Max Weight
BS	Base Station
BEC	Binary Erasure Channel
URLLC	Ultra-reliable Low Latency Communication
PF	Proportional Fair
HD	High Definition
IoT	Internet of Things
eMBB	enhanced Mobile BroadBand
MDP	Markov Decision Process
MATP	Max Age with Throughput Constraints
MMW	Multi-Cell Max Weight
RHC	Receding Horizon Control
BU	Best-Effort Uniform Updating

CHAPTER 1

INTRODUCTION

With the advent of technologies such as 5G and the Internet of Things, information freshness is becoming an important characteristic for a wireless communication network. Many applications require timely updates, i.e. they are delay-constrained, to function in a proper manner. A few examples of applications having delay constraints are:

1. **Wireless communication inside a self driving car**
   Self Driving cars use a plethora of sensors to navigate their way through traffic

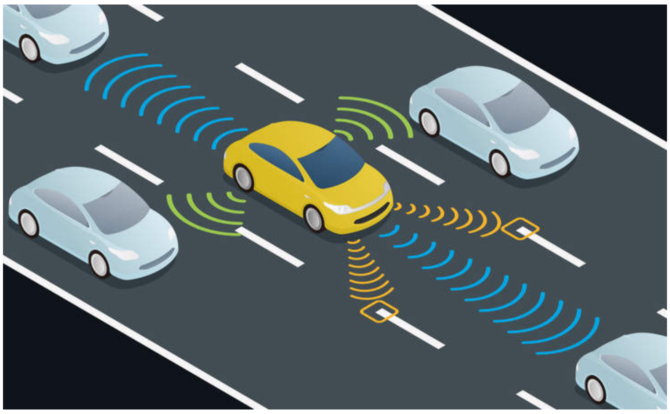

Figure 1.1: Sensing in a self driving car

   and it is extremely important that the central processing unit in the car takes data from all the sensors. Even if one sensor's data is not taken into account, it can lead to a fatal accident. Hence it is important that no sensor's information goes too stale at the central processing node.

2. **Environment sensing in IoT Devices**
   Nowadays, IoT devices are being increasingly used to sense various parameters

Figure 1.2: Sensors for detecting Forest Fires

in the environment, such as temperature, humidity or pollution levels. These devices are also used in detection of natural disasters such as forest fires. We know that most forest fires start small, maybe due to someone throwing something combustible in extremely hot weather. All sensors need to detect these fires at all times and provide timely updates to the central node for early detection of such natural disasters. In such a case, it is important that the information at the central node from all sensors is fresh.

3. **Real time gaming**
   Real Time Gaming is becoming increasingly popular as games like Fortnite and

Figure 1.3: Gamers playing the popular game Fortnite

PUBG gain popularity. Users play these games extremely seriously and would like the delays to be kept to a minimum. It is important that the central server takes updates from all users in a timely manner in order to facilitate the best experience for its users.

4. **Intrusion detection**
   Security systems like CCTV cameras are being used more widely by both public

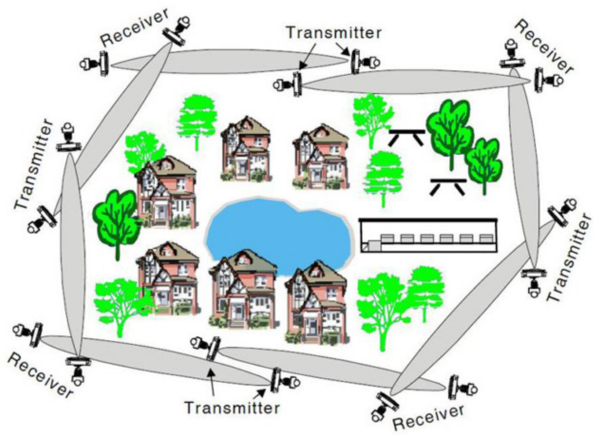

Figure 1.4: Example of an Intrusion Detection System

and private parties to secure an area of importance. There are many such devices operating at the same time ensuring that the area remains safe, or catching a culprit red-handed. Here it is important that information is taken from each and every such device to minimize the chances of crimes. Hence it is imperative that information from each security device is fresh at the central node.

From the above examples, we see that freshness of information is an important consideration for wireless communication networks. To quantify freshness of information, we use the Age-of-Information(AoI) metric, which is defined in the next section.

1.1 Age of Information

Age of Information can be defined as the the time elapsed since the User Equipment(UE) last received a new packet from the Base Station(BS). Suppose we define $u(t)$ to be the timestamp of the last received update at time t. Then the AoI is given by the following equation:

$$h(t) = t - u(t)$$

We use the function h(t) to denote AoI in the rest of the thesis.

We consider only the discrete time case in this thesis. Hence the evolution of AoI for a UE can be defined as follows: If the UE is able to successfully receive a transmission at timeslot t, then its AoI becomes 1 in the next timeslot. If it is not able to receive a transmission successfully at timeslot t, then the AoI of the UE increases by 1, which can be formally written as $h(t + 1) = h(t) + 1$.

Now that we have defined Age-of-Information, it is imperative to design and study scheduling policies dependent on the Age-of-Information metric, so that they can be applied in the many important applications. Recent work on scheduling policies related to age of information is given in the next section.

1.2 Related Work

In Srivastava *et al.* (2019), the problem of minimizing long term max-age was considered, and the optimal Max-Age Policy was proposed for only delay constrained UEs, and the approximately optimal Max-Age with Throughput Constraints Policy was proposed for minimizing the long term max age with one throughput constrained UE. In Kadota *et al.* (2018), the problem of minimizing the long term average age is considered and many policies, including the Max-Weight Policy were analyzed. In

Banerjee *et al.* (2020), the AoI problem was considered in the online setting and the stochastic setting for the multiple base stations case. In the stochastic problem, the Multi-Cell Max Weight policy was proposed, and the lower bound and upper bound on the competitive ratio of the Max-Age policy was found in the online setting. In Lin *et al.* (2012), the receding horizon policy was examined in the online setting, for geographical load balancing. In Feng and Yang (2021), the problem of minimizing the long term average AoI in the energy harvesting context was considered.

1.3 Results in the Thesis

1. The Optimality of the Max-Age Policy and the superior performance of the MATP Policy when compared to other popular policies were verified
2. It was shown that as β is increased, the throughput is not affected until the value of 10,000 is reached. For the value of 10^6 and above, the throughput approaches the maximum possible throughput
3. The AoI distributions of all UEs was exactly same, irrespective of their $p_i's$. UEs with very bad channels affected the distribution to a great degree irrespective of the other UEs
4. The Long Term Average AoI Problem using the Sum of Square Roots Lyapunov Function was analyzed and the following scheduling policy was obtained: Schedule the user with highest value of $p_i h_i$
5. It was verified that the MMW policy has 2-optimality in the i.i.d uniform mobility case. As M and N are varied but $\frac{M}{N}$ is kept constant, the ratios are very close. In the non-uniform mobility setting, the ratios are still less than 2, and the graph is linear with p_i
6. An upper bound of 2N and 3N for the competitive ratio for the Average Age and Max Age cost, respectively, under the Max Age policy in the Online Setting was obtained
7. The Receding Horizon Control Policy in the Online Setting was analyzed using the Potential Function Method but no significant results were found
8. An expression for the competitive ratio for the Best-Effort Uniform Updating Policy under the Energy Harvesting Online Setting was found

The rest of the thesis is organized in the following fashion: Chapter 2 describes the Max-Age and Max-Age with Throughput Constraints Policies and their properties. Chapter 3 analyzes the problem of minimizing the long term average AoI for the system with multiple UEs and multiple Base Stations. Chapter 4 analyzes the Max-Age and

Receding Horizon Control Policies in the online setting and provides an upper bound on their competitive ratios. Chapter 5 analyzes the Best-Effort Uniform Updating Policy under the Energy Harvesting Online Setting and gives an expression for the competitive ratio. In Appendix A, we describe the Lyapunov Drift Optimization Method. In Appendix B, we describe the Amortized Costs Method using the Potential Functions for finding the competitive ratio. In Appendix C, we give proofs related to the Max-Age Policy.

CHAPTER 2

Scheduling Policies for a Single Base Station

2.1 Introduction

In this chapter, we analyze the max-AoI and average AoI metrics with regard to multiple users communicating with a single base station. The chapter is organized in the following fashion:

First we define the system model for our problem. Then we summarize the findings for the average AoI case from Kadota *et al.* (2018). Next, we present the findings of Srivastava *et al.* (2019) for the Max-AoI case considering only delay constraints. We provide simulation results for the above cases. We then present simulation results for the case when one of the users is also throughput constrained. Next, we see simulations about the age frequency distribution of UEs under the MA policy. Finally, we use a sum of square roots function to design a scheduling policy using Lyapunov Drift Optimization.

2.2 System Model

We consider a downlink scheduling problem, with one BS serving N wireless UEs. Each UE is infinitely backlogged. Time is divided into slots. At each timeslot, the BS can transmit to only one UE.

When the BS transmits to UE i, then we assume that the channel is a binary erasure channel, BEC(p_i). This means that when transmitted, the data is sent and decoded successfully by the UE i with probability p_i, and the data is permanently lost with probability $1 - p_i$. Here $p_i > 0 \; \forall i$.

Finally, we use the function $h_i(t)$ to denote the AoI of UE i at timeslot t.

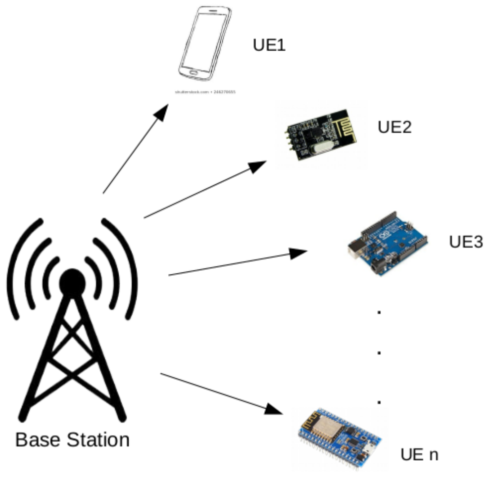

Figure 2.1: System Model

2.3 The Max-Weight Policy

First we study the problem of minimizing the Long Term Average AoI for the above model. Hence we want to solve the following problem:

$$OPT^* = \min_{\pi \in \Pi} \lim_{K \to \infty} \frac{1}{KM} \mathbf{E}[\sum_{k=1}^{K} \sum_{i=1}^{M} \alpha_i h_i(k)]$$

subject to the constraints:

$$\hat{q}_i^{\pi} \geq q_i, \forall i$$
$$\sum_{i=1}^{M} u_i(k) \leq 1, \forall k$$

Here we take the policy π from the set of all feasible policies Π which minimizes the **long term average AoI**. Also α_i is the weight of node/UE i.

The first constraint puts a lower bound on the throughput of each user and the second constraint states that at each timeslot only one user can be scheduled.

We now define the Max-Weight(MW) Policy which can be shown to satisfy the constraints in the problem.

Max-Weight Policy: At each timeslot, schedule the UE which has the highest value of $p_i h_i^2$.

This policy is derived using the method of Lyapunov Drift Optimization. MW is a scheduling policy designed to reduce the expected increase in the Lyapunov Function. It can be said that the MW policy keeps the network in a desirable state by controlling the growth of the Lyapunov Drift function. In section 2.11, we use this method in a similar way to obtain a new policy.

It was shown that the MW policy is 4-optimal in Kadota *et al.* (2018).

2.4 Minimizing the Peak AoI for Delay Constrained UEs

We now consider the problem of minimizing the long term Peak AoI with only delay constrained UEs. Delay constrained UEs do not have throughput as an important consideration but they want the latency constraints to be satisfied.

With hard deadline constraints for each user in the case of URLLC traffic in 5G, minimizing the peak-AoI metric is more practically meaningful than the minimizing of average AoI.

At any slot t, we define $h_{max}(t) = \max_{i \in \{1,2,\ldots,N\}} h_i(t)$ to be the peak instantaneous AoI among all N users. Our objective is to design a scheduling policy $\pi^* \in \Pi$ which minimizes the time averaged expected Peak AoI. Mathematically, we want to optimally solve the following stochastic control problem $\mathbf{P}_{Sched}$:

$$\lambda^* = \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbf{E}(\max_i h_i(t)) \tag{2.1}$$

The only constraint we have in this problem is that the BS can schedule transmission to only one UE in each timeslot.

We now define the Max-Age(MA) Policy, a greedy policy aimed at finding the optimal solution to the above problem.

Max-Age(MA) Policy: At any timeslot, schedule the user having the highest AoI. In other words, at any timeslot t, schedule the user i which maximizes the metric $\max_{i \in \{1,2,\ldots,N\}} h_i(t)$, with ties being broken arbitrarily.

In the next section, we give theorems related to optimality and tail distributions for the MA Policy.

2.5 Optimality of the MA Policy and Other Results

In this section, we look at theorems and proofs relating to the MA Policy. The following results are from the paper by Srivastava *et al.* (2019).

The first theorem proves that the MA policy is optimal in solving the stochastic control problem P_{Sched} mentioned above.

Theorem 2.5.1. *The greedy policy MA is an optimal policy for the problem (2.1). Moreover the long term peak AoI is given by* $\lambda^* = \sum_{i=1}^{N} \frac{1}{p_i}$.

Proof. Please Refer Theorem C.0.1 of Appendix C. □

We can see that the MA policy schedules UEs independent of their channel statistics, which is unlike the MW policy which considers the probability of successful transmission while determining which UE should be scheduled.

Moreover, this is a rare instance where the Bellman equation has an analytic solution, since the cost-to-go has an explicit solution.

The following three theorems show that the MA policy also has the optimal large deviation exponent for the max-age metric. Theorem (2.5.2) proves that the peak-Age process has an exponentially light tail under the action of the MA policy.

Theorem 2.5.2. *Under the action of the MA policy, there exists a constant* $c(N, \boldsymbol{p}) > 0$*, such that, for any fixed time* $t \geq 1$ *and any* $k \geq 2N$*,*

$$\mathbf{P}^{MA}(\max_i h_i(t) \geq k) \leq c(N, \boldsymbol{p}) k^N (1 - p_{min})^k$$

Proof. Please Refer Theorem C.0.2 in Appendix C. □

The next theorem establishes a fundamental performance bound of the peak-AoI tail probability under the action of any scheduling policy.

Theorem 2.5.3. *Under the action of an arbitrary scheduling policy* π*, at any slot* $t \geq k$ *and* $k \geq 1$*, we have:*

$$\mathbf{P}^{\pi}(\max_i h_i(t) \geq k) \geq (1 - p_{min})^k$$

Proof. Please Refer Theorem C.0.3 in Appendix C. □

Finally, combining theorem (2.5.2) and theorem (2.5.3) we can show that the MA policy achieves the optimal large-deviation exponent for the peak-AoI metric.

Theorem 2.5.4. *The MA policy achieves the optimal large-deviation exponent for the max-age metric and the value of the optimal exponent is given by:*

$$-\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{k}\log \mathbf{P}^{MA}(\max_i h_i(t) \geq k) = -\log(1-p_{min})$$

Proof. Please Refer Theorem C.0.4 in Appendix C. □

From theorem(2.5.4), we are able to show that the peak-AoI metric stays within a bounded limit with high probability. This ensures high-probability delay guarantees to URLLC traffic having a strict latency requirement.

We conclude this Section with the following stability result of the Age-process $\boldsymbol{h}(t)_{t\geq 1}$ under the MA policy.

Theorem 2.5.5. *The Markov Chain $\boldsymbol{h}(t)_{t\geq 1}$ is Positive Recurrent under the action of the MA policy.*

Proof. Please Refer Theorem C.0.5 in Appendix C. □

2.6 Simulation Results for the MA Policy

Simulation Set-Up: We simulate a downlink wireless network with N nodes, each with a binary erasure channel. The probability of successful transmission for the i^{th} channel p_i is sampled i.i.d. from a Gaussian distribution in [0, 1]. Each simulation is run for 10^5 slots, and an average of 100 simulations is taken for the plots. For the PF algorithm, the value of ϵ is set to 0.1.

Simulated Policies: In this section, we simulate the following four scheduling policies for the downlink wireless system described below:

1. MA: We schedule the UE which has the highest age metric h_i
2. MW: We schedule the UE which has the highest value of $p_i h_i^2$

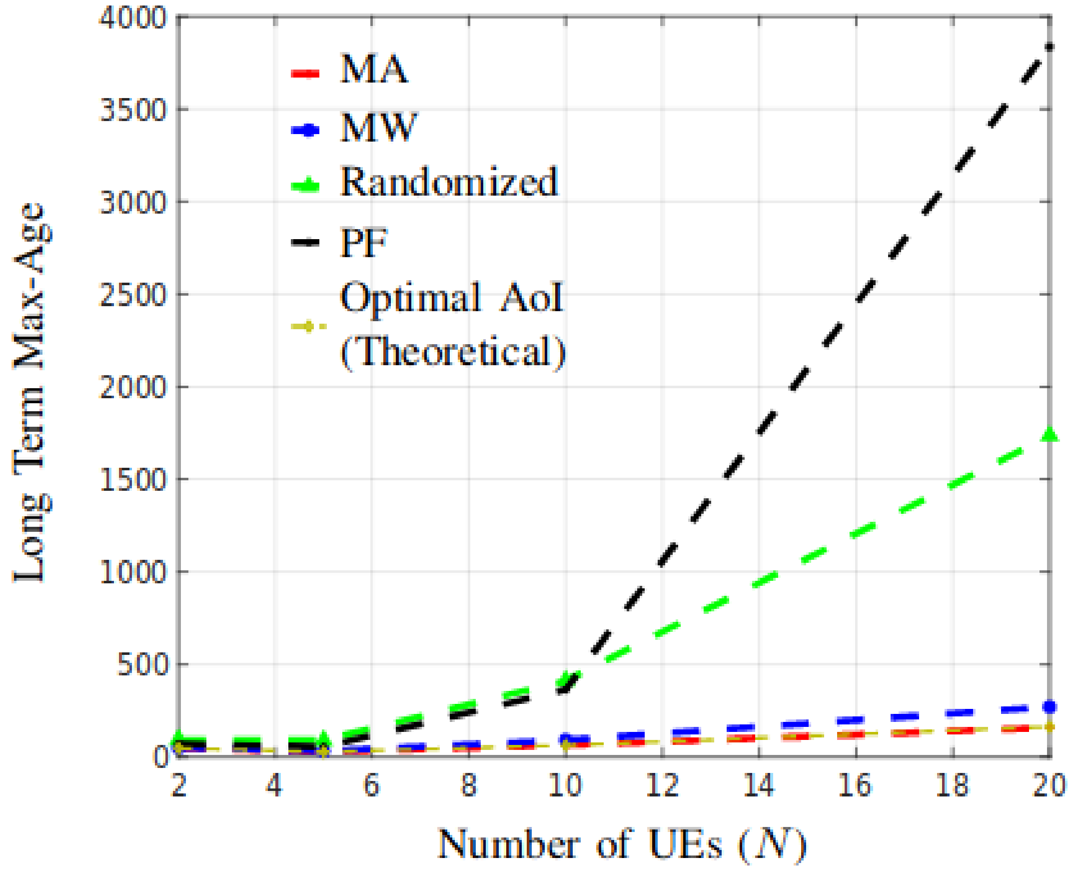

Figure 2.2: Comparative Performance of the Proposed Max-Age (MA) policy with three other Scheduling Policies with varying the number of UEs for the Problem $\mathbf{P}_{Sched}$

3. Randomized Policy: We schedule one user at random in every timeslot

4. Proportional Fair Policy: We schedule the UE which has the highest value of the metric $\frac{p_i}{R_i(t)}$, where the function $R_i(t)$ is the exponentially-smoothed average rate, which is updated at every time slot as: $R_i(t+1) = R_i(t) + \epsilon y_i(t)$. Here $y_i(t)$ is the instantaneous throughput to the UE i at slot t. This is based on the paper by Kim and Han (2005)

Discussion: In Figure 2.2, we have compared the performance of five different scheduling policies on the basis of long-term Max-Age in the set-up described in the System Model. The number of delay-constrained UEs associated with the BS has been varied from 2 to 20. For reference, we have also included the Theoretical Optimal value of AoI, given in Theorem (2.5.1). As expected, we see that the performance of the Max-Age (MA) policy matches with the optimal value. The Max-Weight policy performs slightly worse than the optimal MA policy. However, we find that the randomized and the PF policy performs very poorly in terms of the long-term peak-age metric. The bottom line is that a utility-maximizing policy (such as PF, which maximizes the summation of logarithmic rates of the UEs) may be far from optimality when maximizing freshness of information on the UE side.

2.7 Minimizing the Peak AoI with Throughput Constraints

In this section, we consider a generalization of the above system model, where, in addition to maintaining a small peak- AoI, there is also a throughput constrained UE (denoted by UE 1 WLG), which is interested in maximizing its throughput. This is taken from Srivastava *et al.* (2019).

A possible real life example would be when a person is streaming HD videos at home, which is a throughput constrained activity, and all smart IoT devices are also communicating, with these being delay constrained UEs, and hence they require low latency. In a small-cell residential network, all of these devices are served by a single BS typically located within the house.

In this problem, our objective is to consider a system with one throughput constrained UE and the rest being delay constrained UEs. To take into account the throughput constraints of UE 1, we define a sequence of random variables $\{\bar{a}(t)\}_{t\geq 1}$ such that $\bar{a}(t) = 1$ if the UE does not successfully transmit in timeslot t, and $\bar{a}(t) = 0$ if the UE does successfully transmit in timeslot t. Also we define $\beta \geq 0$ as the non-negative tuning parameter.

We want to solve the following scheduling problem $\mathbf{P}'_{Sched}$ and find an appropriate scheduling policy:

$$\lambda^{**} = \inf_{\pi\in\Pi} \limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbf{E}(\max_i h_i(t) + \beta\bar{a_1}(t))$$

The single-stage cost described above may be understood as follows: the first term $\max_i h_i(t)$ denotes the usual maximum AoI across all UEs. The second term imposes a penalty of β if UE 1 does not receive a packet at the current slot. By suitably controlling the value of β, a tradeoff between the peak-AoI and achievable throughput to UE 1 (serving the eMBB traffic) may be obtained.

The problem $\mathbf{P}'_{Sched}$ is an instance of an infinite state average cost MDP with an additional action-dependent additive per-stage cost term $\beta\bar{a}_1(t)$. Arguing as before, and introducing an additional cost term g_i arising due to the throughput constraint, the Bellman Equation for this problem may be written down as follows:

$$\lambda^{**} + V(\boldsymbol{h}) = \min_i \{p_i V(1, \boldsymbol{h_{-i}} + \mathbf{1}) + (1-p_i)V(\boldsymbol{h} + \mathbf{1}) + g_i\} + \max_i h_i \tag{2.2}$$

where:

$$g_i = \beta, \text{if i} \neq 1 \tag{2.3}$$

$$g_i = \beta(1-p_1), \text{otherwise} \tag{2.4}$$

Here g_i denotes the expected throughput cost when the UE 1, which successfully receive a packet at slot t. The rest of the Bellman equation is the same as the exclusively delay constrained case.

We consider the same differential cost-to-go function $V(\boldsymbol{h}) = \sum_i \frac{h_i}{p_i}$. In this case the RHS expression of equation (2.9) can be written as follows:

$$V(\mathbf{h}) + \sum_j \frac{1}{p_j} + \min_i(-h_i + g_i) + \max_i h_i \tag{2.5}$$

2.8 The MATP Policy

We now define the Max-Age with Throughput Constraints(MATP) Policy as follows:

Max-Age with Throughput Constraints(MATP) Policy: At any slot t, schedule the user i having the highest value of $h_i(t) - g_i$, where g_i is given by equations (2.3) and (2.4).

The MATP policy strikes a balance between minimizing the peak-AoI (through the first term $h_i(t)$), while also ensuring sufficient throughput to the eMBB user (through the second term g_i). As the term β increases, the weight given to the eMBB user dominates the AoI term, which in turn facilitates the eMBB user.

Next, we show that the MATP Policy is approximately optimal in solving the Bellman equation given in equation (2.2).

First we observe that:

$$\begin{aligned}\min_i(-h_i + g_i) &\geq min_i(-h_i) + \min_i g_i \\ &= -\max_i(h_i) + \beta(1 - p_1)\end{aligned}$$

Also, since $g_i \leq \beta \; \forall i$, we have:

$$\begin{aligned}\min_i(-h_i + g_i) &\leq \min_i(-h_i + \beta) \\ &= -max_i h_i + \beta\end{aligned}$$

Now we take $\lambda^{**} = \sum_j \frac{1}{p_j} + \beta$. We see that under the action of the MATP policy, the sup-norm of the difference between the RHS and LHS of the Bellman Equation is bounded by the constant βp_1. In other words, upon denoting the RHS of the Bellman operator of (2.2) by T(.), we have:

$$||V - TV|| \leq \beta p_1$$

Hence, we conclude that the policy MATP approximately solves the Bellman Equation (2.2). We see that the MATP policy does take into account the channel statistics, i.e. the channel information of the eMBB user is taken into account. This is in contrast to the MA policy, which is completely oblivious to channel statistics.

2.9 Numerical Results for the MATP Policy

Simulation Set-Up: We simulate a downlink wireless network with N nodes, each with a binary erasure channel. The probability of successful transmission for the i^{th} channel p_i is sampled i.i.d. from a uniform distribution in [0, 1]. Each simulation is run for 10^5 slots, and an average of 100 simulations is taken for the plots. For the PF algorithm, the value of ϵ is set to 0.1. The first user is said to be delay constrained and the erasure probability for UE 1 is set to 0.8

In figure 2.3, we plot the time averaged throughput of UE 1 as a function of β. We see from the green curve that the maximum possible time averaged throughput for UE

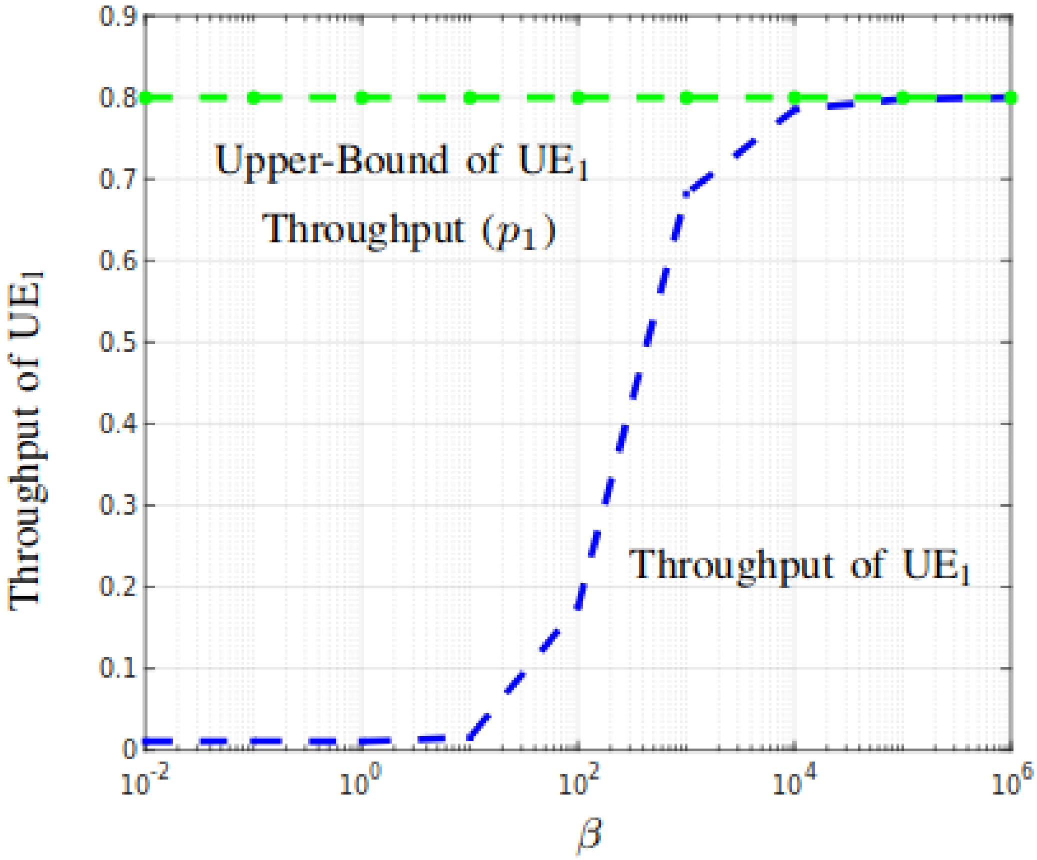

Figure 2.3: Variation of Throughput of UE 1 with the parameter β.

1 is 0.8, which occurs when only UE 1 is scheduled for transmission, and successfully receives information with an erasure probability of 0.8. Now, as we start increasing the value of β from 0.01 to 10, we see that the throughput is almost 0. As we increase the value of β from 100 to 1000 and gradually to 10,000, we see that there is a steep upward curve and at $\beta = 10,000$, the throughput of the eMBB user is close to the maximum possible limit. As we further increase the value of β, we get closer and closer to the upper bound of 0.8, as is clear from the figure.

In figure 2.4, we compare the MATP policy with the following three policies described in section (2.6):

1. MW Policy
2. Randomized Policy
3. Proportional Fair Policy

From the figure we can see that the Proportional Fair Policy performs even worse than the randomized policy. This could mean that PF, which does optimization to satisfy

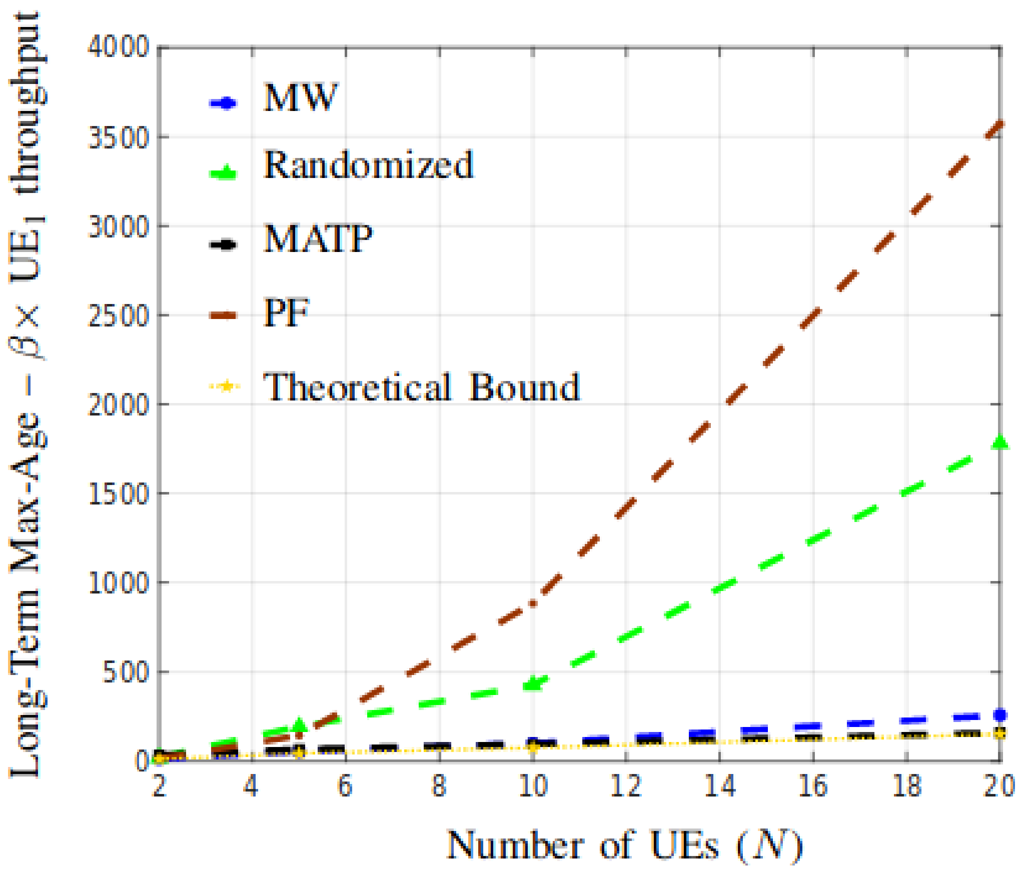

Figure 2.4: Comparative Performance of the Proposed MATP Policy with other well-known scheduling policies for the Problem $\mathbf{P}'_{Sched}$

throughput constraints, does not perform well when tested under this metric. MW performs reasonably well under this metric and is close to the theoretical bound. Finally, we see that MATP closely follows the theoretical bounds, which is given by $\sum_j \frac{1}{p_j} + \beta$. Hence we see that the MATP policy is approximately optimal for solving the stochastic problem $\mathbf{P}'_{Sched}$.

2.10 Age Frequency Distribution of UEs

In this section we analyze using simulations how many times a particular AoI value is reached by any UE under the action of the MA policy. As an example, consider an AoI value of 5. We want to see how many times a particular UE has an AoI of 5 during the time horizon T. We do this for all UEs.

Finally, we want to see if there is a relation between these values for UEs and how they depend on the erasure probabilities and number of UEs.

2.10.1 Variation of Age Distribution with Number of UEs

Simulation Setup: The system model is the same as we have considered in the previous sections in this chapter, where we have one BS communicating with N UEs. We consider a time horizon $T = 10^5$. We take 100 sample paths and add all values together. The probability distributions are randomly chosen for each N. Then we record the number of times any AoI value is reached by a UE and plot it in the following graphs.

Below we have considered three values of N: $\{2, 5, 20\}$. Then we have individually plotted the AoI distribution on a linear Y-axis in three graphs and finally plotted all in one graph with a Y-axis in log scale.

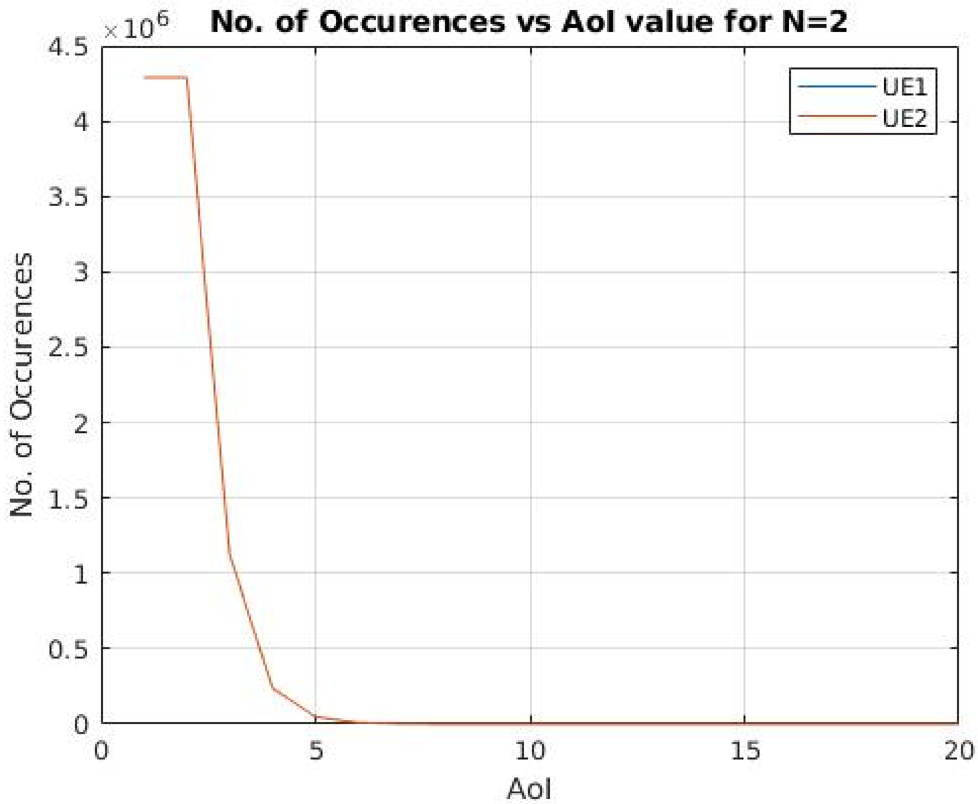

Figure 2.5: AoI value vs Number of Occurrences for $N = 2$

In the figures (2.5), (2.6) and (2.7), we see that the the AoI distributions of the UEs are practically indistinguishable. Also the dropoff is exponential, which supports the large deviation exponent property of the MA policy.

Below are some observations comparing the AoI distributions on the basis of the value of N:

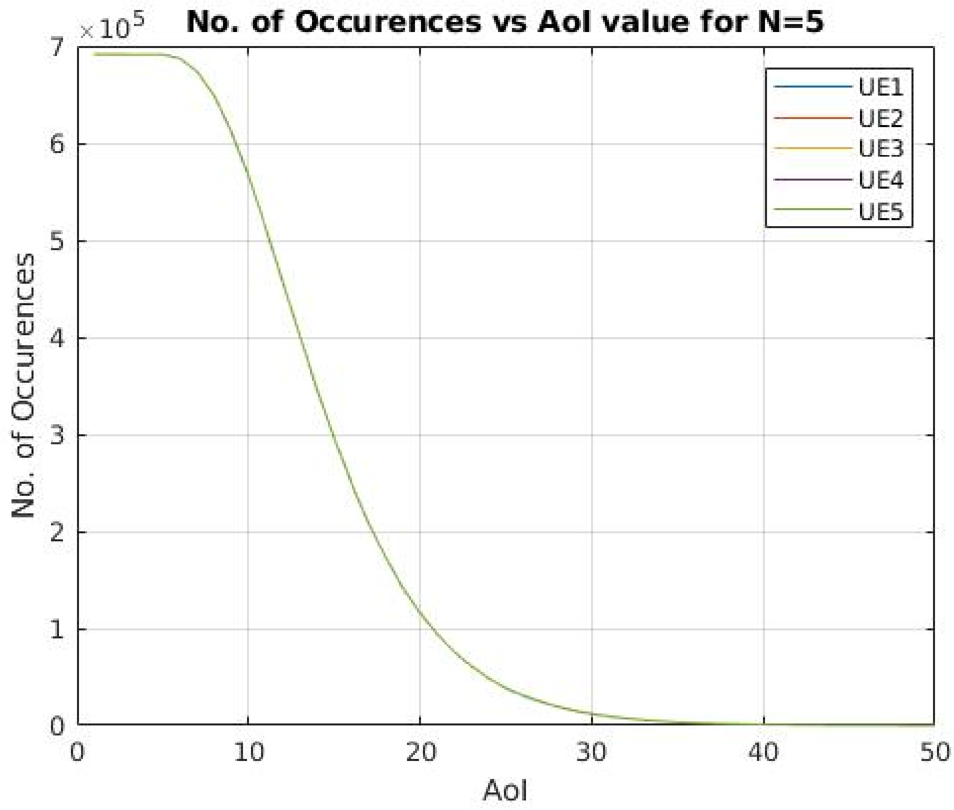

Figure 2.6: AoI value vs Number of Occurrences for $N = 5$

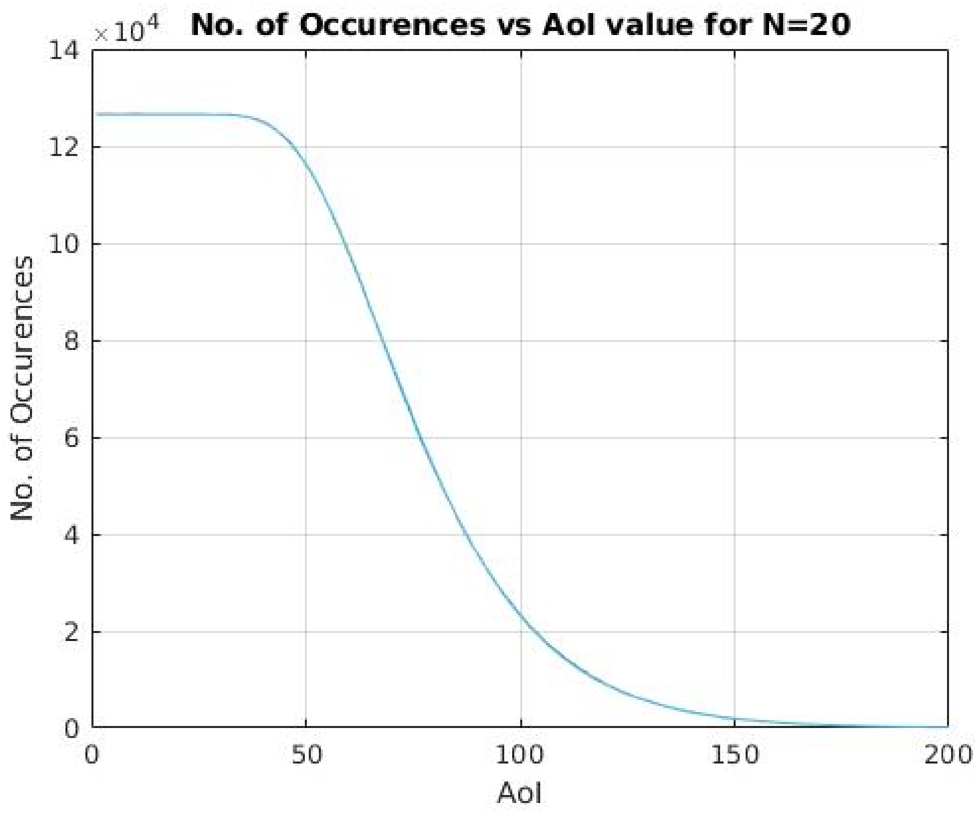

Figure 2.7: AoI value vs Number of Occurrences for $N = 20$

1. For all AoI distributions, the lower AoI values happen most frequently, which is hardly surprising. The number of times these values occur falls with the value of N.
2. As N increases, the AoI distribution has a larger range of AoI, which makes sense, since the MA policy is a round robin policy and hence the AoI of a user increases for a longer time till it is scheduled again as the value of N increases.

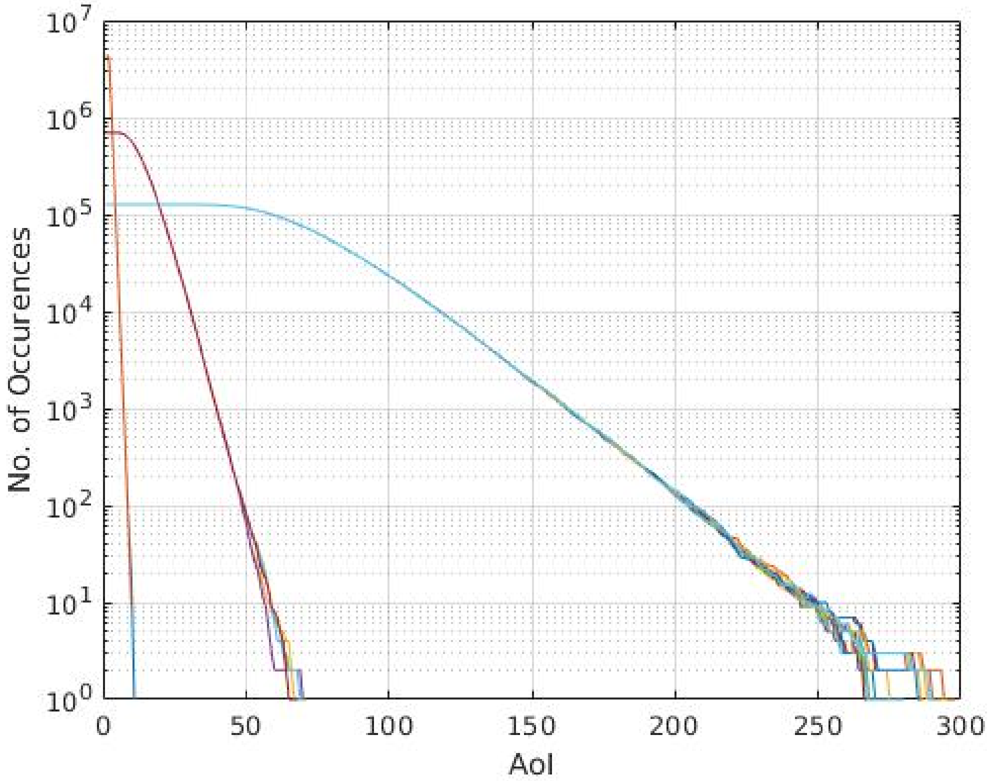

Figure 2.8: AoI value vs Number of occurrences for $N = \{2, 5, 20\}$ on Log Scale

In the figure (2.8) we compare the AoI distributions for all three values of N with the Y-axis on the log scale. $N = 2$ has the highest number of occurrences for AoI of 1, followed by $N = 5$ and $N = 20$. Here it is clearer that at the tail of each AoI distribution, the number of occurrences of each AoI value for larger values of AoI starts to differ for each UE. This is probably due to the finiteness of the simulation.

Hence the following hypothesis is proposed:

Hypothesis: The long term AoI distribution of UEs under the action of the MA policy is the same and is independent of the number of UEs and erasure probability distribution.

2.10.2 Variation of Age Distribution with Distribution of $p_i's$

Simulation Setup: The system has 1 BS communicating with N UEs. Here we have taken $N = 5$. We have considered a time horizon of $T = 10^5$. We take 100 sample paths and add all the values together. The following four probability distributions are considered(these are values of the $p_i's$):

1. $\mathbf{P_1} = \{0.1, 0.1, 0.1, 0.1, 0.1\}$
2. $\mathbf{P_2} = \{0.1, 0.3, 0.5, 0.7, 0.9\}$
3. $\mathbf{P_3} = \{0.9, 0.9, 0.9, 0.9, 0.9\}$
4. $\mathbf{P_4} = \{0.01, 0.02, 0.5, 0.98, 0.99\}$

First we consider how the AoI distribution is dependent on the sum of probability i.e. what happens if all UEs have good channels vs if all UEs have bad channels. Hence to analyze this, we consider only $\mathbf{P_1}, \mathbf{P_2}$ and $\mathbf{P_3}$ in the first plot.

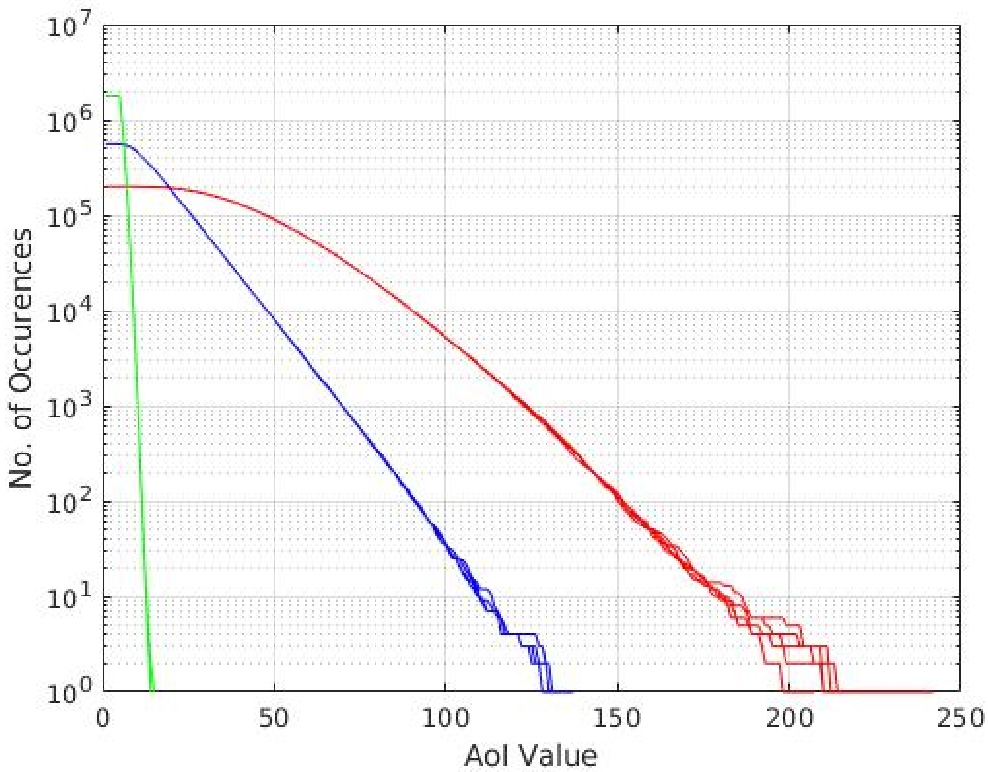

Figure 2.9: AoI value vs Number of occurrences for $N = 5$ on Log Scale for various erasure probability distributions

In figure 2.9, the red curves represent $\mathbf{P_1}$, the blue curves represent $\mathbf{P_2}$ and the green curves represent $\mathbf{P_3}$.

We can see from the plots that when the $p_i's$ are small, i.e. 0.1, then the AoI distribution has a larger range since to complete one cycle of the round robin in the MA policy will take much more time. As we go to $\mathbf{P_2}$ and then to $\mathbf{P_3}$, we see that the range of the AoI distribution becomes smaller and smaller. Even though this is the case, for most of the range of AoI, the UEs have the same number of occurrences. This changes only in the higher part of the range when there are lesser number of samples.

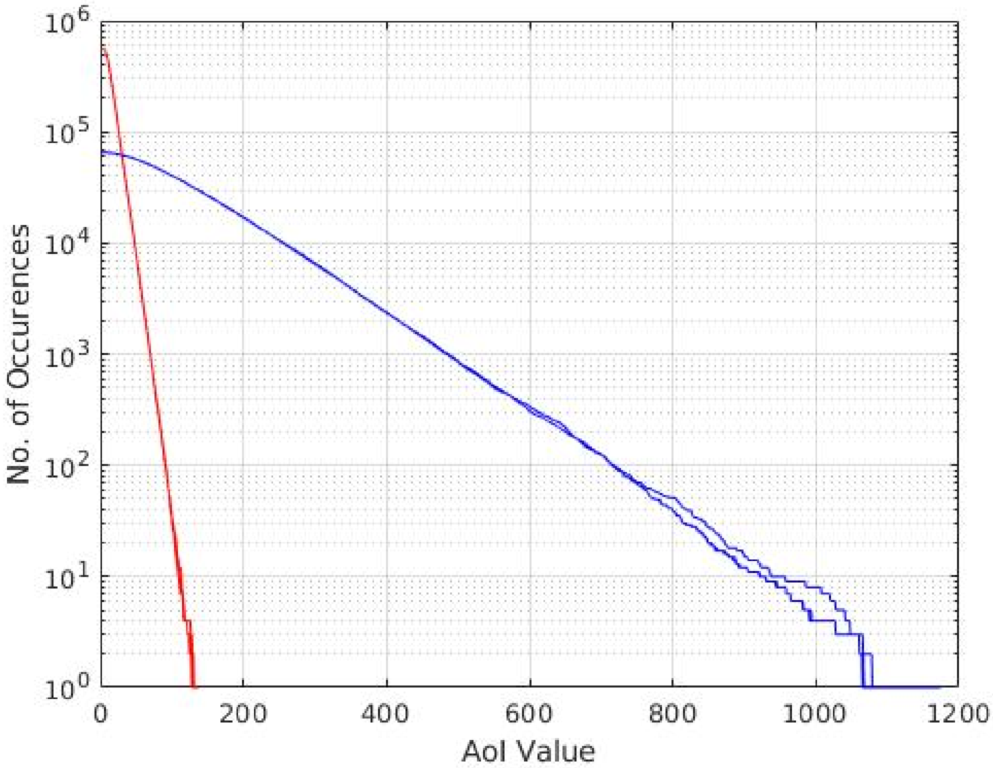

Figure 2.10: AoI value vs Number of occurrences for $N = 5$ on Log Scale for erasure probability distributions with the same sum of $p_i's$

In figure 2.10, the red curves represent $\mathbf{P_2}$ and the blue curves represent $\mathbf{P_4}$.

Here we have some UEs with extremely small probabilities of successful transmission in the probability distribution $\mathbf{P}_4$. Hence we see that even though the sum of $p_i's$ is same, these affect the AoI distribution in a major way. This means that extremely small or extremely large values of $p_i's$ affect the distribution in a major way irrespective of the other probabilities.

2.11 Design of Scheduling Policy using Sum of Square Roots Lyapunov Function

In this section, we move a little from the Peak AoI problem and try to design a scheduling policy with the same method used for the design of the MW policy mentioned in section (2.3). We use the Lyapunov Drift Approach, which has a brief introduction in Appendix A. We try and use a sum of square root Lyapunov function to design a scheduling policy.

Hence the Lyapunov function is defined as:

$$L(\boldsymbol{h}(t)) = \sum_i \sqrt{\frac{h_i(t)}{p_i}}$$

We also define the function $\mu_i(t)$ as a binary control variable which denotes if UE i is served during timeslot t. It is 1 when the UE is served during timeslot t, and 0 if it is not served during timeslot t.

From this we can define $\boldsymbol{\mu}(t) = \{\mu_1(t), ..., \mu_N(t)\}$.

Next we write down the expectation of $\sqrt{h_i(t+1)}$ as a function of $\boldsymbol{h(t)}$ and $\boldsymbol{\mu(t)}$.

First notice that $\sqrt{h_i(t)+1} \leq \sqrt{h_i(t)} + 1$. Hence we have:

$$\implies \mathbf{E}(\sqrt{h_i(t+1)}|\boldsymbol{h}(t), \boldsymbol{\mu(t)}) \tag{2.6}$$

$$= \mu_i(t)p_i + (1 - \mu_i(t)p_i)\sqrt{h_i(t)+1} \tag{2.7}$$

$$\leq \mu_i(t)p_i + (1 - \mu_i(t)p_i)(\sqrt{h_i(t)} + 1) \tag{2.8}$$

$$= 1 + \sqrt{h_i(t)} - \mu_i(t)p_i\sqrt{h_i(t)} \tag{2.9}$$

We know that:

$$\mathbf{E}(\sqrt{h_i(t)}|\boldsymbol{h}(t), \boldsymbol{\mu(t)}) = \sqrt{h_i(t)} \tag{2.10}$$

Hence we subtract equation (2.19) from equation (2.18) and divide by $\sqrt{p_i}$ to obtain:

$$\mathbf{E}(\frac{\sqrt{h_i(t+1)} - \sqrt{h_i(t)}}{\sqrt{p_i}}|\boldsymbol{h}(t), \boldsymbol{\mu(t)}) = 1 - \mu_i(t)\sqrt{h_i(t)p_i}$$

Finally, we sum over all UEs to obtain:

$$\mathbf{E}(L(\boldsymbol{h}(t+1)) - L(\boldsymbol{h}(t))|\boldsymbol{h}(t), \boldsymbol{\mu}(t)) \leq N - \sum_i \mu_i(t)\sqrt{h_i(t)p_i}$$

The above equation represents the Lyapunov drift and is equivalently written as:

$$\Delta(\boldsymbol{h}(t), \boldsymbol{\mu}(t)) \leq N - \sum_i \mu_i(t)\sqrt{h_i(t)p_i}$$

Now we want to minimize the Lyapunov drift at each timeslot so that we are able to maintain stability of the network. Hence we choose the policy to be:

MW Policy for Sum of Square Roots Lyapunov Function: At timeslot t, we schedule the UE which has the highest value of $p_i h_i(t)$.

Hence, the policy obtained by using the square root Lyapunov function is similar to the Max-Weight Policy.

2.12 Summary

In this chapter, we first looked at the average age problem and the MW policy. Then we described the problem of minimizing the long term Peak AoI for delay constrained UEs. We described the optimal MA policy for this problem which also had the optimal Large Deviation Exponent. Next, we compared the performance of the MA policy with other popular policies using simulations.

We then considered the problem of scheduling when one of the users is throughput constrained and described the approximately optimal policy MATP. We then compared this policy with other popular policies and found that it beats the other policies by a significant margin.

Next, we investigate the relations between the age distribution of the UEs using extensive simulations and found interesting properties. Finally we analyzed the MW policy using a sum of square roots Lyapunov function and were able to arrive at a scheduling policy very similar to the one obtained by using a square Lyapunov function.

CHAPTER 3

Scheduling Policies for Multiple Base Stations

3.1 Introduction

In this chapter, we consider a network in which there are multiple base stations and multiple UEs. We describe theorems and policies based on the paper Banerjee *et al.* (2020) and then simulations show simulation results from Matlab. The chapter is organized in the following fashion:

First we define the system model for our problem. Then we provide a description of the policy which we will simulate. Next, we provide details of the simulations. Finally we show the simulations and analyze the simulation results.

3.2 System Model

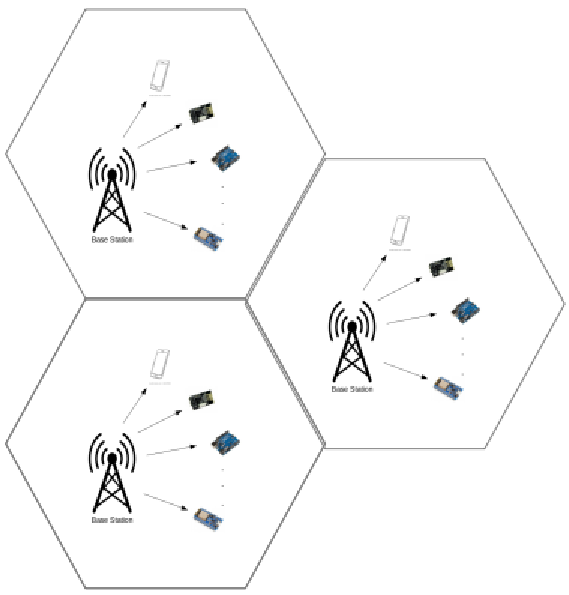

Figure 3.1: System Model for Multiple BSs and Multiple UEs

3.2.1 Channel Model

We consider a system which has M BSs and N UEs. The BSs have designated areas in which they are able to carry out successful communication which we call a cell. Time is divided into slots and at every slot, the BS can beamform and schedule a packet transmission to one of the UEs in its cell. The channel between the BS and UE_i is a BEC(p_i), i.e. p_i is the probability of successful transmission. This value remains the same irrespective of the BS associated with UE_i.

Hence if UE_i is scheduled in timeslot t, then with probability p_i the transmission is successful, and with probability $1 - p_i$, the packet is lost.

3.2.2 Mobility Model

We assume that the UE mobility is modelled by a stationary ergodic process. Now let $C_i(t)$ denote the BS to which UE_i is connected. We also assume that the cells of the BSs are disjoint. This means that no UE will be able to communicate with two or more BSs in the same timeslot. Then $\{C_i(t)_{t\geq 1}\}$ is a stationary ergodic process.

The probability that any UE_i is associated with any BS_j is given by $\mathbf{P}(C_i(t) = j) = \psi_{ij}$. We also assume that the mobility of different UEs is independent.

Finally, the state of the UEs are given by $\boldsymbol{h}(t) = (h_1(t), ..., h_N(t))$. The association of UEs to the BSs is given by $\boldsymbol{C}(t) = (C_1(t), ..., C_N(t))$.

3.3 Problem Definition

We want to design a scheduling policy π as follows: At any timeslot t, π selects a UE in each cell and schedules the transmission of a fresh packet from the BSs to the UEs over the binary erasure channel. The scheduling policies are assumed to be causal.

Now let all policies which satisfy the above constraints belong to the set Π. We want to design a distributed scheduling policy which minimizes the long term average

AoI of all users. The following problem is then posed:

$$AoI^* = \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{N} (\sum_{i=1}^{N} \mathbf{E}^{\pi}(h_i(t))) \tag{3.1}$$

3.4 A Lower Bound

The lower bound for this problem was calculated in Banerjee *et al.* (2020) as the following theorem:

Theorem 3.4.1. *In the stationary setup, the optimal AoI in equation (3.1) is lower bounded by:*

$$AoI^* \geq \frac{1}{2Ng(\psi)} (\sum_{i=1}^{N} \sqrt{\frac{1}{p_i}})^2 + \frac{1}{2}$$

where the quantity $g(\psi)$ denotes the expected number of cells with at least one UE, where the expectation is taken with respect to the stationary occupancy distribution ψ.

It was shown that if the limiting occupancy distribution of each UE is uniform across all BSs i.e. $\psi_{ij} = \frac{1}{M}, \forall i, j$, then:

$$g(\psi^{unif}) = M(1 - (1 - \frac{1}{M})^N)$$

3.5 The MMW Policy

We define the Multi-Cell Max Weight(MMW) Policy which is an extension of the MW policy in the single BS case.

MMW Policy: At every slot, each BS schedules a UE which is in its cell and has the highest value of $p_i h_i^2(t)$.

We have the following theorem:

Theorem 3.5.1. π^{MMW} *is a 2-approximation scheduling policy for statistically identical UEs with i.i.d uniform mobility, i.e.* $p_i = p, \forall i$ *and* $\psi_{ij} = \frac{1}{M}, \forall i, j$.

3.6 Simulation Results for the MMW Policy

3.6.1 Variation of Erasure Probability for Statistically Identical UEs and I.I.D Uniform Mobility

Simulation Setup: We take $N = 2000$ and $M = 500$. We want to first simulate for i.i.d. uniform mobility and see how the ratio of Average AoI under MMW and Optimal Average AoI vary with changing $p_i = p$. We vary p from 0.1 to 1 in steps of 0.1. We simulate this over a time horizon of 10^5 for each p. The ratio is calculated as:

$$Ratio = \frac{\text{Average AoI in Simulation}}{\frac{\sum_{i=1}^{N}(\sqrt{\frac{1}{p_i}})^2}{2NM(1-(1-\frac{1}{M})^N)}} \tag{3.2}$$

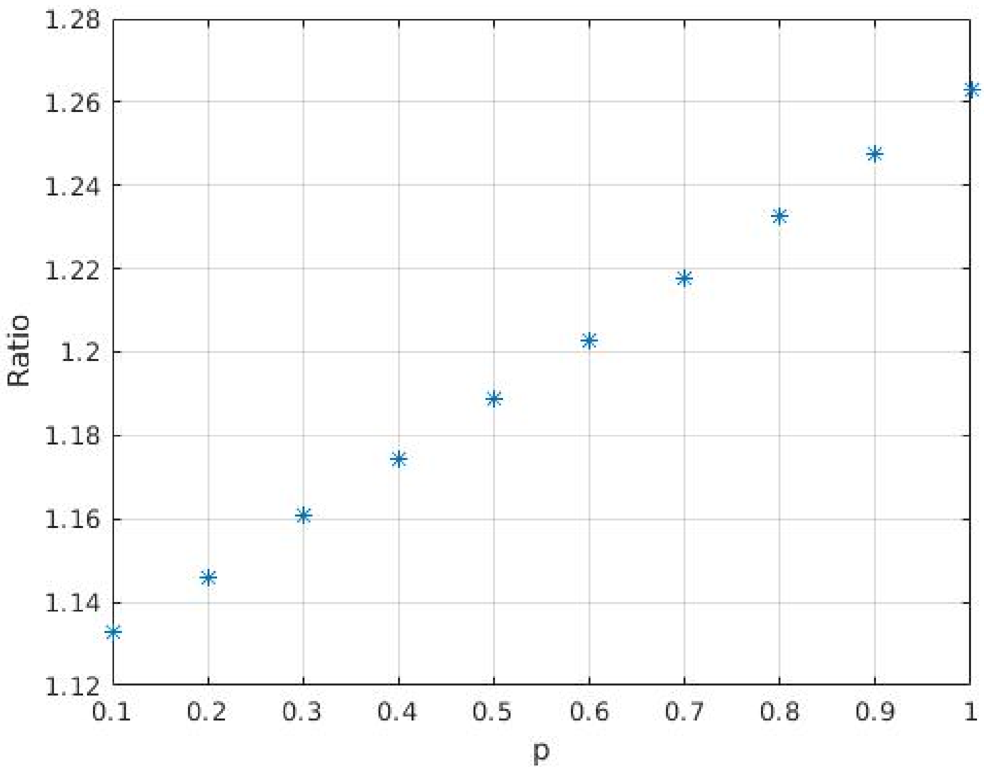

Figure 3.2: Ratio of Average AoI under MMW and Optimal Average AoI vs p for i.i.d Uniform Mobility and statistically identical UEs

In figure 3.1, we have plotted the ratio, given in equation (3.2), against the uniform probability of successful transmission. From theorem 3.5.1, we know that the ratio in this case can be atmost 2. But we see that the ratios are actually very close to 1 for all values of p. Moreover, the ratios are approximately linear with respect to the probabilities p.

3.6.2 Maintaining $\frac{M}{N}$ Constant and Analyzing the Ratio

Simulation Setup: In this subsection, we maintain a constant value for $\frac{M}{N}$, and have taken it to be 3 here. The value of p is 0.5 for all UEs. The time horizon for this simulation is 10^5 slots. The values of M range from 5 to 50 and values of N correspondingly range from 15 to 150.

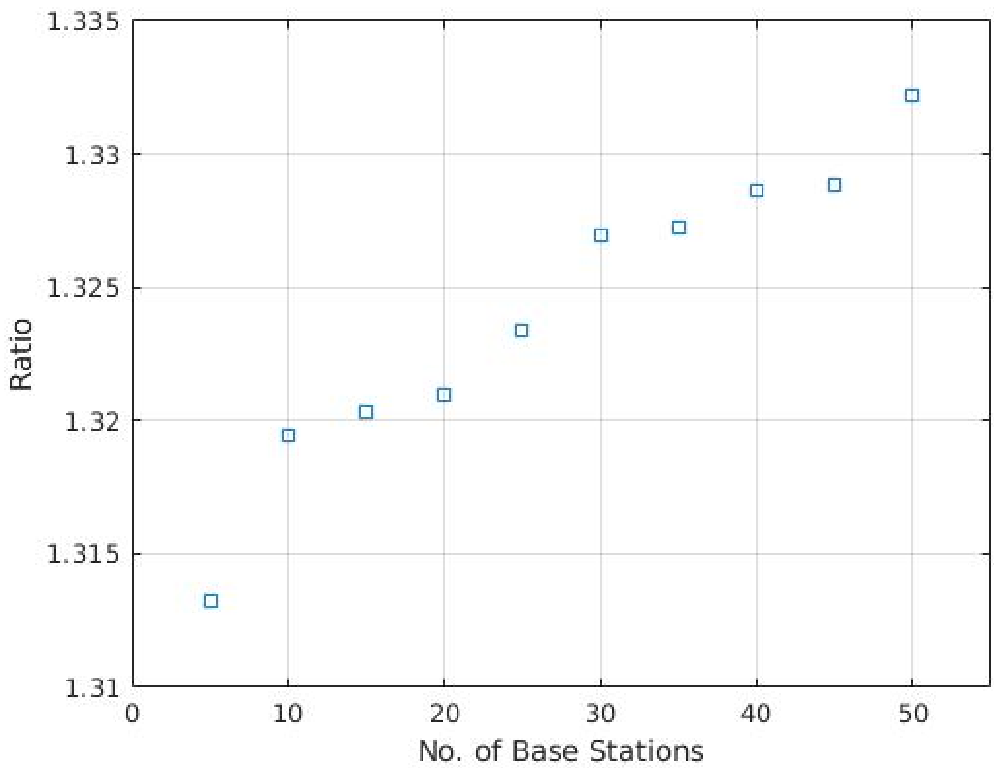

Figure 3.3: Ratio of Average AoI under MMW and Optimal Average AoI vs p for i.i.d Uniform Mobility for fixed $\frac{M}{N}$

In figure 3.2, we see that all ratios are smaller than 2, which is to be expected, due to theorem 3.5.1. We can also see that the ratio is essentially increasing with the number of base stations, but the difference is not very significant.

3.6.3 Analyzing the Ratio for Random Erasure Probabilities

Simulation Setup: We take $M = 2$ and $N = 5$. Next we choose 10 different probability distributions for the $p_i's$. We take a time horizon of 10^5 slots and evaluate the ratios.

The simulation results are given in the form of a table.

Table 3.1: Ratios for Random Probability Distributions under the MMW Policy

p_1	p_2	p_3	p_4	p_5	Ratio
0.8195	0.7955	0.3282	0.1836	0.2940	1.4383
0.4180	0.4464	0.8804	0.5023	0.0137	1.9432
0.4311	0.2391	0.4281	0.6187	0.1680	1.3997
0.1276	0.9524	0.6537	0.2532	0.1882	1.4189
0.4382	0.7412	0.7322	0.2149	0.7920	1.4360
0.7436	0.4132	0.3562	0.8696	0.3512	1.4142
0.9352	0.3156	0.3924	0.1208	0.3908	1.4411
0.3298	0.5880	0.1656	0.9878	0.0200	1.7333
0.0918	0.4204	0.9032	0.6222	0.7976	1.4976
0.1218	0.6680	0.4590	0.4315	0.7266	1.4320

In table 3.1, we see how the ratio is dependent on the probability distribution. We see that all ratios are below 2. There are two ratios which are closer to 2, which are in row 2 and row 8. In row 2, the ratio is 1.9432, which is extremely close to 2, and in row 8 the ratio is 1.7333. We see that these are the only two rows in which we have a p_i which is smaller than 0.1.

To see if smaller $p_i's$ result in higher ratios, we consider the following p_i distribution: $[0.01, 0.9, 0.9, 0.9, 0.9]$. We simulate for 10 sample paths and take the average ratio. The ratio obtained is $Ratio = 2.1329$ which is greater than 2. Hence we can conclude that the MMW Policy does not have 2-optimality for all possible probability(p_i) distributions.

Next we see how the ratio depends on the number of UEs with small p_i. Hence we simulate with the following 5 distributions for $p_i's$:

1. $p_1 = [0.01, 0.9, 0.9, 0.9, 0.9]$
2. $p_2 = [0.01, 0.01, 0.9, 0.9, 0.9]$
3. $p_3 = [0.01, 0.01, 0.01, 0.9, 0.9]$
4. $p_4 = [0.01, 0.01, 0.01, 0.01, 0.9]$
5. $p_5 = [0.01, 0.01, 0.01, 0.01, 0.01]$

In table 3.2, we see that as the ratio keeps decreasing as the number of UEs with small $p_i's$ increases. This can be a property of the MMW Policy, since if only one UE

Table 3.2: Ratios for probability distributions with increasing number of UEs with bad channels

Distribution	Ratio
p_1	2.1205
p_2	1.7562
p_3	1.5585
p_4	1.4308
p_5	1.3162

has a low p_i, and the other UEs have a high p_i, the UEs with a higher p_i have a higher chance of getting scheduled until the UE with low p_i has a very high AoI compared to these UEs with high p_i.

3.6.4 Analyzing the Ratio for I.I.D Non-Uniform Mobility

Simulation Setup: In this simulation, the UEs are restricted to move among only a set of BSs as given below:

There are 50 UEs and 20 BSs. UE_1 to UE_{15} are constrained to move in the cells covered by BS_1 to BS_5. UE_{16} to UE_{30} are constrained to move in the cells covered by BS_6 to BS_{10}. UE_{31} to UE_{45} are constrained to move in the cells covered by BS_{11} to BS_{15}. UE_{46} to UE_{50} are constrained to move in the cells covered by BS_{16} to BS_{20}. Hence the mobilities are not uniform. The $p_i's$ are varied from 0.1 to 1 in steps of 0.1 for all UEs.

In figure 3.3, we see that even though mobility is not uniform, the ratio is still below 2. Also, as seen in subsection 3.6.1, the curve made is approximately linear.

3.7 Summary

In this chapter, we have performed simulations to find properties of the problem of UE mobility in an area covered by multiple base stations.

First, we looked at the construction of the system. Our objective was to minimize the long term average AoI for all UEs. In this regard, we described the lower bound

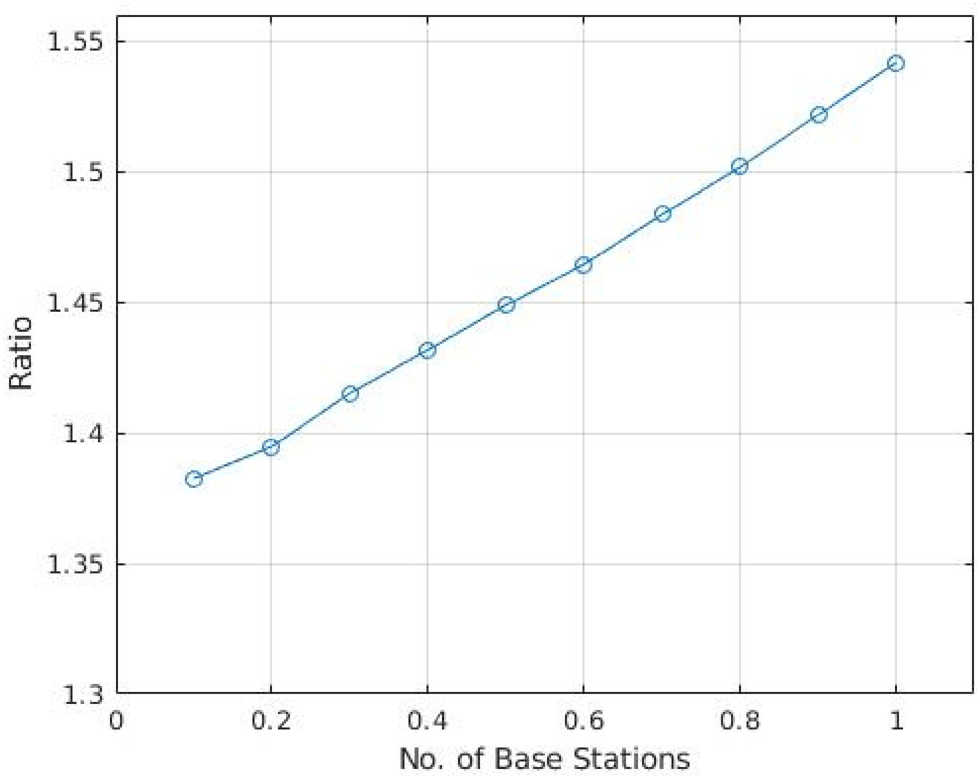

Figure 3.4: Ratio of Average AoI under MMW and Optimal Average AoI for I.I.D Non-Uniform Mobility

for the problem. Next, we described the MMW policy which was a direct extension of the MW policy, for the multiple BS case. This, and other results from the paper Banerjee *et al.* (2020) were stated and explained.

Finally, we performed extensive simulations for the MMW policy and found the variation in the ratio of the average AoI due to the MMW policy, and the optimal policy by varying various parameters such as the distribution of $p_i's$, the number of UEs, the number of base stations, and changing the mobility pattern of the UEs and discussed the trends in the ratio.

CHAPTER 4

Online Scheduling Policies

4.1 Introduction

In this chapter, we analyze scheduling policies in the online adversarial setting. The chapter is organized in the following fashion:

First we define the system model for our problem. Next, we define our problem in the adversarial setting. In sections (4.4) and (4.5) we will find the competitive ratio for the MA policy using the potential function method. In section(4.6) we define the RHC Policy and then try to find the competitive ratio for RHC in section (4.7).

4.2 System Model for MA

We have one BS and N UEs communicating with each other. The BS can transmit to only one UE at every timeslot. The UEs do not communicate with each other. We have taken the system model from Banerjee *et al.* (2020).

4.2.1 Channel Model

The channel state $Ch_i(t)$ of any UE i at any timeslot t can be either $Good$ or Bad.

If the channel at any timeslot is $Good$, then the UE is able to decode the packet successfully. On the other hand, if the channel is Bad, then the packet is lost.

Hence there are N different channels corresponding to the N UEs. We assume that there exists an omniscient adversary which chooses the state of the N channels for every timeslot from a set of 2^N possible channel states per timeslot.

The scheduling policy in consideration is "online" and does not have information about the current state or future states of the channels.

Now we define the cost function for any general scheduling policy over a time horizon of T in the average age and max age case:

$$Cost_{avg}(T) = \frac{1}{N}\sum_{t=1}^{T}\sum_{i=1}^{N} h_i(t)$$
$$Cost_{max}(T) = \sum_{t=1}^{T}\max_i h_i(t)$$

4.2.2 The Performance Metric

Suppose we want to measure the performance of a particular online scheduling policy π. To do so, we use the **Competitive Ratio** η^π which compares the performance of π with an optimal offline policy OPT which has hindsight knowledge, i.e. it has complete knowledge about the past, present and future states of the channel.

In a formal way, we define $\boldsymbol{\sigma} \in \{\{0,1\}^N\}^T$ as a sequence of length T representing the vector of channel states chosen by the adversary for the entire horizon. Then we can define the competitive ratio as follows:

$$\eta^\pi = \sup_{\boldsymbol{\sigma}}\left(\frac{\text{Cost of online policy}\pi\text{on}\boldsymbol{\sigma}}{\text{Cost of OPT on}\boldsymbol{\sigma}}\right)$$

We assume that OPT has complete knowledge about the channel and π has only causal information about the channel.

4.3 Problem Definition

We want to find the competitive ratio of the MA Policy in both the Max age metric minimization and Average age metric Minimization with respect to the OPT Policy. Hence we define the setup which we will use to solve the problem.

We assume a finite time horizon T. In this time horizon, we assume that in the MA policy, the BS is able to carry out a transmission successfully K times. Now, let T_i be the timeslot during which the MA Policy is able to carry out its i^{th} successful transmission for $1 \leq i \leq K$. Now we define $T_0 = 0$ and $T_{K+1} = T$. Finally let the inter-transmission interval be given by $\Delta_i = T_{i+1} - T_i$. We also take $\Delta_0 = 0$.

4.4 Competitive Ratio for MA in the Average Age Case

In this section, we calculate the competitive ratio of the MA policy for the Average Age Cost Metric using the Amortized costs method. For information on this method, please refer Appendix B. We describe the Amortized costs method for finding the competitive ratio in the next subsection.

4.4.1 The Amortized Costs Method

Let the algorithm ALG be the policy for which we want to find the competitive ratio. Again, we define OPT to be the optimal offline policy.

Now we denote by ALG_i to be the cost incurred by ALG during the i^{th} interval. Let the corresponding cost for OPT be OPT_i. We call ALG_i and OPT_i as the "actual costs" for the i^{th} interval.

Now let the potential function be ϕ. So the amortized cost for the i^{th} interval is defined as:

$$a_i = ALG_i + \phi_i - \phi_{i-1}$$

Here ϕ_i can be defined as $\phi_i = \phi(T_i) - \phi(T_{i-1})\ \forall i$.

Now to prove that ALG is c-competitive, we need to show that:

1. For each interval Δ_i, we have $a_i \leq c.OPT_i$
2. $\exists b$ independent of $\boldsymbol{\sigma}$ such that $\forall i$, we have $\phi_i \geq b$.

Here we see that MA is the algorithm ALG in question. Now in the following subsections, we define and calculate the potential function, and calculate the per-interval costs for both MA and OPT so that we can finally calculate the competitive ratio for the MA Policy.

4.4.2 The Potential Function

We define the potential function ϕ using the following rules:

1. The potential function increase by 2 if MA and OPT are able to successfully transmit in a particular timeslot.
2. The potential function increase by 1 if either MA or OPT, but not both, are able to successfully transmit in a particular timeslot.
3. The potential function does not change if both MA and OPT are not able to transmit in a particular timeslot.

We also assume that $\phi(0) = 0$.

4.4.3 Cost of OPT in the i^{th} Interval

In this section we find a lower bound on the cost of the optimal OPT policy. So we consider the interval Δ_i.

First we see that during this interval, there is one UE which consistently sees BAD channels in every slot, since it is the UE which has been scheduled by the MA policy for transmission. We can say that this UE had an AoI of atleast 1 at the start of the interval, and hence the cost just for this UE can be lower bounded by:

$$Cost \geq \sum_{k=1}^{\Delta_i}(1+k)$$

Now, we calculate a lower bound for the other $N-1$ UEs. We can say, from the definition of AoI, that the AoI of each UE is atleast 1. Hence the cost in the interval Δ_i for 1 UE is lower bounded by:

$$Cost \geq \sum_{k=1}^{\Delta_i} 1$$

And hence we can say for $N-1$ UEs, the cost is lower bounded by:

$$Cost \geq (N-1)\sum_{k=1}^{\Delta_i} 1$$

Now combining the UE which consistently sees bad channels and the other $N-1$ UEs, we can write a lower bound on the cost for the opt policy in the interval Δ_i.

$$
\begin{aligned}
OPT_i &\geq \sum_{k=1}^{\Delta_i}(1+k) + (N-1)\sum_{k=1}^{\Delta_i} 1 \\
&= \sum_{k=1}^{\Delta_i}(k) + (N)\sum_{k=1}^{\Delta_i} 1 \\
&= \frac{1}{2}(\Delta_i^2 + \Delta_i) + N\Delta_i \\
&\geq \frac{1}{2}\Delta_i^2 + N\Delta_i
\end{aligned}
$$

4.4.4 Cost of MA in the i^{th} Interval

We take the cost for MA in the interval Δ_i from the paper Banerjee *et al.* (2020). We provide a brief description of how it is obtained below.

We know that the MA policy has scheduled only one user during the i^{th} interval. Hence during the interval, the AoI of each user increases by 1 in each timeslot.

Now at the start of the slot, the AoI of each user in sorted order can be written as:

$$
\{1, 1+\Delta_{i-1}, 1+\Delta_{i-1}+\Delta_{i-2}, ..., 1+\sum_{j=1}^{N-1}\Delta_{i-j}\}
$$

Hence during the k^{th} slot of the i^{th} interval the sorted AoIs are:

$$
\{k, k+\Delta_{i-1}, k+\Delta_{i-1}+\Delta_{i-2}, ..., k+\sum_{j=1}^{N-1}\Delta_{i-j}\}
$$

Hence we can write the cost of MA in the i^{th} interval as:

$$
MA_i = \frac{1}{N}\sum_{k=1}^{\Delta_i} k + \sum_{k=1}^{\Delta_i}\sum_{m=1}^{N-1}(k + \sum_{j=1}^{m}\Delta_{i-j})
$$

The inequalities are solved and we finally get the following upper bound for MA_i:

$$
MA_i \leq N\Delta_i^2 + \frac{1}{2}\Delta_i
$$

4.4.5 Calculation of Potential Function in the i^{th} Interval

For calculation of the competitive ratio for the MA policy, we assume that during every timeslot in the interval i, there is atleast one UE which has a good channel. In a later section, we will see how all UEs having bad channels in some timeslots does not affect our calculations.

This means we can assume that OPT is able to transmit in every timeslot, since it has complete knowledge about the channel for the time horizon T.

We also know that MA is able to complete a successful transmission for only one UE, which occurs at time T_i. Hence from slots $T_{i-1}+1$ to $T_i - 1$, only OPT transmits. And during timeslot T_i both OPT and MA transmit.

Hence from the definition of the potential function, ϕ_i can be written as:

$$\begin{aligned}\phi_i &= 1.(\Delta_i - 1) + 2.1\\ &= \Delta_i + 1\end{aligned}$$

4.4.6 Calculating the Competitive Ratio

Now we have calculated all the components needed to use the Amortized Costs method. Hence we calculate the amortized cost below:

$$\begin{aligned}a_i &= MA_i + \phi_i - \phi_{i-1}\\ &\leq N\Delta_i^2 + \frac{1}{2}\Delta_i + (\Delta_i + 1) - (\Delta_{i-1} + 1)\\ &= N\Delta_i^2 + \frac{1}{2}\Delta_i + \Delta_i - \Delta_{i-1}\\ &\leq N\Delta_i^2 + \frac{1}{2}\Delta_i + \Delta_i\end{aligned}$$

Here the last inequality comes from the fact that $\Delta_i \geq 0 \; \forall i$.

Moreover the second property of the Amortized costs method is satisfied since if we have $b = 0$, then we can easily see that $\phi_i \geq b \; \forall i$. This is because $\phi_i = \Delta_i + 1 \geq 0$.

Hence we need to find c such that $a_i \leq c.OPT_i$.

$$\begin{aligned}a_i &\leq N\Delta_i^2 + \frac{1}{2}\Delta_i + \Delta_i \\ &\leq c.(\frac{1}{2}\Delta_i^2 + N\Delta_i) \\ &\leq c.OPT_i\end{aligned}$$

Here we choose $c = 2N$.

We see that:

$$\begin{aligned}2N.OPT_i &\geq 2N(\frac{1}{2}\Delta_i^2 + N\Delta_i) \\ &= N\Delta_i^2 + 2N^2\Delta_i \\ &\geq N\Delta_i^2 + \frac{1}{2}\Delta_i + \Delta_i \\ &\geq ALG_i\end{aligned}$$

We can write the second inequality since we know that $N \geq 1$. Hence we have shown that $c = 2N$ solves the inequality.

Hence we have shown that the competitive ratio for the MA policy in minimizing the Average Age metric is O(N)-competitive.

Finally, the competitive ratio is given by $\eta^{MA} \leq 2N$.

4.4.7 OPT sees BAD Channels for All Users

In this subsection, we will show that even if there are timeslots in which all UEs see BAD channels, it does not affect the inequalities.

To see this, we assume that $\phi_{i-1} = x \geq 0$ and $\phi_i = y \geq 0$. Now the amortized cost equations still hold in this case, for the following reason:

1. $y \leq \Delta_i + 1$ hence we can conveniently substitute Δ_i instead of y.
2. $\Delta_{i-1} \geq x \geq 1$ as there is at least one transmission in the MA scheduling, hence we can ignore it while still preserving the inequality in the relevant step.

Moreover we can see that $-x \leq 1$ and $y \leq Delta_i + 1$ and adding both, we get,

$y - x \leq \Delta_i$. In this way we can continue with the same proof later.

4.5 Competitive Ratio for MA in the Max Age Case

In this section, we calculate an upper bound to the competitive ratio for the MA policy under the following cost metric:

$$Cost(t) = \max_i h_i(t)$$

We use the same potential function as in section (4.4). Hence ϕ_i has the same expression

4.5.1 Cost of OPT in the i^{th} Interval

In this subsection, we calculate the lower bound on OPT_i. We know that $\max_i h_i(t) \geq h_{avg}(t)$. Hence the lower bound used in the previous section can be used here as well.

Hence we can write:

$$OPT_i \geq \frac{1}{2}\Delta_i^2 + N\Delta_i$$

4.5.2 Cost of MA in the i^{th} Interval

Now we need to find a upper bound on MA_i. Hence we can write:

$$MA_i = \sum_{t=1}^{\Delta_i} \max_i h_i(t) \tag{4.1}$$

Now we know that at the start of the i^{th} interval the maximum possible AoI can be $1 + \sum_{j=1}^{N-1} \Delta_{i-j}$, if a particular UE is not scheduled till timeslot T_{i-1}. Hence we can take it as an upper bound.

Hence we can see that the MA policy would schedule this UE during the i^{th} interval and this UE would get served at timeslot T_i. Using this information we can write an upper bound on the cost during the i^{th} interval:

$$MA_i \le \sum_{k=1}^{\Delta_i}(k + \sum_{j=1}^{N-1} \Delta_{i-j}) \tag{4.2}$$

$$= \sum_{k=1}^{\Delta_i} k + \sum_{k=1}^{\Delta_i}\sum_{j=1}^{N-1} \Delta_{i-j} \tag{4.3}$$

$$= \frac{\Delta_i(\Delta_i + 1)}{2} + \sum_{j=1}^{N-1} \Delta_i\Delta_{i-j} \tag{4.4}$$

Now using the AM-GM inequality, we can bound $\Delta_i\Delta_{i-j}$ as follows:

$$\Delta_i\Delta_{i-j} \le \frac{\Delta_i^2 + \Delta_{i-j}^2}{2}$$

Substituting back in equation (4.4), we get:

$$\begin{aligned} MA_i &\le \sum_{j=1}^{N-1} \frac{\Delta_i(\Delta_i + 1)}{2} + \sum_{j=1}^{N-1} \frac{\Delta_i^2 + \Delta_{i-j}^2}{2} \\ &\le \frac{3}{2}N\Delta_i^2 + \frac{\Delta_i}{2} \end{aligned}$$

Finally using this we calculate the amortized cost for the i^{th} interval.

$$\begin{aligned} a_i &= MA_i + \phi_i - \phi_{i-1} \\ &\le \frac{3}{2}N\Delta_i^2 + \frac{\Delta_i}{2} + \Delta_i - \Delta_{i-1} \\ &\le \frac{3}{2}N\Delta_i^2 + \frac{\Delta_i}{2} + \Delta_i \\ &= \frac{3}{2}N\Delta_i^2 + \frac{3}{2}\Delta_i \end{aligned}$$

4.5.3 Calculation of the Competitive Ratio

We can find the competitive ratio η_{MA} by finding a suitable c which satisfies the inequality: $MA_i \leq c.OPT_i$. We choose $c = 3N$. Hence we can write:

$$\begin{aligned} 3NOPT_i &\geq 3N(\frac{1}{2}\Delta_i^2 + N\Delta_i) \\ &\geq \frac{3}{2}N\Delta_i^2 + 3N^2\Delta_i \\ &\geq a_i \end{aligned}$$

These inequalities hold since $N \geq 1$. Hence we see that $\eta^{MA} \leq 3N$.

Hence the competitive ratio is atmost $O(N)$.

4.6 The Receding Horizon Control Policy

It is well known that wireless channels with block fading can be estimated quite accurately for a few slots in the future. Hence we consider a relaxed adversarial model, where at any slot t, the BS can estimate the channels perfectly for a window of the next $w \geq 0$ slots. Here w is an adjustable system parameter that can be adaptively tuned by the policy in accordance with the scale of time-variation of the channels(e.g., fading block length).

We assume that the channel states are binary-valued and chosen by an omniscient adversary. We now define the Receding Horizon Control Policy which uses the information about the w future slots.

Receding Horizon Control: At every timeslot t, we schedule the user that minimizes the total AoI cost for the next w timeslots.

4.7 Competitive Ratio for RHC

The potential function method was used to find a competitive ratio for the Max Age policy and a simple potential function did the trick.

To be able to apply this method to find a competitive ratio for the RHC policy, we need to find a method to quantify the effect of the window size w.r.t the OPT policy.

It is very easy to see that the following potential function: ϕ increases by 1 on successful transmission and doesn't change when there is no transmission does not work here since both RHC and OPT would transmit when successful transmission is possible.

We know that if only one user has a good channel in a particular timeslot, both the RHC and the OPT policy would both transmit and there would be no difference. But when there are two or more good channels in the timeslot, it is possible that RHC is as good as the OPT policy but if that is not the case, how do we quantify the difference in the potential function not knowing any numerical values?

We can easily conclude the following:

1. RHC will transmit successfully in every slot where there is atleast one user with a good channel.
2. If only one user has a good channel, then RHC and OPT transmit to the same user.
3. The only place where a discrepancy can occur is when there are two or more users with a good channel. In this case OPT and RHC might choose different users for transmission.

Using this we design the following model:

Let Δ_i denote the time interval between the $i-1^{th}$ and i^{th} time that the user chosen for transmission does not match with OPT. Let there be K such intervals and let the horizon size be T. Now we define the potential function: ϕ increases by 1 every time OPT matches RHC.
Hence ϕ_i is defined as the value of the potential function at the end of the i^{th} interval.

Now we use the amortized costs method, where amortized cost at the end of the i^{th} interval is defined as:

$$a_i = RHC_i + \phi_i - \phi_{i-1}$$

We want to find the competitive ratio η such that:

1. For any event, $a_i \leq c.OPT_i$
2. $\exists$ a constant b such that $\forall i, \phi_i \geq b$

We know that $OPT_i \geq \frac{1}{2}\Delta_i^2 + N\Delta_i$ in the Max Age Policy. But the same OPT_i does not work here since the type of interval doesn't satisfy the assumptions taken in the interval for the MA Policy. Hence we need to find a suitable bound for our defined intervals.

There will be two cases:

1. $\Delta_i < N$
2. $\Delta_i > N$

We can calculate OPT_i for both cases and choose the competitive ratio which is larger.

From the above equation, $a_i \leq c.OPT_i$, knowing that $RHC_i = OPT_i + (.)$ we can see that:

$$\eta \geq 1 + \frac{(.)}{OPT_i}$$

Hence we see that the larger competitive ratio will be for a smaller OPT_i. We now calculate it for the first case.

We can use a similar lower bound for OPT as in the paper but since we know that Δ_i out of N users transmit,

$$\begin{aligned} OPT_i &\geq \Delta_i \sum_{k=1}^{\Delta_i} 1 + (N - \Delta_i) \sum_{k=1}^{\Delta_i} k \\ &= \Delta_i^2 + (N - \Delta_i) \frac{\Delta_i(\Delta_i + 1)}{2} \end{aligned}$$

From the above definition of the potential function, we can see that $\phi_i - \phi_{i-1} = 1$.

Now we just need to estimate RHC_i.

We know that in each interval Δ_j, the difference in AoI cost occurs only in the last time slot. This difference is less than or equal to $\Delta_i - 1$. This gets added in every time slot after the slot $1 + \Delta_1 + ... + \Delta_j$. This is also added Δ_i times in our interval. All such previous discrepancies in the cost of RHP and OPT are added and we get the following:

$$
\begin{aligned}
RHC_i &\leq OPT_i + \Delta_i((\Delta_1 - 1) + (\Delta_1 + \Delta_2 - 1) + ... + (\sum_{k=1}^{i}(\Delta_k)) - 1) \\
&= OPT_i + \Delta_i(\sum_{k=1}^{i}(i+1-k)\Delta_k) - i\Delta_i \\
&= OPT_i + i\Delta_1\Delta_i + (i-1)\Delta_2\Delta_i + ... + \Delta_i^2 - i\Delta_i \\
&\leq OPT_i + i(\frac{\Delta_i^2 + \Delta_1^2}{2}) + ... + \Delta_i^2 - i\Delta_i \\
&= OPT_i + (\frac{i(i+1)}{4} + \frac{1}{2})\Delta_i^2 + \sum_{k=1}^{i-1}\frac{i+1-k}{2}\Delta_k^2 - i\Delta_i \\
&\leq OPT_i + (\frac{i(i+1)}{4} + \frac{1}{2})\Delta_i^2 + i\sum_{k=1}^{i-1}\Delta_k^2 - i\Delta_i \\
&\leq OPT_i + (\frac{i(i+1)}{4} + \frac{1}{2})\Delta_i^2 + i^2\Delta_i^2 - i\Delta_i \\
&= OPT_i + \frac{5i^2 + i + 2}{2}\Delta_i^2 - i\Delta_i
\end{aligned}
$$

Finally we can find η.

$$
\begin{aligned}
\eta &\geq 1 + \frac{\frac{5i^2+i+2}{2}\Delta_i^2 - i\Delta_i}{\Delta_i^2 + (N - \Delta_i)\frac{\Delta_i(\Delta_i+1)}{2}} \\
&= 1 + \frac{\frac{5i^2+i+2}{2}\Delta_i - i}{\Delta_i + (N - \Delta_i)\frac{\Delta_i+1}{2}}
\end{aligned}
$$

This expression contains i and hence needs to be processed further. Hence it is very probable that the window size for RHP will come into picture, but it is difficult to incorporate.

4.8 Summary

In this chapter, we first looked at the online problem of minimizing the average and max cost for the MA Policy. We found the competitive ratio of 2N and 3N respectively using the amortized costs method.

We then considered the RHC Policy, and found an expression for the competitive ratio using the amortized costs method.

CHAPTER 5

Scheduling Policies involving Energy Harvesting Nodes

5.1 Introduction

In this chapter, we define the problem of scheduling transmissions for a single UE which can do energy harvesting. This means that the UE can charge its battery from environmental sources like solar energy or wind energy. In this case the UE has to use energy conservatively, so that it can minimize its AoI. The chapter is organized in the following fashion:

First we define the system model for our problem. We then discuss the problem definition. Next we define our problem in the adversarial setting. Finally, we try to find the competitive ratio for the problem proposed.

5.2 System Model

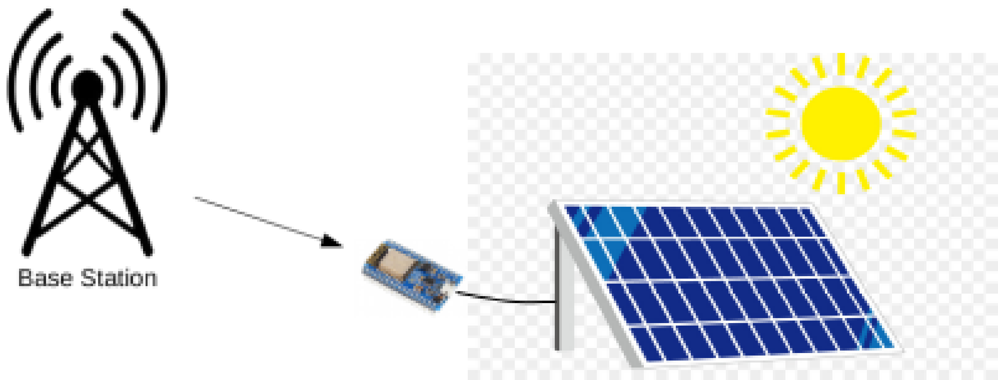

Figure 5.1: System Model for Energy Harvesting Online Setting

For this problem, we have a system which contains one UE and one BS. The UE gives status updates to the BS but does not receive any information from the BS.

The UE has an infinite-sized battery. Each status update costs one unit of battery energy. Moreover the UE can send a status update to the BS only if the battery has atleast one unit of energy.

The channel between the UE and the BS is assumed to be perfect, i.e. any update sent by the UE is received and decoded correctly at the BS with probability 1.

5.3 Problem Definition

There exists an omniscient adversary, which schedules the energy arrivals. It aims to schedule the energy arrivals in such a way that the cost of the policies are maximized. In this regard, we want to test the Best-effort Uniform Updating Policy(BU), which was described in the paper Feng and Yang (2021). The BU Policy is described below:

Best-effort Uniform Updating Policy: The sensor is scheduled to update the status at $s_n = n, n = 1, 2, ...$ The sensor sends an update at s_n if $E(s_n^-) \geq 1$. Otherwise, the sensor keeps silent until the next scheduled status updating time point.

We want to find a competitive ratio for the BU Policy when compared to an optimal offline policy which can predict all energy arrivals in the time horizon T.

The maximum number of transmissions possible for any policy upto time t is given by:

$$N_t \leq \sum_{k=1}^{t} e_k$$

where e_k is the amount of energy received by the UE in the k^{th} slot.

5.4 A Lower Bound for the OPT Policy

We know for any policy that the following lower bound holds:

$$LB = \max_{t \leq T} \frac{t}{2} + \frac{t^2}{2(1 + \sum_{i=0}^{t} e_i)}$$

We will use this lower bound for the OPT policy.

5.5 An Upper Bound for the BU Policy

We consider the time upto timeslot t. We know that N_t energy arrivals have occurred. We want to find for what pattern of energy arrivals is the cost the highest. That will be the upper bound for the BU Policy since the channel is perfect.

Hence the highest cost will occur when there is one inter-arrival interval which is as large as possible since the function of cost in each interval is of the form $H_\tau(H_\tau + 1)$. Hence we will have N_t arrivals at the start and one arrival at the end of time t. In this case the cost is given by:

$$
\begin{aligned}
BU &\leq N_t.(1)^2 + (t - N_t)^2 \\
&= t^2 - 2N_t t + N_t + N_t^2 \\
&\leq t^2 + N_t + N_t^2 \\
&\leq t^2 + (\sum_{i=0}^{t} e_i)(1 + \sum_{i=0}^{t} e_i)
\end{aligned}
$$

Now we want the highest possible cost hence we take $\max_{t \leq T} t^2 + (\sum_{i=0}^{t} e_i)(1 + \sum_{i=0}^{t} e_i)$

5.6 An Upper Bound for the Competitive Ratio

Using the above two results, we find an upper bound on the competitive ratio for the BU policy.

$$
\begin{aligned}
\eta &\leq \frac{\max_{t \leq T}(t^2 + (\sum_{i=0}^{t} e_i)(1 + \sum_{i=0}^{t} e_i))}{\max_{t \leq T}(\frac{t}{2} + \frac{t^2}{2(1+\sum_{i=0}^{t} e_i))}} \\
&\leq \min_{t \leq T} \frac{t^2 + (\sum_{i=0}^{t} e_i)(1 + \sum_{i=0}^{t} e_i)}{\frac{t}{2} + \frac{t^2}{2(1+\sum_{i=0}^{t} e_i)}}
\end{aligned}
$$

Hence this is the expression we obtained for the competitive ratio in the energy harvesting online setting.

5.7 Summary

In this chapter, we have considered a unique problem involving an adversarial network with energy harvesting. The UE does not receive status updates from the BS, and hence cannot make use of the BS's knowledge while scheduling updates.

We considered the Best effort Uniform Updating Policy, and found an expression for an upper bound on the competitive ratio.

APPENDIX A

Lyapunov Drift Optimization

This note on Lyapunov Drift Optimization is taken from the notes of Hajek (2012).

We consider an irreducible discrete-time Markov process X on a countable state space S, with one-step transition probability matrix P.

Now if f is a function on S, Pf represents the function obtained by multiplication of the vector f by the matrix P: $Pf(i) = \sum_{j \in S} p_{ij} f(j)$. If f is non-negative, Pf is well defined with the understanding that $Pf(i) = +\infty$ for some, or all i. It is also known that $Pf(i) = \mathbf{E}(f(X(t+1))|X(t) = i)$.

Let V be a non-negative function on S be the Lyapunov Function. Then the drift vector of $V(X(t))$ is given by:

$$\begin{aligned} d(i) &= \mathbf{E}(V(X(t+1))|X(t) = i) - V(i) \\ &= \sum_{j:j \neq i} p_{ij}(V(j) - V(i)) \end{aligned}$$

Hence we can also write it as $d = PV - V$.

Then we have the following theorem, which is the **Foster Lyapunov Stability Criterion**:

Theorem A.0.1. *Suppose $V : S \to \mathbf{R}_+$ and let C be a finite subset of S.*

1. *If $\{i : V(i) \leq K\}$ for all K, and if $PV - V \leq 0$ on $S - C$, then X is recurrent.*
2. *If $\epsilon > 0$ and b is a constant such that $PV - V \leq -\epsilon + bI_C$, then X is positive recurrent.*

APPENDIX B

Online Algorithms and the Potential Function Method

B.1 Online Algorithms

An online algorithm A is presented with a request sequence $\boldsymbol{\sigma} = \{\sigma(1), ..., \sigma(m)\}$. The algorithm A has to serve each request online, that is, without knowledge of future requests.

1. When serving request $\sigma(t), 1 \leq t \leq m$, the algorithm does not know any request $\sigma(t')$ with $t' > t$.
2. Serving requests incurs costs, and the goal is to serve the entire request sequence so that the total cost is as small as possible.

B.2 Competitive Analysis

Optimal Offline Algorithm: An optimal offline algorithm knows the entire request sequence in advance and can serve it with minimum cost. It is often abbreviated as OPT.

In competitive analysis, an online algorithm A is compared to an optimal offline algorithm in the following way:

Given a request sequence, let $C_A(\boldsymbol{\sigma})$ denote the cost incurred by A and $C_{OPT}(\boldsymbol{\sigma})$ by OPT. The algorithm is called c-competitive if there exists a constant a such that:

$$C_A(\boldsymbol{\sigma}) \leq c.C_{OPT}(\boldsymbol{\sigma}) + a$$

for all request sequences $\boldsymbol{\sigma}$.

In this case, the factor c is called the competitive ratio of algorithm A.

B.3 The Potential Function Method to find the Competitive Ratio

Suppose the online algorithm for which we want to find the competitive ratio is named ALG. ALG attempts to minimize the cost of servicing the request sequence $\boldsymbol{\sigma}$.

Let OPT be the optimal offline algorithm for the same problem as ALG.

Now suppose both ALG and OPT are processing a request sequence $\boldsymbol{\sigma}$. In order to process a request in $\boldsymbol{\sigma}$, each of these algorithms perform a sequence of computational operations.

With any request sequence, we can assign a unique sequence of operations. If we have two such sequences, one for ALG and one for OPT, we can combine both sequences into one compound sequence that specifies the actions of both algorithms.

In this grand sequence, we may interleave the actions of ALG and OPT in any order, provided that ALG's actions involved in processing the $j + 1^{th}$ request do not precede OPT's actions involved in the processing of the j^{th} request, and vice versa.

1. Any partition of this sequence into segments is called an **Event Sequence**. Each such segment is called an **Event**.
2. In many cases, an event will simply be all the operations associated with ALG's response to a request or OPT's response to a request.

Configuration: An algorithm's configuration is its state with respect to the outside world.

B.3.1 The Potential Function

Let S_{ALG} and S_{OPT} be the set of possible configurations for ALG and OPT. Now let ϕ be a function such that:

$$\phi : S_{ALG} X S_{OPT} \rightarrow \mathbf{R}$$

Any such function ϕ is called a potential function.

Now let $e_1, e_2, ..., e_n$ be any event sequence corresponding to some request sequence.

For each $i = 1, 2, ..., n$, let ϕ_i be the value of ϕ just after the i^{th} event.

Now let ϕ_0 be the value just before the start of the sequence. ϕ_0 is a constant depending only on the individual configurations of ALG and OPT.

B.3.2 Finding the Competitive Ratio

Finally, suppose we want to prove that ALG is c-competitive by using a potential function argument. There are two methods to do so:

1. Amortized Costs Method
   Denote by ALG_i the cost incurred by ALG in the i^{th} event. Denote by OPT_i the cost incurred by OPT in the i^{th} event. These are called the actual costs for the i^{th} event. Then:

   $$a_i = ALG_i + \phi_i - \phi_{i-1}$$

   is called the amortized cost for the i^{th} event. To prove ALG is c-competitive, we need to show that:
   (a) For each event e_i, $a_i \leq c.OPT_i$.
   (b) $\exists$ b independent of $\boldsymbol{\sigma}$ such that $\forall i, \phi_i \geq b$.

2. Interleaving Moves Method
   We find ϕ satisfying the following conditions:
   (a) If only OPT is active during event e_i, and pays x for this move, then $\Delta\phi = \phi_i - \phi_{i-1} \leq cx$, that is, ϕ increases by atmost cx.
   (b) If only ALG is active during event e_i, and pays x for this move, then $\Delta\phi \leq -x$, that is, ϕ decreases by atleast x.
   (c) There exists b independent of $\boldsymbol{\sigma}$ such that $\forall i$, $\phi_i \geq b$.

APPENDIX C

Proofs Related to the MA Policy

The proofs are taken from Srivastava *et al.* (2019) and constitute the work of Prof. Abhishek Sinha.

Theorem C.0.1. *The greedy policy MA is an optimal policy for the problem (2.1). Moreover the long term peak AoI is given by* $\lambda^* = \sum_{i=1}^{N} \frac{1}{p_i}$.

Proof. The stochastic control problem under investigation is an instance of a countable-state average-cost MDP with a finite action space. The state **h**(t) of a system at a slot t given by the current AoI vector of all users, i.e., **h**(t) = $(h_1(t), h_2(t), ..., h_N(t))$. The per-stage cost at time t is $\max_{i\in 1,2,...,N} h_i(t)$, which is unbounded, in general. Finally, the finite action space $\mathbf{A} = 1, 2, ..., N$ corresponds to the index of the user scheduled at a given slot.

Let the optimal cost for the problem P_{Sched} be denoted by λ^* and the differential cost-to-go from the state **h** be denoted by V(**h**). Then, following the standard theory of average cost countable state MDP, we consider the following Bellman Equation:

$$\lambda^* + V(\boldsymbol{h}) = \min_i p_i V(1, \boldsymbol{h_{-i}} + \mathbf{1}) + (1 - p_i)V(\boldsymbol{h} + \mathbf{1}) + \max_i h_i \quad \text{(C.1)}$$

Here the vector $\boldsymbol{h_{-i}}$ denotes the N-1 dimensional vector which has all coordinates except the i_{th} coordinate and $\mathbf{1}$ is the all-one vector. The i_{th} coordinate is 1 in the equation.

This Bellman equation can be divided into two parts: the first one is obtained from the cases where UE i is scheduled in a particular time slot, and can information can be successfully transmitted to UE i, or it can be lost.

With probability p_i, the transmission will be successful, which means that the i_{th} coordinate will have an AoI of 1 in the next timeslot and the other UEs will increase their AoI by 1 in the next timeslot. This gives us: $p_i V(1, \boldsymbol{h_{-i}} + \mathbf{1})$. With probability

$1-p_i$, the transmission will be unsuccessful. In this case all the UEs will increase their AoI by 1. This gives us the part: $(1-p_i)V(\boldsymbol{h}+\boldsymbol{1})$. We want to minimize this sum of these two parts.

The second part is $\max_i h_i$ which denotes the the cost for the current step.

Now we will show that the following:

$$V(\boldsymbol{h}) = \sum_j \frac{h_j}{p_j} \tag{C.2}$$

$$\lambda^* = \sum_j \frac{1}{p_j} \tag{C.3}$$

constitute the solution for the Bellman equation(2.2).

First we simplify the following expression so that we can later simplify the RHS.

$$\begin{aligned}
&\implies p_i V(1, \boldsymbol{h_{-i}}+\boldsymbol{1}) + (1-p_i)V(\boldsymbol{h}+\boldsymbol{1}) \\
&= p_i(\sum_{j\neq i} \frac{h_j+1}{p_j}) + \frac{p_i}{p_i} + (1-p_i)\sum_j \frac{h_j+1}{p_j} \\
&= 1 + \sum_j \frac{h_j+1}{p_j} - p_i \frac{h_i+1}{p_i} \\
&= \sum_j \frac{h_j}{p_j} + \sum_j \frac{1}{p_j} - h_i
\end{aligned}$$

Now using the above expression we show that the RHS and LHS are equal.

$$\begin{aligned}
&\implies LHS \\
&= \min_i p_i V(1, \boldsymbol{h_{-i}}+\boldsymbol{1}) + (1-p_i)V(\boldsymbol{h}+\boldsymbol{1}) + \max_i h_i \\
&= \min_i(\sum_j \frac{h_j}{p_j} + \sum_j \frac{1}{p_j} - h_i) + \max_i h_i \\
&= \sum_j \frac{h_j}{p_j} + \sum_j \frac{1}{p_j} - \max_i h_i + \max_i h_i \\
&= \sum_j \frac{h_j}{p_j} + \sum_j \frac{1}{p_j} \\
&= \lambda^* + V(\boldsymbol{h}) \\
&= RHS
\end{aligned}$$

Hence the solution given by equations (2.3) and (2.4) constitute the solution to the Bellman equation.

Hence the MA policy is an optimal solution to the problem P_{Sched} and the long term peak AoI is given by $\lambda^* = \sum_j \frac{1}{p_j}$.

□

Theorem C.0.2. *Under the action of the MA policy, there exists a constant* $c(N, \boldsymbol{p}) > 0$*, such that, for any fixed time* $t \geq 1$ *and any* $k \geq 2N$*,*

$$\mathbf{P}^{MA}(\max_i h_i(t) \geq k) \leq c(N, \boldsymbol{p}) k^N (1 - p_{min})^k$$

Proof. First we define the following probabilities:

$$\max_i p_i = p_{max}$$

$$\min_i p_i = p_{min}$$

Next we use the union bound to get the LHS to a form which we can calculate.

$$\implies \mathbf{P}(\max_i h_i(t) \geq k) \tag{C.4}$$

$$= \mathbf{P}(\bigcup_i h_i(t) \geq k) \tag{C.5}$$

$$\leq \sum_i \mathbf{P}(h_i(t) \geq k) \tag{C.6}$$

Now we want to get an inequality w.r.t. expression (2.7). Hence we notice that for any $h_i(t)$ to be greater than or equal to k, it must not have transmitted for the previous k slots before time t.

Also we see that between any two successful transmissions of UE j, all others UEs need to have successful transmissions atleast once, since after the first transmission of UE j, all other UEs have higher AoI. Hence we see that during the k slots upto timeslot t, atmost N-1 UEs can have a successful transmission. Hence this probability can be written as:

$$\mathbf{P}(h_i(t) \geq k) \leq \sum_{j=0}^{N-1} \binom{n}{k} p_{max}^j (1-p_{min})^{k-j}$$
$$\leq \binom{k}{N-1} (1-p_{min})^k (\frac{1-p_{min}}{p_{max}+p_{min}-1})((\frac{p_{max}}{1-p_m in})^N - 1)$$

Now we know that $\binom{k}{N-1} \leq \frac{k^N}{(N-1)!}$. Hence we define the constant $c'(N, \boldsymbol{p})$ as follows:

$$c'(N, \boldsymbol{p}) = \frac{1-p_{min}}{(N-1)!(p_{max}+p_{min}-1)}((\frac{p_{max}}{1-p_m in})^N - 1)$$

Hence we can write the inequality as:

$$\mathbf{P}(h_i(t) \geq k) \leq c'(N, \boldsymbol{p}) k^N (1-p_{min})^k$$

Finally, we add this inequality over all UEs:

$$\implies \sum_i \mathbf{P}(h_i(t) \geq k) \quad \leq \sum_i c'(N, \boldsymbol{p}) k^N (1-p_{min})^k = c(N, \boldsymbol{p}) k^N (1-p_{min})^k$$

Here $c(N, \boldsymbol{p}) = \sum_i c'(N, \boldsymbol{p})$. Hence we can finally write:

$$\mathbf{P}(\max_i h_i(t) \geq k) \leq c(N, \boldsymbol{p}) k^N (1-p_{min})^k$$

Hence we have proved the theorem. □

Theorem C.0.3. *Under the action of an arbitrary scheduling policy π, at any slot $t \geq k$ and $k \geq 1$, we have:*

$$\mathbf{P}^\pi(\max_i h_i(t) \geq k) \geq (1-p_{min})^k$$

Proof. Let $i^* = argmin_i p_i$. Then

$$\mathbf{P}^\pi(\max_i h_i(t) \geq k) \tag{C.7}$$
$$\geq \mathbf{P}^\pi(h_{i^*}(t) \geq k) \tag{C.8}$$
$$\geq (1-p_{min})^k \tag{C.9}$$

We obtain expression (2.6) from expression (2.5) using the fact that the max AoI has to be greater than k which includes the case where i^* has AoI greater than k, which would imply max AoI is greater than k and also the case where i^* has AoI less than k.

The next inequality follows from the fact that one of the cases in which expression(2.7) occurs is when there were k successive erasures for UE i^* just before time t, which occurs with probability $(1-p_{min})^k$. It can happen in other ways as well which makes expression(2.6) greater than expression(2.7).

Hence for any policy π we satisfy the given inequality.

□

Theorem C.0.4. *The MA policy achieves the optimal large-deviation exponent for the max-age metric and the value of the optimal exponent is given by:*

$$-\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{k}\log\mathbf{P}^{MA}(\max_i h_i(t)\geq k) = -\log(1-p_{min})$$

Proof. To prove this result, we first consider the result from theorem(2.5.2):

$$\mathbf{P}^{MA}(\max_i h_i(t)\geq k)\leq c(N,\boldsymbol{p})k^N(1-p_{min})^k$$

Next we take the log on both sides and divide them by k to obtain:

$$\frac{1}{k}\log\mathbf{P}^{MA}(\max_i h_i(t)\geq k)\leq\frac{\log c(N,\mathbf{p})}{k}+N\frac{\log k}{k}+\log(1-p_{min})$$

Next, we take limit with respect to t since theorem(2.5.2) is valid for all $t\geq 1$:

$$\lim_{t\to\infty}\frac{1}{k}\log\mathbf{P}^{MA}(\max_i h_i(t)\geq k)\leq\frac{\log c(N,\mathbf{p})}{k}+N\frac{\log k}{k}+\log(1-p_{min})$$

Finally we take limit with respect to k:

$$\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{k}\log\mathbf{P}^{MA}(\max_i h_i(t)\geq k)\leq\log(1-p_{min}) \tag{C.10}$$

Now we show the opposite relation from theorem(2.5.3):

$$\mathbf{P}^{\pi}(\max_i h_i(t)\geq k)\geq(1-p_{min})^k$$

We take log and divide by k on both sides:

$$\frac{1}{k}\log \mathbf{P}^{\pi}(\max_i h_i(t) \geq k) \geq \log{(1-p_{min})}$$

Next we see that the above bound is valid for any $t \geq k$, for any fixed $k \geq 0$. Hence we can take the limit as t tends to infinity:

$$\lim_{t\to\infty}\frac{1}{k}\log \mathbf{P}^{\pi}(\max_i h_i(t) \geq k) \geq \log{(1-p_{min})}$$

Finally we see that the bound is valid for any $k \geq 0$, hence we can take the limit with respect to k:

$$\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{k}\log \mathbf{P}^{\pi}(\max_i h_i(t) \geq k) \geq \log{(1-p_{min})}$$

Since the above bound is true for any policy π, it is true for the MA policy. Hence we can write:

$$\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{k}\log \mathbf{P}^{MA}(\max_i h_i(t) \geq k) \geq \log{(1-p_{min})}$$

From equations (2.8) and (2.9) we see that $LHS \geq RHS$ and $LHS \leq RHS$ are both true. This is only possible if $LHS = RHS$. Hence we have shown that for the MA policy, we have:

$$-\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{k}\log \mathbf{P}^{MA}(\max_i h_i(t) \geq k) = -\log(1-p_{min})$$

□

Theorem C.0.5. *The Markov Chain $\boldsymbol{h}(t)_{t\geq 1}$ is Positive Recurrent under the action of the MA policy.*

Proof. We can clearly see that $\{\boldsymbol{h}(t)\}_{t\geq 1}$ is a countable state Markov Chain under the action of the MA policy.

We want to show that the Markov Chain is Positive Recurrent. For this we choose the following as the Lyapunov function:

$$h_{avg}(t) = \frac{1}{N}\sum_{i=1}^{N} h_i(t)$$

Using this Lyapunov function, we do the drift analysis.

First, we define $i^*(t) = argmax_i h_i(t)$. We break ties arbitrarily here. Hence the MA policy transmits a fresh packet to the $i^*(t)^{th}$ user at time slot t. Hence there is a successful transmission with probability $p_{i^*(t)}$ and unsuccessful with probability $1 - p_{i^*(t)}$.

Hence when the transmission is unsuccessful, since the AoI of all users increases by 1, we have:

$$h_{avg}(t+1) = h_{avg}(t) + 1$$

When the transmission is successful, the AoI of the scheduled UE i^* decreases to 1, and the AoI of all other users increases by 1. Hence we can write:

$$Nh_{avg}(t+1) = Nh_{avg}(t) + N - 1 + 1 - h_{i^*}(t)$$

Now we know that $h_{i^*}(t) = \max_i h_i(t) \geq h_{avg}(t)$, we can write:

$$h_{avg}(t+1) \leq (1 - \frac{1}{N})h_{avg}(t) + 1$$

Now we can upper bound the Lyapunov Drift as follows:

$$\begin{aligned}
\Delta(h_{avg}(t)) &= \mathbf{E}(h_{avg}(t+1) - h_{avg}(t)) \\
&\leq (1 - p_{i^*(t)})(h_{avg}(t) + 1) + p_{i^*(t)}((1 - \frac{1}{N})h_{avg}(t) + 1) - h_{avg}(t) \\
&= 1 - \frac{p_{i^*(t)}}{N}h_{avg}(t) \\
&\leq 1 - \frac{p_{min}}{N}h_{avg}(t) \\
&= 1 - \frac{p_{min}}{N^2}\sum_{i=1}^{N} h_i(t)
\end{aligned}$$

Here $p_{min} = \min_i p_i$.

The above upper bound shows that if any of the N UEs has an AoI of more than $\frac{2N^2}{p_{min}}$, then the drift is strictly less than -1. Hence whenever the state of the network is outside the N-dimensional box $[0, \frac{2N^2}{p_{min}}]^N$, then our Lyapunov function is strictly negative. Hence using the Foster-Lyapunov Theorem for the stability of Markov Chains, we conclude that the Markov Chain is Positive Recurrent. □

REFERENCES

1. **Banerjee, S.**, **R. Bhattacharjee**, and **A. Sinha**, Fundamental limits of age-of-information in stationary and non-stationary environments. *In 2020 IEEE International Symposium on Information Theory (ISIT)*. IEEE, 2020.

2. **Feng, S.** and **J. Yang** (2021). Age of information minimization for an energy harvesting source with updating erasures: Without and with feedback. *IEEE Transactions on Communications*.

3. **Hajek, B.**, *An Exploration of Random Processes for Engineers*. 2012. URL `http://www.ifp.illinois.edu/~hajek/Papers/randomprocDec11.pdf`.

4. **Kadota, I.**, **A. Sinha**, and **E. Modiano**, Optimizing age of information in wireless networks with throughput constraints. *In IEEE INFOCOM 2018-IEEE Conference on Computer Communications*. IEEE, 2018.

5. **Kim, H.** and **Y. Han** (2005). A proportional fair scheduling for multicarrier transmission systems. *IEEE Communications letters*, **9**(3), 210–212.

6. **Lin, M.**, **Z. Liu**, **A. Wierman**, and **L. L. Andrew**, Online algorithms for geographical load balancing. *In 2012 international green computing conference (IGCC)*. IEEE, 2012.

7. **Srivastava, A.**, **A. Sinha**, and **K. Jagannathan**, On minimizing the maximum age-of-information for wireless erasure channels. *In 2019 International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOPT)*. IEEE, 2019.

LIST OF PAPERS BASED ON THESIS

1. **Srivastava, A.,A. Sinha, andK. Jagannathan**, On minimizing the maximum age-of-information for wireless erasure channels.*In 2019 International Symposium on Model-ing and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOPT)*. IEEE, 2019.