Auction Inference And Optimization

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THESIS CERTIFICATE

This is to certify that the thesis titled **Auction Inference And Optimization**, submitted by **Anant Shah**, to the Indian Institute of Technology, Madras, for the award of the degree of **Bachelor and Master of Technology**, is a bona fide record of the research work done by him under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

In this thesis we look at the sample-complexity in non-truthful mechanisms. Hartline and Taggart (2019) introduce the problem of non-truthful sample complexity and identify a parameterized family of mechanisms with either all-pay, winners-pay bids and truthful payment semantics for which they show polynomial sample complexity bounds. They consider the welfare and revenue objective for non-truthful mechanisms when the value distributions are bounded in the range $[0, 1]$. They also look at the revenue objective for truthful mechanisms for unbounded regular value distributions. We consider the revenue objective for non-truthful mechanisms and show polynomial sample complexity bounds for regular potentially unbounded distributions.

Along with this we provide a multiplicative bound on the revenue estimation error for a rank-by-bid position auction with either winners-pay-bids or all-pay semantics. Chawla *et al.* (2017) obtain an additive bound on the revenue estimation error when values are bounded in the range $[0, 1]$. They provide a revenue estimator definition which is nothing but a weighted order statistic. We perform a thorough empirical analysis on the properties of the estimation error and verify its dependence on various parameters such as number of agents, number of samples, etc.

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CHAPTER 1

INTRODUCTION

Mechanism Design is a branch of economic theory that provides a framework to analyze outcomes in strategic settings. Obtaining mechanisms with "favourable" outcomes is one of the main goals in this field. For example, in the setting of auctions (Myerson (1981)), the seller would want to design the rules so that buyers bid their true value for the item (Vickrey (1961)). Such a mechanism where the buyers report their true values is known as a truthful mechanism. In a single-parameter mechanism design problem, the agents have one value for receiving the service. Just as truthfulness is a property desired in equilibrium, a different goal of the designer could be to maximize the welfare or revenue. Revenue maximizing mechanisms do require knowledge about the distribution of agents. For example, in a single-item single-agent setting, posting a price which depends on the distribution of the agent is the revenue optimal mechanism (Myerson (1981)). Thus to design optimal mechanisms, having knowledge about the distribution is imperative.

There is vast literature on the design of mechanisms from data in the truthful setting. The data that is obtained is the actual value of the agent and the goal is to further design a truthful mechanism, the data of which will again correspond to the values. Dhangwatnootai *et al.* (2010) consider a prior-independent setting and propose a single sample mechanism where reserve prices are stochastically chosen from the available data and obtain near optimal revenue approximations. The goal of a prior independent mechanism is to design mechanisms which work well for all type of distributions. Elkind (2007) show that an auction with reserve price type auction can be learned in polynomial time for finite support distributions. Cole and Roughgarden (2014) who show that polynomial many truthful samples, polynomial in the number of bidders and approximation factor, are necessary and sufficient to obtain an approximation to the optimal revenue. Morgenstern and Roughgarden (2015) consider a general statistical learning approach to learning learning approximately optimal auctions from data. The problems in truthful sample complexity were largely resolved by Devanur *et al.* (2016), Gonczarowski and Nisan (2017) and Guo *et al.* (2019).

However, most practical applications run non-truthful mechanisms i.e a mechanism where truth telling is not an equilibrium. Thus the assumption of access to truthful data might not hold in these settings. Typical auctions used in practice such as i.i.d rank based position auctions (cf. Jansen and Mullen (2008)) with winners-pays-bid semantics (eg : Paes Leme *et al.* (2020)) or all-pay semantics (all participants pay whether they are allocated the item or not) are not truthful. The equilibrium strategy of a player or the bid function is not the identity function. Another benefit of non-truthful mechanisms is that they are more robust, in terms of performance, to the prior distributions of the agents (Feng and Hartline (2018)). Thus the question in non-truthful mechanism design is whether the optimal revenue can be achieved with equilibrium bid samples.

Hartline and Taggart (2019) consider the problem of non-truthful sample complexity and state the goals of the problem clearly. They consider an environment where agents could come from different populations and make no assumption on the feasible allocation rule. They reduce the problem to i.i.d rank-based position auctions, a scenario where the equilibrium bid distribution is well behaved and can be utilized for estimation. Their mechanism utilizes two sets of samples : design time samples and run time samples. Design time samples are obtained by running a mechanism from a family of mechanisms and then these samples are used to select the parameters of the mechanism to be run. Run time samples are obtained when this chosen mechanism is run. The environment considered is a batched environment and hence a final allocation decision is made based on a batch of samples or the run time samples. They define the problem of non-truthful sample complexity as follows

Definition 1. *(Hartline and Taggart (2019)) The problem of non-truthful sample complexity is to identify in a parameterized family of winner-pays-bid (or all-pay) mechanisms and polynomials p_{design} and p_{run} such that with n-agent environments and desired loss ϵ :*

- *C1 : With $m_{design} = p_{design}(n, \epsilon^{-1})$ design-time samples of profiles of Bayes-Nash equilibrium bids from any mechanism in the family, parameters of the designed mechanism can be selected*
- *C2 : With $m_{run} = p_{design}(n, \epsilon^{-1})$ run-time samples of profiles of Bayes-Nash equilibrium bids in the selected mechanism, the selected mechanism can be run*
- *C3 : The expected performance, in agents' values and the m_{run} run-time samples of the selected mechanism, is at most ϵ less than that of the Bayesian optimal mechanism*

They show polynomial time sample complexity for the following environments :

- Winner-pays-bid and all-pay mechanisms, additive welfare approximation, and bounded value distributions
- Winner-pays-bid and all-pay mechanisms, additive revenue approximation, and bounded and regular value distributions
- Truthful mechanisms, multiplicative revenue approximation, and (unbounded) regular value distributions

We show polynomial time sample complexity bounds for winner-pays-bid and all-pay mechanisms, a multiplicative revenue approximation for unbounded regular value distributions. We do use the normalization that the monopoly revenue is one but our results still hold in the case the monopoly revenue of agent i is R_i^*. As the optimal surrogate values are expected order statistics, Hartline and Taggart (2019) show that estimating the revenue of multi-unit auctions up to an ϵ error suffices. We show multiplicative estimation error bounds for regular unbounded distributions with the normalization mentioned above.

As mentioned above, the parameters for the optimal parameterized mechanism is the expected order statistics. Thus the problem reduces to estimating the order statistics from the design time equilibrium bid samples. Chawla *et al.* (2017) consider the more general problem of estimating the per-agent revenue of a counterfactual auction from the equilibrium bid distribution of an incumbent auction. Such scenarios are typically encountered in the A/B testing of auctions (eg : Kohavi *et al.* (2009), Chatham *et al.* (2004)) where typically a website runs one form a service and mixes in a different service with some probability to estimate the gains/losses had the website used the latter. A similar setting is used in Randomized Controlled Trials where a different version is evaluated by splitting the users into two groups, the Treatment Group and the Controlled group. The difference in the outcomes of the two groups convey the effects of the different version. As far as A/B testing of auctions are concerned, the setting Chawla *et al.* (2017) consider is the following : auction A is currently being run while auction B is run a few number of times randomly to see how the agents react. They assume that the agents are aware of this mixing and hence bid according to an auction which is the convex combination of A and B. The goal is to estimate the revenue of auction B from the data obtained from auction C. The traditional approach to revenue estimation by Guerre *et al.*

(2000) involves inverting the equilibrium bid distribution to obtain the value of the agent. This inversion is done based on the empirical bid distribution.

In this thesis, we thoroughly investigate the behavior of the estimator proposed by Chawla *et al.* (2017) for all the dependent variables. We also observe the benefits of truncation in extreme incumbent / counterfactual auction settings. We also compare the method proposed to classical approaches of smoothing the bid function (k-nearest neighbors smoothing) and observe that no smoothing is optimal.

CHAPTER 2

PRELIMINARIES

In the single-parameter independent private model for mechanism design, there are n agents each with a value v_i drawn from a distribution, corresponding to the population of that agent F_i. The model is described in the quantile space where the results and intuitions are much more transparent (eg : Hartline (2013)). The quantile q_i of an agent in population i with value v_i is the probability that a randomly sampled value from F_i is smaller than the value v_i of that agent i.e $q_i = F_i(v_i)$. We can thus express the value of an agent in a population as a function of the quantile i.e $v_i(q) = F_i^{-1}(q)$. The profile of values for the agents is denoted by $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ and the profile of quantiles is denoted by $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$. An allocation $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ where $x_i \in \{0, 1\}$ is an indicator for agent i being served. The space of feasible allocations could be constrained and is denoted as $\mathcal{X} \subset \{0, 1\}^n$. Depending on whether agent i wins or not, she can be assigned a non-negative payment p_i. The utility of the agent is linear in the allocation and payment as $v_i(q_i)x_i - p_i$.

A mechanism $(\boldsymbol{x}, \boldsymbol{p})$ for this problem maps a vector of bids, $\boldsymbol{b}$, to a feasible allocation $\boldsymbol{x} \in \mathcal{X}$ and a payment vector $\boldsymbol{p}$ where the i^{th} index corresponds to the payment for agent i. A mechanism consists of allocation algorithms $\tilde{\boldsymbol{x}}(\boldsymbol{b})$ which maps the bid profiles to a feasible allocation and a payment rule $\tilde{\boldsymbol{p}}(\boldsymbol{b})$ which maps the bid profiles to a payment vector. The two payment formats that we consider are the *winners-pay-bid* format wher, as the name suggests, agents pay only if they receive the service i.e $\tilde{\boldsymbol{p}}_i(\boldsymbol{b}) = b_i\tilde{\boldsymbol{x}}_i(\boldsymbol{b})$ and the *all-pay* format where agents pay irrespective of if they have received the service or not i.e $\tilde{\boldsymbol{p}}_i(\boldsymbol{b}) = b_i$. A strategy for an agent in a mechanism is a mapping from her value to the bid. The strategy of the i^{th} agent is denoted by $s_i(.)$ and $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ is a strategy profile. Mechanisms with these payment formats do not have truth-telling as an equilibrium i.e all agents bidding their value is not an equilibrium strategy.

In this thesis we deal with sample-complexity in non-truthful mechanisms. For the same we analyze non-truthful mechanisms in Bayes-Nash equilibrium. As mentioned earlier, a strategy for agent i is denoted by $s_i(.)$. It maps the agents quantile q_i to a bid, and thus with a uniformly drawn quantile, induces a bid distribution for a particular strategy. Given a quantile q_i for agent i, a distribution over the quantiles of other agents induces an interim allocation rule for agent i, which essentially denotes the probability with which agent i receives the service. Agent i's interim allocation rule is $x_i(q_i) = \mathbf{E}_{\boldsymbol{q}_{-i}}[\tilde{x}_i(\boldsymbol{s}(\boldsymbol{q}))]$. In a similar vein, the interim payment rule is $p_i(q_i) = \mathbf{E}_{\boldsymbol{q}_{-i}}[\tilde{p}_i(\boldsymbol{s}(\boldsymbol{q}))]$. When the agents values are independently distributed, Myerson (1981) gave a characterization of the interim allocation and payment at a Bayes Nash Equilibrium

Theorem 1. *(Myerson (1981)) For independently distributed agents, interim allocation and payment rules are induced by a Bayes-Nash equilibrium with onto strategies if and only if for each agent i*

- *(monotonicity) allocation rule $x_i(q_i)$ is monotone non-decreasing in q_i*
- *(payment identity) payment rule $p_i(q_i)$ satisfies $p_i(q_i) = v_i(q_i)x_i(q_i) - \int_0^{q_i} x_i(r)v_i'(r)dr + p_i(0)$*

While we consider the revenue objective, Hartline and Taggart (2019) study both the welfare and revenue objective. The welfare of a mechanism is $\mathbf{E}[\sum_i v_i(q_i)x_i(q_i)]$. The optimal mechanism for welfare allocates the value-maximizing feasible set. This allocation rule is monotone and implementable via payments from Theorem 1.As far as the revenue of a mechanism is considered, the characterization provided by Myerson (1981) allows us to write the revenue as a weighted sum of single-agent posted pricing mechanisms. In a single-agent posted pricing mechanism, an agent is offered the item for a price and then depending on her value chooses to buy that item or not for that particular price. Clearly the agent will not buy the item if her value is below the price and will buy the item if her value is above the price. Formally, the revenue curve $R_i(q_i)$ for a given value distribution specifies the revenue of a single-agent posted pricing mechanism for a posted price of $v(q_i)$. We have that $R_i(q_i) = v_i(q_i)(1 - q_i)$ as the probability that the agents value is greater than $v_i(q_i)$ is $1 - q_i$. A classical result characterizes the expected payment made by an agent at BNE

Lemma 1.1. *(Myerson (1981), Bulow and Roberts (1989)) In Bayes-Nash Equilibrium, the expected payment of an agent satisfies*

$$\mathbf{E}_{q_i}[p_i(q_i)] = R_i(0)x_i(0) + \mathbf{E}_{q_i}[R_i(q_i)x_i'(q_i)] = R_i(1)x_i(1) - \mathbf{E}_{q_i}[R_i'(q_i)x_i(q_i)]$$

The family of auctions that we consider are the rank-based position auctions. More specifically, we consider the i.i.d environment where the value of agents comes from the same distribution. A position auction with n agents is defined by the tuple of weights $(w_1, w_2, \ldots, w_n)$ with the constraint that $1 \geq w_1 \geq w_2 \geq \cdots \geq w_n$. A position auction assigns agents to positions (potentially randomly) and the agent at the i^{th} position is allocated with probability w_i. A rank-based position auction assigns positions to agents based on a rank. A rank-by-bid position auctions ranks agents based on their bids and assigns positions accordingly. Chawla and Hartline (2013) show that for rank-by-bid i.i.d position auctions with all-pay or winnerss-pay semantics, the equilibrium is symmetric, unique and efficient i.e if $\boldsymbol{s}$ is a BNE strategy profile, then $s_i(.) = s(.)$ for all $i \in [n]$. The equilibrium is also such that the agents' values and bids are in the same order i.e an agent with a higher value will also have a corresponding higher bid. Multi-unit auctions are a special case of these position auctions in the sense that they have a constraint on the number of items they allocate. A k-unit(constraint of k-unit) multi-unit auction would have the following position weights : $w_l = 1$ for all $l \in \{1, 2, \ldots, k\}$ and $w_l = 0$ for all $l \in \{k+1, \ldots, n\}$. The highest k-bids-wins position auctions sorts the agents bids and gives the k-items to the top k bidders. For a position environment, we can define its marginal weights $\boldsymbol{w'} = (w_1', w_2', \ldots, w_{n-1}')$ where $w_i' = w_i - w_{i+1}$. We define $w_0' = 1 - w_1$ and $w_n' = w_n$. Note that on the support $\{0, 1, \ldots, n\}$, the marginals induce a probability distribution. A rank-by-bid position auction with weights $\boldsymbol{w}$ can be thought of as a convex combination of highest-bids-wins k-unit auctions where the convex combination weights are given by the marginals. The rank-by-bid multi-unit allocation rule just depends on the rank of an agent amongst other agents and not the exact bids itself. We can write the allocation probability of an agent with quantile q in a k-highest bids-wins position auction as

$$x_k(q) = \sum_{i=0}^{k-1} \binom{n-1}{i} q^{n-1-i}(1-q)^i$$

This is because, to be given the item, the agent with just needs to have one of the highest k quantiles of n agents. But this is nothing but that at most $k-1$ agents out of the remaining $n-1$ agents have a quantile greater than q. The probability an agent has quantile greater than q is $(1-q)$. The per-agent revenue of such an auction is $P_k = \mathbf{E}_q[R(q)x'_k(q)]$.

The allocation rule of a rank-by-bid position auction is nothing but the convex combination of allocation rules of k-highest-bids-wins auction. Let the allocation rule of the rank-by-bid position auction with weights $\boldsymbol{w}$ be $x(.)$, then

$$x(q) = \sum_{k=1}^{n-1} w'_k x_k(q)$$

We can also obtain the per-agent revenue in a rank-by-bid mechanism with weights $\boldsymbol{w}$, in terms of the per-agent revenues of k-highest-bids-wins auctions by revenue equivalence (Myerson (1981)). Denoting the per-agent revenue as P_x,

$$P_x = \sum_{k=1}^{n-1} w'_k P_k$$

As far as the allocation rule of any rank-by-bid position auction is concerned, the following bound on the derivative of the allocation rule $x'(q)$ is shown

Lemma 1.2. *(Chawla* et al. *(2017)) The maximum slope of the allocation rule x of any n-agent rank-based auction is bounded by n : $\sup_q x'(q) \leq n$. The maximum slope of the allocation rule x_k for the n-agent highest k-bids-win auction is bounded by*

$$\sup_q x'_k(q) \in [\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}] \frac{n-1}{\sqrt{\min\{k-1, n-k\}}} = \Theta(\frac{n}{\sqrt{\min\{k, n-k\}}})$$

We utilize this result while proving an upper bound on the estimated error.

2.1 Surrogate Ranking Mechanism

Hartline and Taggart (2019) consider a parameterized family of mechanisms, the Surrogate

Ranking Mechanism, for which they show polynomial time sample complexity to be ϵ close to the optimal welfare/ revenue.

Definition 2. *A Surrogate Ranking Mechanism(SRM) is parameterized by nT surrogate values $\mathbf{\Psi}$, with $\Psi_i = \{\psi_i^1 \geq \psi_i^2 \cdots \geq \psi_i^T\}$ for all $i \in [n]$. The input to the mechanism is a profile of bids.*

- *A surrogate value for each agent i is calculated as*
  - *Draw $T - 1$ run-time samples from the agent's bid distribution*
  - *Calculate the rank r_i of the agent's bid relative to these samples*
  - *Select the agent's surrogate values $\psi_i = \psi_i^{r_i}$ according to the agent's sample rank*
- *For space $\mathcal{X}$ of feasible allocations, the algorithm allocates to maximize the surrogate surplus $arg\, max_{x \in \mathcal{X}} \sum_{i=1}^{n} \psi_i x_i$*
- *Payments are assigned according to any standard payment format*

The idea is to identify "good" surrogate values which approximate the optimal mechanism. Intuitively, since the agents are competing against agents from its own population, they bid as if they are playing in an i.i.d position auction environment. Thus in some sense the Surrogate Ranking Mechanism gives a reduction from non i.i.d agents to an i.i.d setting. Formally,

Theorem 2. *(Hartline and Taggart (2019)) For any profile of value functions $\boldsymbol{v}$ and surrogate values $\mathbf{\Psi}$, the unique stationary equilibrium of the winner-pays-bid(resp. all-pay) SRM is given by each agent i bidding according to the unique and efficient BNE of the i.i.d winner-pays-bid(resp. all-pay) position auction with weights corresponding to the characteristic weights of the i^{th} agent induced by the SRM and value function v_i.*

They show that the surrogate values of the revenue optimal surrogate ranking mechanism are nothing but expected order statistics. This is then a good approximation to the revenue optimal mechanism. The representation error of this parameterized family of mechanisms is small. Intuitively, representation error arises due to the fact that the actual data that we test on does not come from our target function. Our target function here is the optimal mechanism for the agents which would have some sort of a reserve price while the bid data we get is from the ranking mechanism. There is another form of error known as the generalization error. This corresponds to the classical training vs test error in a learning setup. The goal is to learn the optimal surrogate values by running a mechanism from a family of mechanisms. They show that the there exists a set of surrogate values which approximates the optimal welfare or revenue.

Theorem 3. *(Hartline and Taggart (2019)) There exists a surrogate ranking mechanism with winners-pays-bids, all-pay or truthful-payment semantics which attains a* $(1 - O(\sqrt[3]{n/T}))$*-fraction of the optimal welfare in stationary equilibrium. With regular distributions, there exists such a mechanism which attains a* $(1 - O(\sqrt[3]{n/T}))$*-fraction of the optimal revenue in stationary equilibrium.*

2.2 Revenue Inference

The analysis to find the sample complexity for a revenue approximation involves estimating the optimal surrogate values. This in turn involves estimating the revenue of multi-unit auctions. Chawla *et al.* (2017) give an upper bound on the estimation error when the values of the agents are bounded in the range $[0, 1]$. They consider an i.i.d rank-by-bid position auction environment with either all-pay or winners-pay-bid semantics. The typical inference procedure involves estimating the value distribution from the equilibrium bids, as a known value distribution allows the designer to optimize for revenue. Note that the mapping from the value to the bids at equilibrium maximizes expected utility over the bid distribution of the other agents. The assumption is that the value distribution, the allocation rule, and consequently the bid function, are monotone, continuously differentiable and invertible.

For example, we can obtain the symmetric equilibrium bid function $b(q)$ for the winners-pay-bid rank-by-bid position auction with allocation rule $x(q)$ by taking the derivative of the utility with respect to the equilibrium bid and setting it to zero. For the two settings that we consider, the symmetric equilibrium bid satisfies

- **Winners-Pay-Bid** : $v(q) = b(q) + \frac{x(q)b'(q)}{x'(q)}$
- **All-Pay** : $v(q) = \frac{b'(q)}{x'(q)}$

At a high level, they estimate the bid function using samples from the equilibrium bid distribution and utilize this to estimate the revenue. Note that the allocation rule is known as the designer knows what rank-based position auction is being run. A key result utilized in their theoretical analysis is a bound on the error of this estimated bid function. Csorgo (1983) and Csorgo and Revesz (1978) obtain a bound on the weighted statistical error in the bids based on which the following error bound in the shown in the case the normalization is $\sqrt{N}$.

Lemma 3.1. *(Chawla* et al. *(2017)) Suppose that b and b' exist on $(0,\, 1)$ and $\sup_{q\in(0,\,1)} q(1-q)b'(q) < \infty$. Then the density-weighted uniform mean absolute error of the empirical quantile function $\hat{b}(.)$ on $q \in [\delta_N,\, 1-\delta_N]$ with $\delta_N = \frac{25 \log\log N}{N}$ is bounded almost surely as*

$$\mathbf{E}_{\hat{\boldsymbol{b}}}[\sup_{q\in[\delta_N, 1-\delta_N]} |\sqrt{N}(b^{'}(q))^{-1}(b(q)-\hat{b}(q))s|] < 1 + 16\frac{\log\log N}{\sqrt{N}} \sup_q q(1-q)b'(q).$$

Note that the symmetric bid function for an all-pay auction with allocation rule $x(q)$ satisfies $b'(q) = v(q)x'(q)$. We utilize the normalization $\sup_{q\in(0,1)} v(q)(1-q) = 1$ and hence under this normalization $\sup_{q\in(0,\,1)} q(1-q)b'(q) = \sup_{q\in(0,\,1)} q(1-q)v(q)x'(q) < \infty$.

Now consider the scenario where we want to estimate the per-agent revenue of a rank-by-bid position auction Y, by running an incumbent auction rank-by-bid position auction X (as is done in a typical A/B testing scenario). For the symmetric equilibrium bid function of an all-pay auction with allocation rule $x(q)$ when the agents have a value function $v(q)$ can be expressed as $b'(q) = v(q)x'(q)$. Thus for such an incumbent auction, we can write the per-agent revenue for an auction with allocation rule $y(q)$ as

$$P_y = \mathbf{E}_q[y'(q)(1-q)v(q)] = \mathbf{E}_q[y'(q)(1-q)\frac{b'(q)}{x'(q)}] = \mathbf{E}_q[Z_y(q)b'(q)]$$

where $Z_y(q) = (1-q)\frac{y'(q)}{x'(q)}$. This function $Z_y(.)$ can take very large values at the extreme quantiles. For example when the incumbent auction is the 1-unit auction and the counterfactual auction is the $(n-1)$ auction, $Z_y(q)$ is unbounded at $q = 0$. To avoid this divergence at the extremes, the estimator utilizes truncation to a quantile range $[\delta, 1-\delta]$, where δ is the truncation parameter. This truncation induces a bias-variance tradeoff in the eventual revenue estimator. Also note that the $Z_y(.)$ function only depends on the allocation rules of the two auctions. It is independent of the distribution of the population as these are rank-based auctions and hence the allocation for an agent only depends on the rank of its quantile among other participating agents. The incumbent auction is run N times to obtain an estimate of the equilibrium bid function of an agent. Note that the bids obtained are in best response to the distribution of the the other agents. These N bids are then sorted in ascending order as $\hat{b}_1 \leq \hat{b}_2 \cdots \leq \hat{b}_N$. The estimated bid function is defined as follows

Definition 3. *The estimated empirical bid function for the N sorted bids, $\hat{b}_1 \leq \hat{b}_2 \cdots \leq \hat{b}_N$, is defined as*

$$\hat{b}(q) = \hat{b}_i \quad \forall \quad q \in [\frac{i-1}{N}, \frac{i}{N})$$

This corresponds to a piecewise-constant function. Replacing the estimated bid function in the per-agent revenue equation, we obtain the estimated truncated per-agent revenue as

Definition 4. *The estimator of the per-agent revenue $\hat{P}_y$, for an auction with allocation rule y, and N sampled equilibrium bids $\hat{b}_1 \leq \hat{b}_2 \cdots \leq \hat{b}_N$ from an auction with allocation rule x is :*

$$\hat{P}_y = \sum_{i=\delta_N N}^{N-\delta_N N} (Z_y(\frac{i-1}{N} - Z_y(\frac{i}{N})))\hat{b}_i + \delta_N \frac{y'(1-\delta_N)}{x'(1-\delta_N)} \hat{b}_N$$

where $Z_y(q) = (1-q)y'(q)/x'(q)$ and $\delta_N = \max\{25 \log\log N, n\}/N$ is the truncation parameter.

The estimator is a weighted order statistic of the bids. As we see in the next chapter, while the weighted order statistic form is used to derive their estimation error bounds, while performing simulations, the estimator in its current form is numerically unstable due to the rounding errors from the large $Z(.)$ functions. We now look at the behavior of the estimator for extreme incumbent/counterfactual auctions and verify their theoretical bounds.

CHAPTER 3

ESTIMATOR BEHAVIOR AND STABILITY

3.1 Estimator Stability

The revenue estimator proposed is a weighted order statistic of the sampled bids. The $Z(.)$ function can take very large values while the bids are bounded in the range $[0, 1]$ based on the assumption that the values of the agents lie in the range $[0, 1]$. The estimator proposed is the following

$$\hat{P}_y = \sum_{i=\delta_N N}^{N-\delta_N N} [Z(\frac{i-1}{N}) - Z(\frac{i}{N})]\hat{b}_i + \delta_N \frac{y'(1-\delta_N)}{x'(1-\delta_N)}\hat{b}_N$$

While this estimator is used to derive the inference bounds, the above estimator is numerically unstable. Due to the large values of $Z(.)$, this estimator leads to floating point errors while performing simulations. Since $Z(.)$ can take large values, the difference of $Z(.)$ can lead to large negative values which causes a loss in precision. Thus by re-writing the estimator as a difference of the bids, we obtain

$$\hat{P}_y = \sum_{i=\delta_N N}^{N-\delta_N N} Z_y(\frac{i}{N})(\hat{b}_{i+1} - \hat{b}_i)$$

Note that these two estimators are exactly the same, just that one is written as a weighted order statistic of the bids while the other is written as a weighted sum of the difference of the bids. Note that intuitively, this re-written estimator seems to be more stable as $Z(.)$ can still take large values but this is compensated with the small difference in consecutive sampled bids.

Verification : We verify the numerical instability by comparing the two estimators with end-point corrections. We calculate the truncated sum for estimator L and M, with end-point corrections. We are essentially calculating the same summation hence the values should be the

	$N = 10^3$	$N = 10^4$	$N = 10^5$	$N = 10^6$
$\|L - M\|$	$2.33 * 10^{-8}$	$8.21 * 10^{-2}$	$1.05 * 10^{6}$	$4.39 * 10^{12}$

Table 3.1: Absolute error in the truncated estimator L and M, when $n = 10$, $x(q) = 1 - (1 - q)^{n-1}$, $y(q) = q^{n-1}$

same. However, if they are different, then this means that the source of error is the accuracy of computation. Thus we calculate

$$L = \sum_{i=\delta_N N}^{N-\delta_N N} Z_y(\frac{i}{N})(\hat{b}_{i+1} - \hat{b}_i)$$

$$M = \sum_{i=\delta_N N}^{N-\delta_N N} (Z_y(\frac{i-1}{N}) - Z_y(\frac{i}{N}))\hat{b}_i + Z_y(\frac{N - \delta_N N}{N})\hat{b}_{N-\delta_N N+1} - Z_y(\frac{\delta_N N - 1}{N})\hat{b}_{\delta_N N}$$

Setting : The number of agents are $n = 10$, all of them coming from the $Beta(2, 2)$ distribution. The incumbent auction is the $(n-1)$-unit auction while the counterfactual auction is the 1-unit auction.

Observation : We observe that for small N, estimator L and M perform similarly as expected but as N increased the difference of the summations starts to diverge. This means that as we increase N, we start including quantiles for which $Z(.)$ is very large, hence accuracy errors from $(Z_y(\frac{i-1}{N}) - Z_y(\frac{i}{N}))$ start adding up leading to this behaviour.

3.2 Estimator Behavior

Using the re-written estimator we perform simulations to show the properties of the inference error.

3.2.1 Simulation Methodology

We perform Monte-Carlo simulations to calculate the mean absolute error of the estimated revenue $\hat{P}_B$ from the true revenue of the counterfactual auction P_B. We generate a uniform quantile grid in the range $[0, 1]$ where the interval length is Δ. The counterfactual auction is denoted as B and the incumbent auction as C. The allocation rules $x_C(q)$ and $x_B(q)$, their derivatives $x'_C(q)$ and $x'_B(q)$, and $R(q)$ are calculated analytically on the grid. The true revenue is calculated using $R(q)$ and $x'_B(q)$, and then performing numerical integration on the grid

$$P_y = \frac{\Delta}{1+\Delta} \sum_{l=0}^{\frac{1}{\Delta}} (1 - \ell\Delta) v(\ell\Delta) x'_B(\ell\Delta)$$

The equilibrium bids for the incumbent all-pay auction C are calculated in a similar manner using numerical integration on the grid. We utilize the fact that the equilibrium bids for an all-pay auction satisfy $v(q) = \frac{b'(q)}{x'(q)}$. For a particular simulation, we generate a sample of N bids, with replacement, from the bid distribution of the incumbent auction and sort them as $\hat{b}_1 \leq \hat{b}_2 \leq \cdots \leq \hat{b}_N$. Using the definition of $Z(.)$,

$$Z(q) = (1 - q) \frac{x'_B(q)}{x'_C(q)}$$

we estimate the revenue as

$$\hat{P}_B = \sum_{i=\delta_N N}^{N - \delta_N N} Z(\frac{i}{N}) (\hat{b}_{i+1} - \hat{b}_i)$$

where $\delta_N = \frac{\max\{25 \log \log N, n\}}{N}$ is the truncation parameter. We repeat this simulation by performing a random draw of the bids 2000 times and obtain the mean absolute error of the estimated revenue.

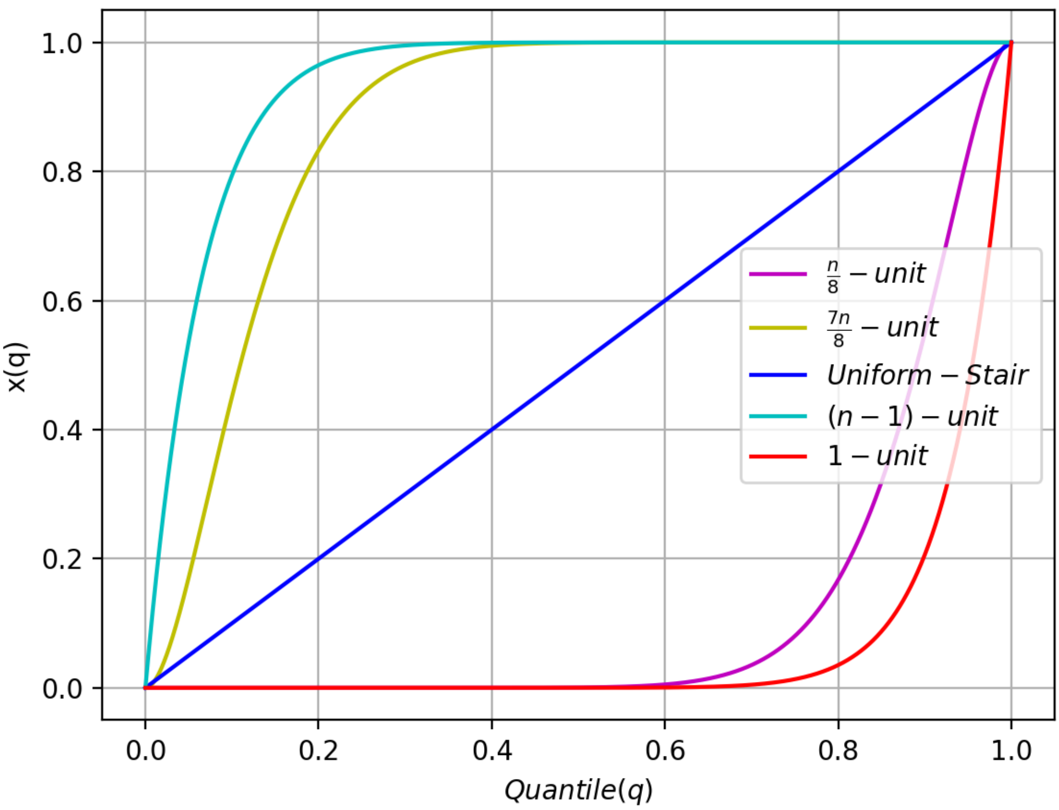


Figure 3.1: Allocation Rules for different auctions when $n = 16$

3.2.2 Simulation Results

In this section, we verify the dependence of the inference error bounds on various parameters. One of the auction we use for our simulations is the Uniform Stair auction.

Definition 5. *The uniform-stair auction is an n-agent position auction defined by the weights* $\boldsymbol{w} = (1, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0)$.

The allocation rule for the uniform stair auction is the uniform stair allocation rule $x(q) = q$. We consider a set of five auctions for our study :

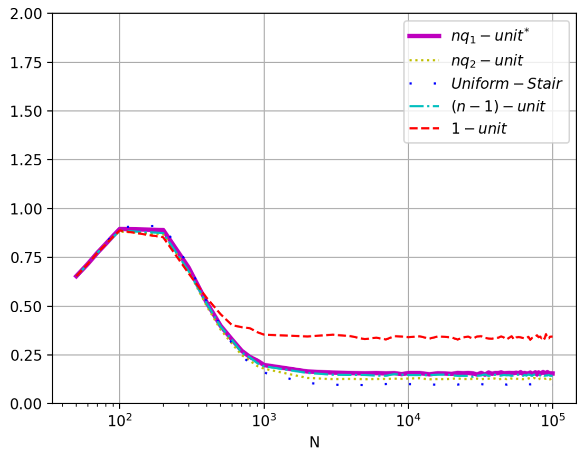
- 1-unit auction : Extreme low supply auction
- $\frac{n}{8}$-unit auction : Low supply auction
- Uniform Stair auction

- $\frac{7n}{8}$-unit auction : High supply auction
- $(n-1)$-unit auction : Extreme high supply auction

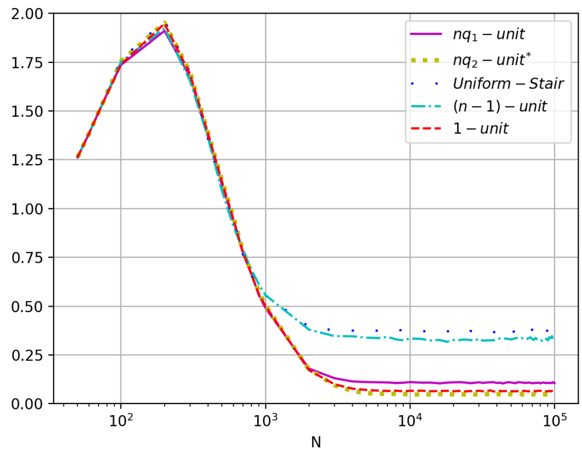
The allocation rule for each of these auctions when $n = 16$ is shown in Figure 3.1. The overview of our experiments is as follows. We first test the dependence of the estimation error on N, the number of samples of the incumbent auction, and n the number of agents, in the case the counterfactual auction is mixed in with some probability ϵ. We test the dependence of the estimation error on the mixture probability ϵ. We show the benefits of truncation in the case the number of agents are large but the number of samples are small. We then compare the estimation method to classical A/B-testing methods and finally verify that the results shown are robust to different distributions.

Dependence of Inference Error on N : Figure 3.2 shows the dependence of the normalized truncated mean absolute error on the number of bid samples N, for a fixed $n = 16$ and the agents come from the $F = Beta(2, 2)$ distribution. The mean error is normalized by $\sqrt{N}$ and the counterfactual auction is mixed in with probability $\epsilon = 10^{-3}$. We consider three settings, when the counterfactual auction is the low supply $\frac{n}{8}$-unit auction, high supply $\frac{7n}{8}$-unit auction and the Uniform Stair auction and for each case plot the estimation error for the five incumbent auctions under consideration. The theoretical bound tells us that the estimator has a $1/\sqrt{N}$ dependence. When we mix the counterfactual auction with some probability, there is no need for truncation. In fact truncation causes a loss in estimation in such a setting. We observe that indeed the estimation error has a $1/\sqrt{N}$ as the normalized error is a constant. The magnitudes of the error vary due to the different dependence on n in different cases. Note that in in certain settings, a different incumbent auction is better at estimating the revenue of an auction than that particular auction itself. For example in Figure 3.2 (a), the Uniform Stair auction is better at estimating the revenue of the $\frac{n}{8}$ auction than the $\frac{n}{8}$ auction itself. This is because the bid distribution of the Uniform Stair auction is better to estimate revenue than the skewed bid distribution of the low supply auction. Note that while the theoretical bound is greater than one at small N, our empirical analysis shows that the inference error is non-trivial even in these extreme settings.

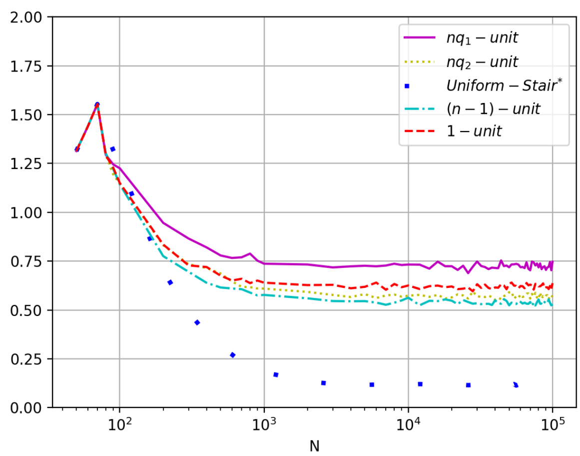
Dependence of Inference Error on n : Figure 3.3 shows the dependence of the normalized



(a) Counterfactual Auction : $\frac{n}{8}$-unit auction

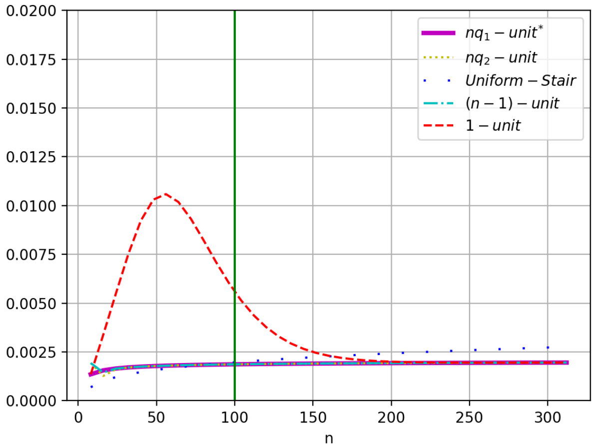


(b) Counterfactual Auction : $\frac{7n}{8}$-unit auction

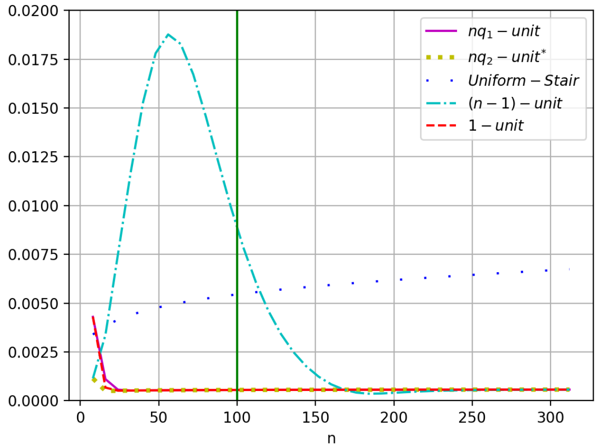


(c) Counterfactual Auction : Uniform Stair Auction

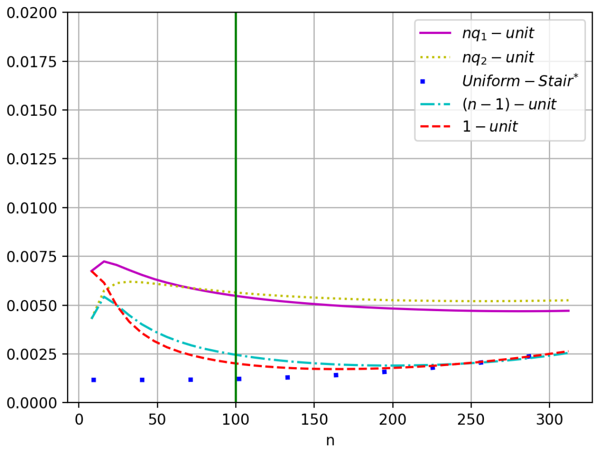
Figure 3.2: Dependence of the normalized truncated mean absolute error on the number of bid samples for different counterfactual auctions and different incumbent auctions when the counterfactual auction is mixed in with some probability



(a) Counterfactual Auction : $\frac{n}{8}$-unit



(b) Counterfactual Auction : $\frac{7n}{8}$-unit



(c) Counterfactual Auction : Uniform Stair Auction

Figure 3.3: Dependence of the truncated mean absolute error on the number of agents for different counterfactual auctions and different incumbent auctions when the counterfactual auction is mixed in with some probability

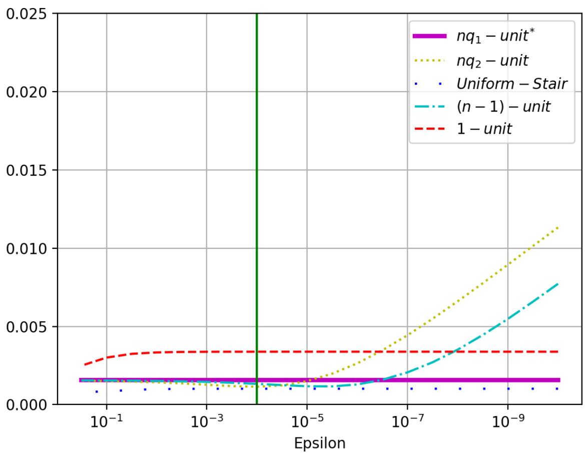
truncated mean absolute error on the number of agents n, for a fixed number of bid samples $N = 10000$, when the agents come from the $F = Beta(2, 2)$ distribution and when the counterfactual auction is mixed with probability $\epsilon = 10^{-3}$. The theoretical analysis gives us the following worst bounds for each of the settings we consider

- Uniform Stair Counterfactual : $O(\frac{log(n)}{\sqrt{N}})$
- Low Supply Counterfactual : $O(\frac{\sqrt{n}\log(n/\epsilon)}{\sqrt{N}})$
- High Supply Counterfactual : $O(\frac{\sqrt{n}\log(n/\epsilon)}{\sqrt{N}})$

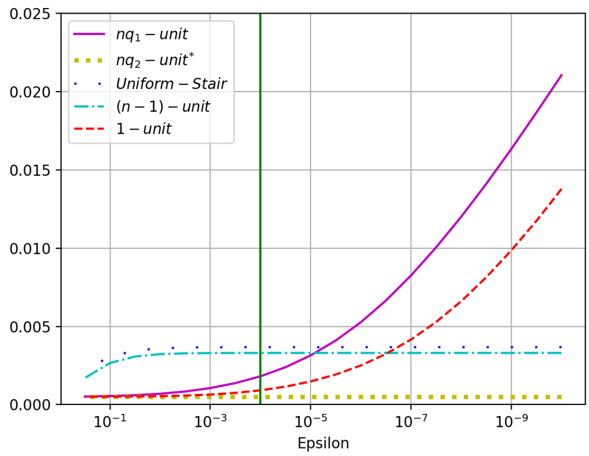
Note that this bound is valid only for $N > n^2$, which corresponds to the region to the left of the solid green vertical line in Figure 3.3. The empirical analysis shows that the inference error is better than the theoretical bounds obtained. We see a trend upwards in Figure 3.3 (c) in the case the incumbent is the counterfactual for large n as the truncation depends on n for a fixed N in this setting which leads to a loss in bid-data.

Dependence of Inference Error on ϵ : Figure 3.4 shows the dependence of the truncated mean absolute error on the mixture probability ϵ, for a fixed number of bid samples $N = 10000$, fixed number of agents $n = 16$ when the agents come from the $F = Beta(2, 2)$ distribution. Recall that in this mixture setting, the counterfactual auction is run with probability ϵ and the incumbent auction is run with probability $(1 - \epsilon)$. The theoretical bound tells us that in the region $N > \frac{1}{\epsilon}$ (corresponding to the region to the left of the solid green line in Figure 3.4), the inference error is bounded as $O(log(1/\epsilon))$, and in the region $N < \frac{1}{\epsilon}$ it is bounded as $n^2 \log N/\sqrt{N}$ which dominates the other bound. Note that when $N > \frac{1}{\epsilon}$, our empirical study shows a sub-logarithmic bound in the case the counterfactual is the low-supply and high-supply auction while it shows a $\Theta(\log(1/\epsilon))$ bound in the case the counterfactual auction is the Uniform Stair. Chawla *et al.* (2017) show that there is a Universal B-Test mechanism which is a mixture of the 1-unit and $(n - 1)$-unit auction. The property of such an auction is that mixing this with any other position auction makes it possible to infer the revenue of that position auction. Figure 3.5 depicts the behavior when we utilize the Universal B-Test. Comparing this to Figure 3.4, there is not much improvement in terms of the inference error.

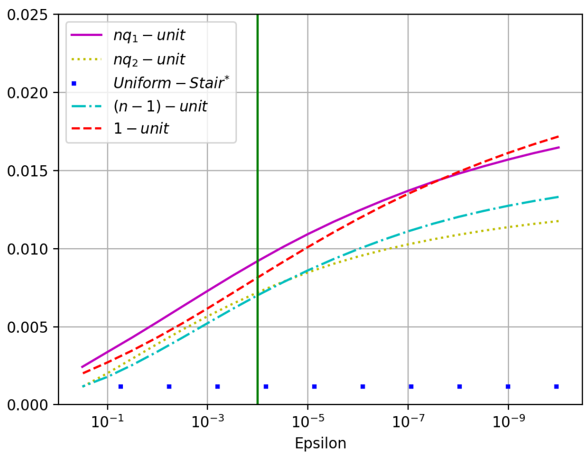
Truncation vs Un-truncation :



(a) Counterfactual Auction : $\frac{n}{8}$-unit



(b) Counterfactual Auction : $\frac{7n}{8}$-unit



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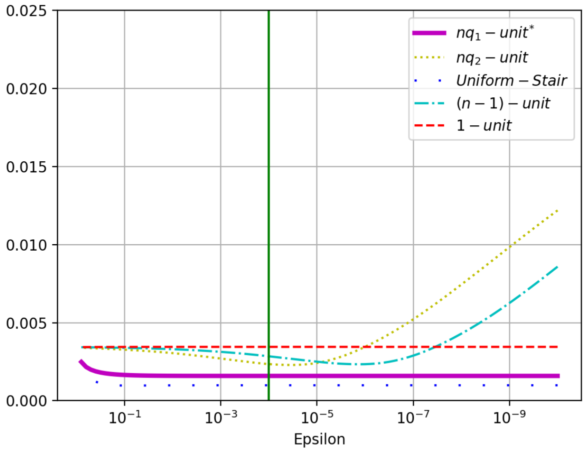
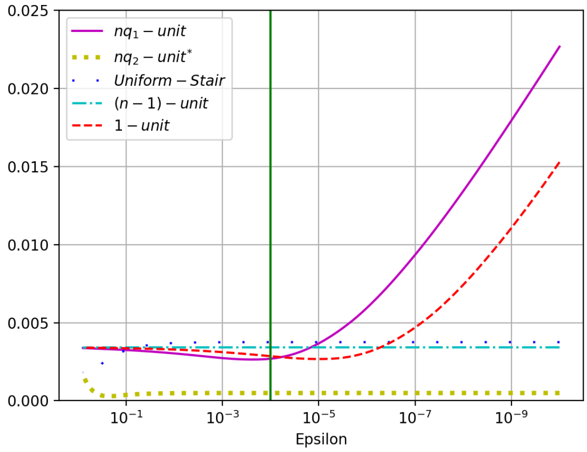
(c) Counterfactual Auction : Uniform Stair

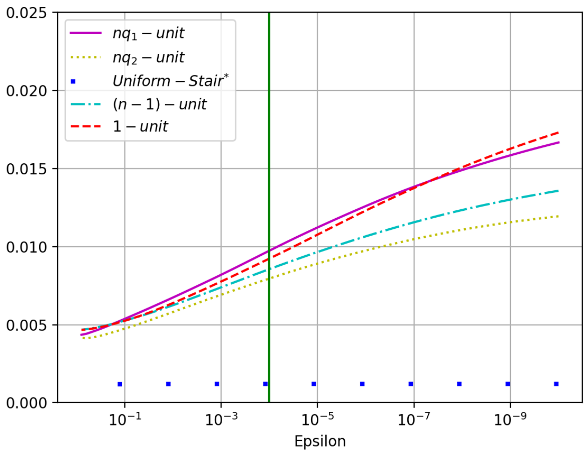
Figure 3.4: Dependence of the truncated mean absolute error on the mixture probability ϵ for different counterfactual auctions and different incumbent auctions when the number of bid samples and number of agents are fixed



(a) Counterfactual Auction : $\frac{n}{8}$-unit



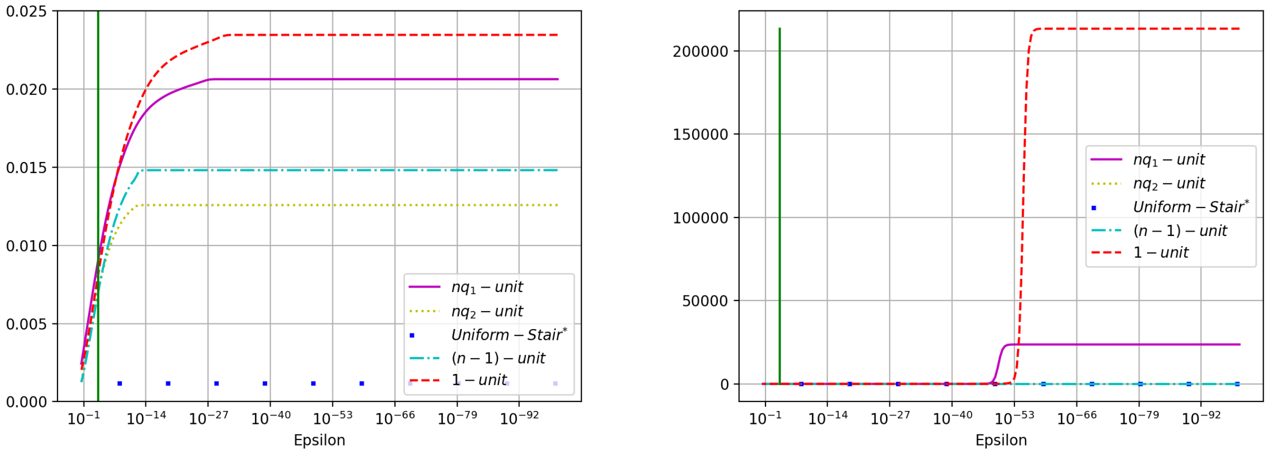
(b) Counterfactual Auction : $\frac{7n}{8}$-unit



(c) Counterfactual Auction : Uniform Stair

Figure 3.5: Dependence of the truncated mean absolute error on the mixture probability ϵ when we mix the Universal A/B test mechanism for different counterfactual auctions and different incumbent auctions when the number of bid samples and number of agents are fixed

We now look at the benefit of truncation when the counterfactual is not mixed in with any probability. As a motivation, we first observe the behavior of the estimator error for extremely small mixture probabilities. Figure 3.6 compares the truncated and un-truncated estimator errors when the counterfactual auction is the Uniform Stair auction, the number of agents are $n = 16$, the number of samples are $N = 10000$ and the agents come from the $F = Beta(2, 2)$ distribution. We observe that when the incumbent are the low-supply auctions, $n/8$-unit and 1-unit, the un-truncated estimation error is of the order 10^5 for extremely small mixture probabilities, while the truncated estimator is bounded and less than the trivial bound of one even at these extremely small mixture probabilities. An interesting point to note is that while the theoretical bounds suggest that at approximately $\epsilon = 1/N$, the better bound becomes $O(n^2 \log N/\sqrt{N})$ (solid green line in Figure 3.6), simulation evidence suggests that this behavior happens at much smaller ϵ.

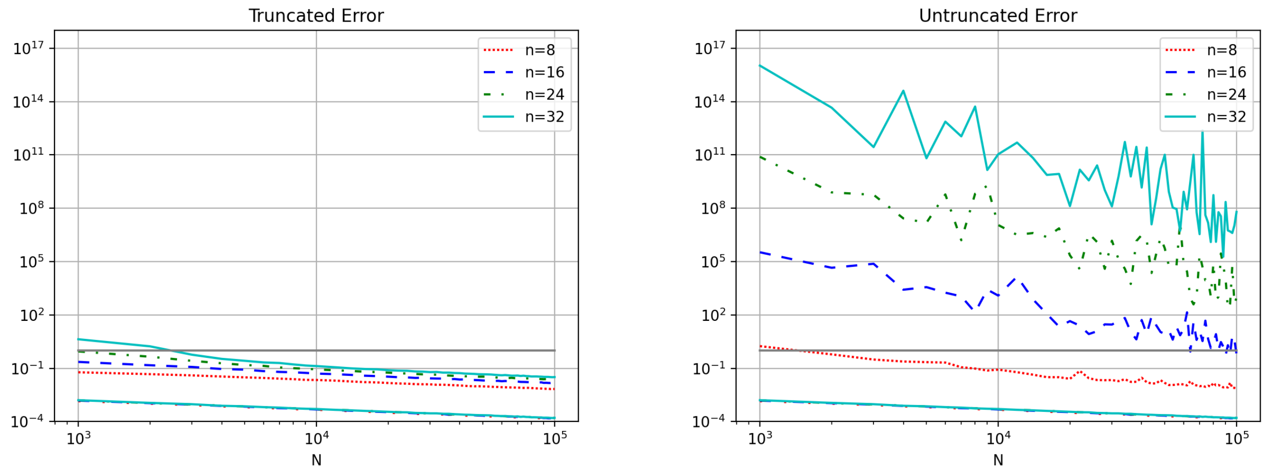


(a) Truncated Error

(b) Untruncated Error

Figure 3.6: Comparison of the truncated error and untruncated error when the counterfactual auction (Uniform Stair) is mixed in the incumbent auction with some probability ϵ

We now compare the truncated and un-truncated error when the counterfactual auction is not mixed in with any probability. Figure 3.7 depicts a comparison of the two, when the incumbent auction is the $n/8$-unit auction and the counterfactual auction is the $7n/8$-unit auction, by plotting the estimation error against N for different number of agents. The agents come from the $F = Beta(2, 2)$ distribution. We also show the estimation error in the case the counterfactual and incumbent auction are the same, depicted by the solid lines for each n. We expect this "counterfactual error" to be $\Theta(1/\sqrt{N})$ where the constants depend on n. The estimator is just the average of the sampled bids and hence we do not expect a drastic estimation error. We now come to the $n/8$-unit incumbent and $7n/8$-unit counterfactual setting. The reason we chose these auctions is because they capture the "worst-case" scenario. The dark solid line in each plot corresponds to the trivial bound of one. We can clearly see the benefit of truncation as the truncated error bounds are less than one while the un-truncated error bounds are of very high order. For $n = 8$, even the un-truncated estimator performs well but truncation is required as n increases for a given N. Another point to note is that for a given n, as N increases, the un-truncated estimator and truncated estimator begin to perform similarly. We can also verify the $\sqrt{N}$ dependence of the estimation error from our plots.

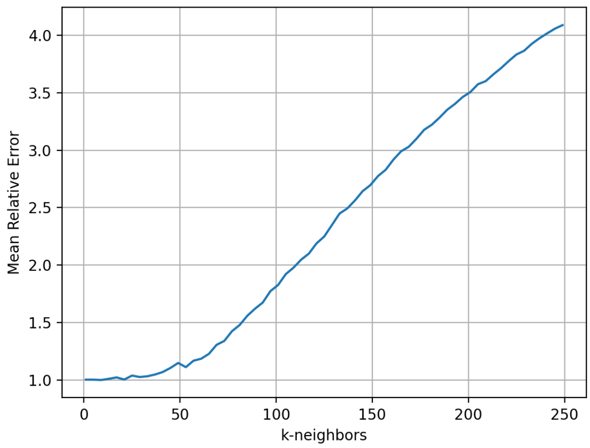


(a) Truncated Error

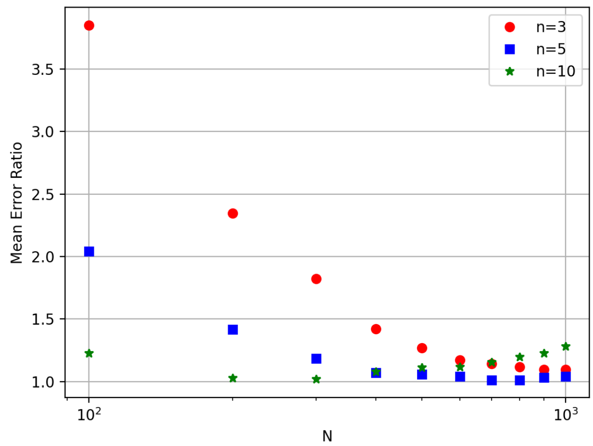
(b) Un-truncated Error

Figure 3.7: Log-log plot to show the estimation error as a function of the number of samples N for different number of agents. The solid black line is the trivial error bound while the solid colored lines correspond to the scenario the incumbent auction is the counterfactual auction itself

We compare the estimation mechanism proposed to a classical method which involves smoothing the bid distribution. The natural smoothing approach we employ is to consider the k-nearest neighbors to each bid in sorted order of bids. The classical approach, which asks for a uniform bound on the error in estimates of values to plug into the revenue estimator, would tune k depending on the bid distribution. We consider the scenario where the incumbent auction is the 1-unit auction (no mixing of the counterfactual auction), the counterfactual auction is the $(n-1)$-unit auction, the number of bid samples $N = 1000$, the number of agents $n = 5$ and the agents value comes from $F = Beta(2, 2)$ distribution. Our empirical evidence (Figure 3.8 (a)) suggests that the optimal smoothing is no smoothing. Recall that the estimator does not require any smoothing dependent on the bid distribution. The truncation parameter used for the estimator does not depend on the type of auction run, but how does it compare to the optimal truncation parameter ? To verify the same, we consider a $(n-1)$-unit counterfactual auction and 1-unit incumbent auction for small values of n. Figure 3.8 (b) shows that for the chosen values of n, the truncation of the estimator is at most four times the error obtained using the optimal truncation.



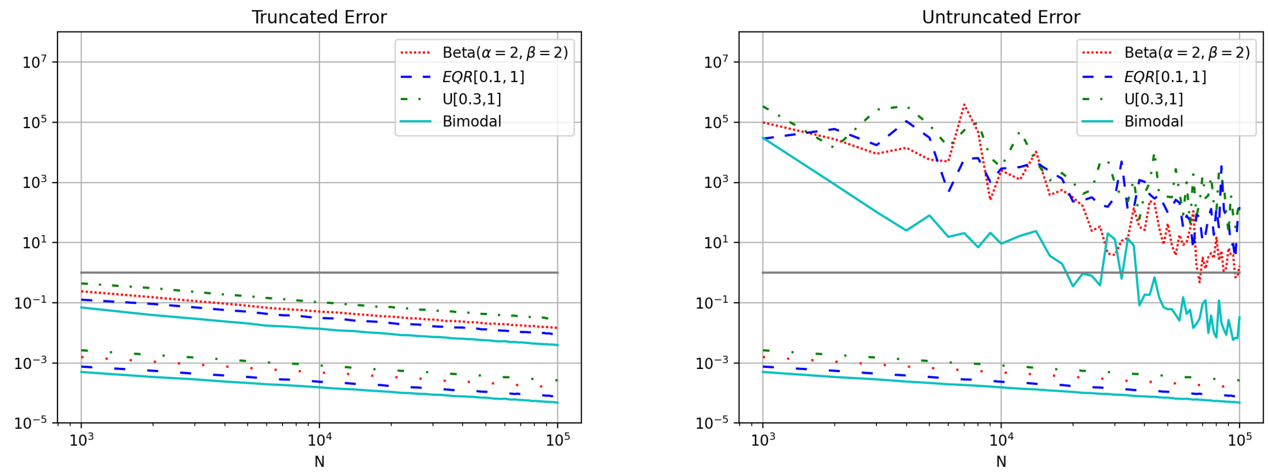
(a) k-nearest neighbor smoothing



(b) Optimal Truncation

Figure 3.8: The figure on the left corresponds to the mean relative error as a function of the amount of smoothing (k-nearest neighbors). The figure on the right corresponds to the ratio of the error using the truncation presccribed and the error obtained from the optimal truncation

Distribution Robustness : Currently all of our experiments are run when the agents come from the distribution $F = Beta(2, 2)$. We verified the robustness of our empirical results by running the experiments for a variety of distributions. The distributions we considered are the equal-revenue distribution on $[0.1, 1]$, the uniform distribution on $[0.3, 1]$ and a bi-modal distribution. Figure 3.9 shows the truncation vs un-truncation experiment for $n = 16$ and the behavior is same as that of the $F = Beta(2, 2)$ distribution.



(a) Truncated Error

(b) Un-truncated Error

Figure 3.9: Log-log plot to show the estimation error as a function of the number of samples N for different distributions.

CHAPTER 4

NON-TRUTHFUL SAMPLE COMPLEXITY

In this chapter, we obtain a sample complexity result for regular distributions, and a multiplicative revenue objective. Our main result is

Lemma 3.2. *For agents with regularly distributed values having a monopoly revenue of 1 leading to potentially unbounded values, there are families of winner-pays-bid and all-pay mechanisms that satisfy conditions C1, C2 and C3 with* $p_{run}(n, \epsilon^{-1}) = O(n\epsilon^{-3})$ *and* $p_{design}(n, \epsilon^{-1}) = \tilde{O}(n^6\epsilon^{-14})$ *for multiplicative loss and the revenue objective.*

Along with this, we obtain multiplicative bounds on the revenue estimation error. To obtain a multiplicative bound, we obtain an upper bound on the term $|\hat{P}_y - P_y|/P_y$, where P_y is the per-agent revenue of the position auction in an i.i.d environment and $\hat{P}_y$ is the estimated revenue by running an incumbent auction.To obtain an upper bound on the desired term, we upper bound the numerator and lower bound the denominator for a setting where the monopoly revenue is one.

Lemma 3.3. *Under the assumption the agents are regular, the revenue curve is normalized such that* $\max_q R(q) = 1$ *and the incumbent rank-based auction* x *is such that its top two position weights are the same, the mean relative error in estimating the revenue of a rank-based auction with allocation rule y using N samples from the bid distribution for an all-pay rank based auction with allocation rule x is bounded as below. Here n is the number of positions in the two auctions, and* $\hat{P}_y$ *is the estimator with* δ_N *set to* $\max(25 \log\log N, n)/N$.

$$\frac{\mathbf{E}_{\hat{b}}[|\hat{P}_y - P_y|]}{P_y} \leq O(\frac{n^3 \log N}{\sqrt{N}})$$

When the distribution are similar, we can obtain a better multiplicative bound.

4.1 Multiplicative Bounds

First, to obtain a lower bound, we obtain lower bounds on the value of the revenue curve at quantile values which are integer multiples of $1/n$, in terms of other such similar quantile values. The bound follows from the concavity of the revenue curve due to the regularity assumption. The quantile convention we use for the lower bound below is that a lower quantile corresponds to a higher strength in the population i.e $v(q) = F^{-1}(1-q)$.

Lemma 3.4. *For all $k_C, k_B \in \{1, \dots, n-1\}$, regular distributions, $R(\frac{k_B}{n}) \geq \frac{1}{n}R(\frac{k_C}{n})$.*

Proof. Case 1 : $k_C > k_B$. Since the revenue curve is concave, $\frac{R(\frac{k_B}{n})}{\frac{k_B}{n}} \geq \frac{R(\frac{k_C}{n})}{\frac{k_C}{n}}$. Thus, $\frac{R(\frac{k_B}{n})}{R(\frac{k_C}{n})} \geq \frac{k_B}{k_C}$. Based on the current case, $k_B \geq 1, k_C \leq n$, thus $\frac{R(\frac{k_B}{n})}{R(\frac{k_C}{n})} \geq \frac{1}{n}$.

Case 2 : $k_C < k_B$. Note that, $1 - \frac{k_B}{n} \geq \frac{1}{n}(1 - \frac{k_C}{n})$. Since the revenue curve is concave, $\frac{R(\frac{k_B}{n})}{1-\frac{k_B}{n}} \geq \frac{R(\frac{k_C}{n})}{1-\frac{k_C}{n}}$, thus $\frac{R(\frac{k_B}{n})}{R(\frac{k_C}{n})} \geq \frac{1}{n}$. Hence the claim is proved.

Finally when $k_C = k_B$, we have $R(\frac{k_C}{n}) = R(\frac{k_B}{n})$ and thus trivially $R(\frac{k_B}{n}) \geq \frac{1}{n}R(\frac{k_C}{n})$.

□

For a multiplicative bound, we want a lower bound on the denominator which is the per-agent revenue in a general rank-based position auction y. Our approach is to obtain a multiplicative bound on the estimated revenue of multi-unit auctions and then show that as any position auction y is a convex combination of multi-unit auctions the same bound follows. The lemma below obtains lower and upper bounds on the per-agent revenue in a multi-unit in terms of the revenue curve.

Lemma 3.5. *For all $k \in \{1, \dots, n-2\}$, the revenue of the k-highest-bids-win auction in an n-agent i.i.d regular environment satisfy*

$$\frac{1}{2}\frac{k}{k+1}R\left(\frac{k+1}{n}\right) \leq P_{(k:n)} \leq R\left(\frac{k}{n}\right)$$

.

For $k = n-1$,

$$\frac{1}{4}R\left(1-\frac{1}{n}\right) \le P_{(n-1:n)} \le R\left(1-\frac{1}{n}\right)$$

Proof. Let us denote the per-agent revenue of the k-highest-bids-win auction in an n-agent environment as $P_{(k:n)}$. For any k, the expected revenue of the auction $Rev[k] = nP_{(k:n)}$.

1. Upper-Bound : For any $k \in \{1, 2, \ldots, n-1\}$, consider another auction with n agents such that each agent is served with ex-ante probability at most k/n, but without a supply constraint i.e this auction could give out items to all agents based on the ex-ante constraint. This corresponds to posting a price of $V(\frac{k}{n})$ for each agent and the number of items sold in expectation is k. Clearly, the revenue of the k-unit auction is less than the revenue obtained from the agents by posting this price. Thus,

$$Rev[k] \le n\,R(\frac{k}{n})$$

2. Lower-Bound : By revenue equivalence,

$$nP_{(k:n)} = k\mathbf{E}[v_{(k+1:n)}]$$

For any value z, we can bound the expected $(k+1)^{th}$ order statistic as

$$\mathbf{E}[v_{(k+1:n)}] \ge zPr\{E_z\}$$

where E_z is the event that at least $(k+1)$ agents have value greater than z. Choosing $z = V(\frac{k+1}{n})$ and observing that the probability that at least $k+1$ agents have a value greater than $V(\frac{k+1}{n})$ is at least $\frac{1}{2}$, we get

$$Rev[k] \ge k.V(\frac{k+1}{n}).\frac{1}{2} = \frac{1}{2}\frac{k}{k+1}nR(\frac{k+1}{n})$$

For the case $k = n-1$, the revenue of this k-unit auction will at least be k times a price of $V(1-\frac{1}{n})$, times the probability that at least n-agents have a value greater than $V(1-\frac{1}{n})$. Thus

$$Rev[n-1] \ge (n-1)V(1-\frac{1}{n})(1-\frac{1}{n})^n \ge nR(1-\frac{1}{n})\frac{1}{4}$$

Hence the claim follows

□

Note that $P_{(1:n-1)}$ and $P_{(1:n)}$ are the per agent revenues for a single unit $(n-1)$ and n agent auction respectively. Bulow and Klemperer (1996) showed that for regular, i.i.d single agent environments, the expected revenue of the highest-bids-wins auction with $(n+1)$-agents is at least that of the optimal auction with n agents. We now bound the single unit auction per-agent revenue in a $(n-1)$-agent setting by the corresponding per-agent revenue in the n-agent setting.

Lemma 3.6. *For i.i.d regular single-item environments, the per-agent revenue of the highest-bids-wins in a $(n-1)$-agent setting is bounded in terms of the per-agent revenue of the highest-bids-wins in a (n)-agent setting as*

$$(1-\frac{1}{n-1})P_{(1:n)} \leq P_{(1:n-1)} \leq (1+\frac{1}{n-1})P_{(1:n)}$$

Proof. Consider a n-agent auction with $(n-1)$ real agents and one fake agent. We run the optimal 1 unit auction on these agents. The revenue of this auction will be $\frac{n-1}{n}OPT(1,n)$ as the revenue contribution will only be from the real agents. Note that this auction will act as a lower bound for the optimal one unit $(n-1)$-agent auction as this is an auction which allocates one item to $(n-1)$ agents. Thus, we get $OPT(1,n-1) \geq \frac{n-1}{n}OPT(1,n)$. Thus utilizing the result of Bulow and Klemperer (1996), we get that the expected revenue of a second-price auction with n-agents is a $\frac{n}{n-1}$ approximation to the optimal revenue. Hence we have

$$(n-1)P_{(1:n-1)} \geq \frac{n-2}{n-1}OPT(1,n-1) \geq \frac{n-2}{n}OPT(1,n) \geq (n-2)P_{(1:n)}$$

Thus we get $P_{(1:n-1)} \geq \frac{n-2}{n-1}P_{(1:n)}$. To obtain a corresponding upper bound, we simply apply

$$nP_{(1:n)} \geq OPT(1,n-1) \geq (n-1)P_{(1:n-1)}$$

and thus $P_{(1:n-1)} \leq \frac{n}{n-1}P_{(1:n-1)}$.

□

In our setting, we consider unbounded value distributions which could potentially lead to equilibrium bids. The lemma below shows bounds on the equilibrium bid function in multi-unit auctions.

Lemma 3.7. *For any regular value distribution, the upper bound on the bid for a k-unit auction($k \in \{2, 3, \ldots, n-2\}$), both winners-pay-bid and all-pay semantics, in a n-agent i.i.d regular environment is given by*

$$\frac{1}{2}V(\frac{k}{n-1}) \leq \lim_{v\to\infty} b(v) \leq V(\frac{k-1}{n-1})$$

. For $k = n-1$, the bid in the limiting case is,

$$\frac{1}{4}V(1-\frac{1}{n-1}) \leq \lim_{v\to\infty} b(v) \leq V(1-\frac{1}{n-1})$$

For $k = 1$, the bid in the limiting case is at least $(n-1)P_{(1:n-1)}$.

Proof. For a k-unit highest-bids-win winners-pay-bid auction, by revenue equivalence, for a bid function $b(.)$,

$$b(v) = \mathbf{E}[v_{(k+1:n)} | v_{(k+1:n)} \leq v]$$

Thus in the limiting case,

$$\lim_{v\to\infty} b(v) = \mathbf{E}[v_{(k:n-1)}]$$

Now if we consider a $(k-1)$-highest bids win auction in a $(n-1)$-agent environment,

$$(n-1)P_{(k-1:n-1)} = (k-1)\mathbf{E}[v_{(k:n-1)}]$$

where $P_{(k:n)}$ is the per-agent expected revenue of a k-unit auction in an n-agent environment. Thus,

$$\lim_{v\to\infty} b(v) = \frac{n-1}{k-1}P_{(k-1:n-1)} \leq \frac{n-1}{k-1}R(\frac{k-1}{n-1}) = V(\frac{k-1}{n-1})$$

where the second inequality comes from Part 1 of Lemma 3.5. For a lower bound, $P_{(k-1:n-1)} \geq \frac{k-1}{2k}R(\frac{k}{n-1})$, which again comes from Lemma 3.5, and the lower bound for the limiting bid follows. When $k = n-1$, using Lemma 3.5, we can bound $P_{(n-2:n-1)} \geq \frac{1}{4}R(1-\frac{1}{n-1})$ and the claim follows. For the case $k = 1$,

$$\lim_{v \to \infty} b(v) = \mathbf{E}[v_{(1:n-1)}] \geq \mathbf{E}[v_{(2:n-1)}]$$
$$\geq (n-1)P_{(1:n-1)}$$

For a k-unit highest-bids-win all-pay auction, by revenue equivalence and a bid function $b(.)$,

$$b(v) = \mathbf{E}[v_{(k+1)}|v_{(k+1)} < v]Pr[v_{(k+1)} < v]$$

In the limiting case,

$$\lim_{v \to \infty} b(v) = \lim_{v \to \infty} \mathbf{E}[v_{(k+1)}|v_{(k+1)} < v]Pr[v_{(k+1)} < v] = \lim_{v \to \infty} \mathbf{E}[v_{(k+1)}|v_{(k+1)} < v]$$

and the proof follows as in the case for winner-pays-bid setting. □

We consider the model where the bidders are i.i.d and their samples are drawn from continuous distributions. The auction is a rank-by-bid auction with either the all-pay format or the winner pays bid format. A rank-by-bid auction can be written as a convex combination of k-highest bids win auctions, where the weights for each k-unit auction are given by $w'_k = w_k - w_{k+1}$. This rank-by-bid auction can be viewed as sampling a k from the distribution **w'** $= (w'_1, w'_2, \ldots, w'_n)$ and then running that k-unit auction. While the previous lemma obtained upper bounds for maximum bids in multi-unit auctions where $k \in \{2, 3, \ldots, n-1\}$, the maximum bid in a single unit auction can be unbounded (eg : equal revenue distribution).

Lemma 3.8. *The maximum bid in a general position auction with its top two position weights being the same, with either all-pay or winner-pays-bid semantics, is bounded.*

Proof. **Bids for an all-pay setting** : Consider an all-pay rank-by-bid auction with position weights **w** $= (w_1, w_2, \ldots, w_n)$. Since this is an all-pay setting and using the fact that an rank-by-bid auction y can be written as a convex combination of multi-unit auctions, we get

$$\lim_{v \to \infty} b^{y-all-pay}(v) = \lim_{v \to \infty} \sum_{k=1}^{n} w'_k b^{k-all-pay}(v)$$
$$= \lim_{v \to \infty} \sum_{k=2}^{n} w'_k b^{k-all-pay}(v)$$

where $b^{y-all-pay}(.)$ is the bid function of the all-pay auction y and $b^{k-all-pay}(.)$ is the bid function for the k-unit all pay auction. The second equality comes from the assumption $w_1 = w_2$. Lemma 3.7 tells us that the maximum-bid is bounded for all $k \in \{2, 3, \ldots, n-1\}$ and hence the maximum bid for the all-pay auction is bounded.

Bids for a winner-pays-bid setting : Let $b^{y-win-pay}(.)$ be the bid function in a winner-pays-bid setting. By revenue equivalence,

$$b^{y-win-pay}(v) Pr[v > v_j \forall j \neq i] w_1 \leq \sum_{k=1}^{n} w'_k b^{k-all-pay}(v)$$

This is an inequality because we have not included the interim payment in the winner-pays-bid auction in the case it is assigned lower positions. Thus in the limiting case,

$$\lim_{v \to \infty} b^{y-win-pay}(v) \leq \lim_{v \to \infty} \sum_{k=1}^{n} \frac{w'_k}{w_1} b^{k-all-pay}(v)$$
$$= \lim_{v \to \infty} \sum_{k=2}^{n} \frac{w'_k}{w_1} b^{k-all-pay}(v)$$

where the last equality comes from the assumption $w_1 = w_2$ and since Lemma 3.7 tells us that the limit exists for each individual term, thus the limit exists for the winners-pay-bid setting as well.

□

In the unbounded setting, while the result above gives us an upper bound on the equilibrium bids for a class of position auctions, we now show a lower bound on the equilibrium bid for any position auction.

Lemma 3.9. *Under the normalization*

$$\max_q R(q) = 1$$

for regular agents, the maximum bid in an all-pay auction y, in a position environment with weights $(w_1, w_2, \dots, w_n)$ is lower bounded by

$$b_{max}^{y-all-pay} \geq \frac{w_1}{4n}$$

where n is the number of i.i.d agents.

Proof.

$$\begin{aligned}
b_{max}^{y-all-pay} &= \sum_{k=1}^{n} w'_k b_{max}^{k-all-pay} \\
&= w'_1(n-1)P_{(1:n-1)} + \sum_{k=2}^{n-2} w'_k \frac{n-1}{k-1} P_{(k-1:n-1)} + w'_{n-1}\frac{n-1}{n-2}P_{(n-2:n-1)} \\
&\geq w'_1 \frac{n-1}{4} R(\frac{2}{n-1}) + \sum_{k=2}^{n-2} w'_k \frac{n-1}{2k} R(\frac{k}{n-1}) + w'_{n-1}\frac{n-1}{4(n-2)}R(1-\frac{1}{n-1}) \\
&\geq \frac{w'_1}{4} + \sum_{k=2}^{n-2} \frac{w'_k}{2k} + \frac{w'_{n-1}}{4(n-2)} \\
&\geq \frac{w_1}{4n}
\end{aligned}$$

where the first equality follows from Lemma 3.8, the second equality from Lemma 3.7, the third equality from the definition of a revenue curve, the first inequality from the worst case analysis of the $(n-1)$-unit auction and the second inequality from the fact that for the normalization $\max_q R(q) = 1$, $R(\frac{k}{n}) \geq \frac{\min\{k,n-k\}}{n}$ for all $k \in \{1, 2, \dots, n-1\}$.

□

We have seen some simple results which bound the equilibrium bid function and the per-agent revenue in a multi-unit auction. We now work towards obtaining an upper bound on the numerator for the normalization where the monopoly revenue is one. The quantile convention we use for the upper bound below is that a lower quantile corresponds to a lower strength in the population i.e $v(q) = F^{-1}(q)$. We employ the following lemma to derive estimation error bounds for our setting

Lemma 3.10. *(Chawla* et al. *(2017)) The per-agent counterfactual revenue of a rank-based auction with allocation rule y can be expressed in terms of the bid function b of an all-pay mechanism x as :*

$$P_y = \mathbf{E}_{q \notin \Lambda}[-Z'_y(q)b(q)] + Z_y(1-\delta_N)b(1-\delta_N) - Z_y(\delta_N)b(\delta_N) + \mathbf{E}_{q \in \Lambda}[Z_y(q)b'(q)]$$

where $Z_y(q) = (1-q)y'(q)/x'(q)$, extreme quantiles are $\Lambda = [0, \delta_N] \cup [1-\delta_N, 1]$, and the truncation parameter is $\delta_N \in [0, \frac{1}{2}]$.

Utilizing this, it is straightforward to show that the estimation error when the counterfactual auction is the k-unit auction is

$$\begin{aligned} |\hat{P}_y - P_y| \le\ & |\mathbf{E}_{q \notin \Lambda}[-Z'_y(q)(\hat{b}(q) - b(q))]| + |\mathbf{E}_{q \in \Lambda}[Z_y(q)b'(q)]| \\ & + |Z_y(1-\delta_N)(b(1-\delta_N) - \hat{b}_N)| + |Z_y(\delta_N)b(\delta_N)| \end{aligned} \tag{4.1}$$

where $\hat{b}(.)$ is the estimated bid function. Bounding each of these terms will give an upper bound on the estimation error.

4.1.1 Bounding the error terms from moderate quantiles

Lemma 3.11. *For Z_k and Λ defined, under the normalization $\max_q R(q) = 1$, the error from the moderate quantiles in the estimator $\hat{P}_k$ is bounded as*

$$\mathbf{E}_{\hat{\boldsymbol{b}}}[|\mathbf{E}_{q\notin\Lambda}[Z_k'(q)(\hat{b}(q)-b(q))]|] \le \frac{4n^2\log N}{\sqrt{N}}(1+\frac{16n\log\log N}{\sqrt{N}}) \tag{4.2}$$

under the assumption that the top two positions in the incumbent auction have the same weight.

Proof. Let us look at the error contribution from the moderate quantiles

$$|\mathbf{E}_{q\notin\Lambda}[Z_k'(q)(\hat{b}(q)-b(q))]| \le \mathbf{E}_{q\notin\Lambda}[|\frac{Z_k'(q)}{Z_k(q)}|]\sup_q|Z_k(q)(\hat{b}(q)-b(q))| \tag{4.3}$$

Note that $Z_k(q) = (1-q)\frac{x_k'(q)}{x'(q)}$. Chawla *et al.* (2017)(Lemma 3.6) show that $Z_k(q)$ is single peaked and hence the first part of the above equation is bounded by $4n\log N$. Now let us look at the second term in the equation,

$$\sup_q|Z_k(q)(\hat{b}(q)-b(q)) = \sup_q|(1-q)\frac{x_k'(q)}{x'(q)}(\hat{b}(q)-b(q))| \tag{4.4}$$

Note that for an all-pay auction, the symmetric bid function satisfies $v(q) = \frac{b'(q)}{x'(q)}$. Thus,

$$\begin{aligned}
\sup_q|Z_k(q)(\hat{b}(q)-b(q))| &= \sup_q|(1-q)\frac{x_k'(q)v(q)}{b'(q)}(\hat{b}(q)-b(q))| \\
&\le \sup_q|(1-q)v(q)x_k'(q)|\sup_q|\frac{1}{b'(q)}(\hat{b}(q)-b(q))| \\
&\le \sup_q|(1-q)v(q)|\sup_q x_k'(q)\sup_q|\frac{1}{b'(q)}(\hat{b}(q)-b(q))| \\
&= \sup_q x_k'(q)\sup_q|\frac{1}{b'(q)}(\hat{b}(q)-b(q))|
\end{aligned} \tag{4.5}$$

where the last inequality comes from the normalization we are using.

Utilizing Lemma 3.1

$$
\begin{aligned}
\mathbf{E}_{\hat{b}}[\sup_q |\frac{1}{b'(q)}(\hat{b}(q) - b(q))|] &\leq \frac{1}{\sqrt{N}}(1 + \frac{16 \log \log N}{\sqrt{N}} \sup_q q(1-q)b'(q)) \\
&\leq \frac{1}{\sqrt{N}}(1 + \frac{16 \log \log N}{\sqrt{N}} \sup_q q(1-q)v(q)x'(q)) \\
&\leq \frac{1}{\sqrt{N}}(1 + \frac{16 n \log \log N}{\sqrt{N}})
\end{aligned} \tag{4.6}
$$

where the first inequality follows from the error of the estimated bid function, the second inequality from the equilibrium bid function for an all-pay auction, the third inequality from the fact that $\sup_q (v(q)(1-q)) = 1$ and $\sup_q x'(q) < n$. Note that we have $\sup_{q \in (0,1)} q(1-q)b'(q) \leq \sup_{q \in (0,1)} x'(q) < \infty$ for all-pay auctions. Thus the expected error bound from the moderate quantiles is given by

$$
\begin{aligned}
\mathbf{E}_{\hat{\boldsymbol{b}}}[|\mathbf{E}_{q \notin \Lambda}[Z_k'(q)(\hat{b}(q) - b(q))]|] &\leq \frac{4n \log N}{\sqrt{N}} \sup_q \{x_k'(q)\}(1 + \frac{16n \log \log N}{\sqrt{N}}) \\
&\leq \frac{4n^2 \log N}{\sqrt{N}}(1 + \frac{16n \log \log N}{\sqrt{N}})
\end{aligned} \tag{4.7}
$$

□

4.1.2 Bounding error from the extremal quantiles

We now bound the error contribution from the extremal quantiles. For the same, we derive basic results relating the allocation rule of a position auction to the corresponding symmetric equilibrium bid function.

Theorem 4. *(Chawla* et al. *(2017)) For any n-agent rank based mechanism with allocation rule y and $\delta < 1/n$, the allocation rule y' satisfies*

1. $\sup_{q<\delta} y'(q) \leq ey'(\delta)$
2. $\sup_{q>1-\delta} y'(q) \leq ey'(1-\delta)$

Lemma 4.1. *For Z_y and Λ, if $\delta_N \leq 1/n$, the second error term in the estimator $\hat{P}_y$ is bounded as*

$$\mathbf{E}_{q\in\Lambda}[Z_y(q)b'(q)] \leq e\delta_N y'(\delta_N) + e\delta_N y'(1-\delta_N)$$

Proof. Using the definition, $Z'_y(q) = (1-q)y'(q)/x'(q)$ and $v(q) = b'(q)/x'(q)$ for an all-pay auction, we get

$$\mathbf{E}_{q\in\Lambda}[Z_y(q)b'(q)] = \mathbf{E}_{q\in\Lambda}[(1-q)v(q)y'(q)] \leq \mathbf{E}_{q\in\Lambda}[y'(q)] \leq e\delta_N y'(\delta_N) + e\delta_N y'(1-\delta_N)$$

where the first inequality comes from the fact that $\max_q R(q) = 1$ and the second inequality follows from Theorem 4. □

Lemma 4.2. *For Z_y and Λ, if $\delta_N \leq 1/n$, the fourth error term of the estimator $\hat{P}_y$ is bounded as*

$$Z_y(\delta_N)b(\delta_N) \leq e\delta_N y'(\delta_N)$$

Proof. Consider the integral, which just follows from integration by parts

$$\int_{q=0}^{\delta_N}(1-q)b'(q)dq = (1-\delta_N)b(\delta_N) + \int_{q=0}^{\delta_N} b(q)dq$$

Thus we get,

$$(1-\delta_N)b(\delta_N) \leq \int_{q=0}^{\delta_N}(1-q)v(q)x'(q)dq \leq \int_{q=0}^{\delta_N} x'(q)dq \leq \delta_N \sup_{q<\delta_N} x'(q) \leq e\delta_N x'(\delta_N)$$

where the first inequality follows from the fact that x is an all-pay auction, the second inequality follows from our normalization $\max_q R(q) = 1$, and the last inequality follows from Theorem 4. Hence we get

$$Z_y(\delta_N)b(\delta_N) \leq e\delta_N y'(\delta_N)$$

where this follows from the definition of $Z_y(\delta_N)$ □

We now bound the third error term.

Lemma 4.3. *Under the normalization*

$$\max_q R(q) = 1$$

, for $\hat{q} < 1 - \delta_N$, the equilibrium bid function satisfies

$$b(1-\delta_N) - b(\hat{q}) \le \frac{1}{\delta_N}(x(1-\delta_N) - x(\hat{q}))$$

Proof. By assumption $\max_q R(q) \le 1$ and by definition $R(q) = (1-q)v(q)$ and $v(q) = \frac{b'(q)}{x'(q)}$, thus $b'(q) \le \frac{x'(q)}{1-q}$. We can conclude that, for any $\hat{q} < 1 - \delta_N$,

$$\begin{aligned} b(1-\delta_N) - b(\hat{q}) &\le \int_{q=\hat{q}}^{1-\delta_N} \frac{x'(q)}{1-q} dq \\ &\le \frac{1}{\delta_N}(x(1-\delta_N) - x(\hat{q})) \end{aligned}$$

□

We now bound the third term in the error by analyzing a certain set of functions corresponding to allocation rules for rank-based auctions. The allocation rule and its derivative for the n-agent k-unit auction are

$$x_k(q) = \sum_{i=0}^{k-1} \binom{n-1}{i} q^{n-i-1}(1-q)^i$$

$$x_k'(q) = (n-1)\binom{n-2}{k-1} q^{n-k-1}(1-q)^{k-1}$$

For $k \in [n-1]$, define the function $f_k(.)$ as

$$f_k(q) = \frac{x_k'(q)}{1-q}$$

Note that for $k = 1$, $f_1(q) = (n-1)\frac{q^{n-2}}{1-q}$ and

$$f_k(q) = (n-1)\binom{n-2}{k-1} q^{n-k-1}(1-q)^{k-2} \quad \forall k \in \{2, 3, \ldots, n-1\}$$

We are interested in the behavior of $f_k(.)$ in the range $[1-\frac{1}{n}, 1]$. For all $k \in \{3, \ldots, n-2\}$, $f_k(.)$ has a unique maxima at $q^* = \frac{n-k-1}{n-3}$. It is a monotonically increasing function for $q < q^*$ and monotonically decreases for $q > q^*$.

Lemma 4.4. *For $k \in \{3, 4, \ldots, n-2\}$ unit and $\delta < 1/n$, the function $f_k(.)$ satisfies*

- $\sup_{q>1-\delta} f_k(q) = f_k(1-\delta)$
- $\sup_{q<1-\delta} f_k(q) = f_k(\delta)$

Proof. For all $k \in \{3, 4, \ldots, n-2\}$, $f_k(q)$ attains a unique maxima at $q^* = (n-k-1)/(n-3)$ where $\frac{1}{n} < q^* < 1 - \frac{1}{n}$. In the quantile range $q \in [q^*, 1]$, we have $f'_k(q) \leq 0$ while in the quantile range $q \in [0, q^*]$, we have $f'_k(q) \geq 0$. Since $\delta < 1/n$, we have $1 - \delta > q^*$ and hence $f(1-\delta) = \sup_{q>1-\delta} f_k(q)$. Also $\delta < q^*$ and hence $f(\delta) = \sup_{q<1-\delta} f_k(q)$.

□

Lemma 4.5. *For $k \in \{2, 3, \ldots, n-1\}$ units and $\delta < 1/n$, the function $f_k(q)$ satisfies*

$$\sup_{q>1-\delta} f_k(q) \leq e f_k(1-\delta)$$

Proof. For $k \in \{3, 4, \ldots, n-2\}$ units, the claim follows from Lemma 4.4. For $k = 2$ units, $f_2(q) = (n-1)(n-2)q^{n-3}$

$$\begin{aligned} f_2(1-\delta) &= (n-1)(n-2)(1-\delta)^{n-3} \\ &\geq (n-1)(n-2)(1-\frac{1}{n})^{n-3} \\ &\geq (n-1)(n-2)(1-\frac{1}{n})^{n-1} \\ &\geq \frac{1}{e}(n-1)(n-2) = \frac{f_2(1)}{e} \end{aligned}$$

$f_2(q)$ is a monotonically increasing function and hence the claim follows for $k = 2$ units. For $k = n-1$ units, $f_{n-1}(q) = (n-1)(1-q)^{n-3}$ is a monotonically decreasing function and hence $\sup_{q>1-\delta} f_{n-1}(q) = f_{n-1}(1-\delta)$. The claim follows.

□

Lemma 4.6. *Any n-agent rank-based mechanism, where the top two ranked agents are allocated with the same probability, having an allocation rule x and $\delta < 1/n$, the allocation rule derivative x' satisfies*

$$\sup_{q>1-\delta} \frac{x'(q)}{1-q} \leq e\frac{x'(1-\delta)}{\delta}$$

Proof. Let $(w_1, w_2, \dots, w_{n-1})$, where $w_1 \geq w_2 \geq \cdots \geq w_{n-1}$, denote the allocation probabilities for the rank based mechanism where the i^{th} ranked agent is allocated with probability w_i. Note that for a n-agent rank based mechanism where the top two ranked agents are allocated with the same probability ($w_1 = w_2$), we have

$$\frac{x'(q)}{1-q} = \sum_{k=2}^{n-1} w'_k f_k(q)$$

where $\sum_{k=2}^{n-1} w'_k = w_1 \leq 1$. Using Lemma 4.5, we have

$$\begin{aligned}
\sup_{q>1-\delta} \frac{x'(q)}{1-q} &\leq \sup_{q>1-\delta} \sum_{k=2}^{n-1} w'_k f_k(q) \\
&\leq e \sum_{k=2}^{n-1} w'_k f_k(1-\delta) \\
&\leq e \sum_{k=2}^{n-1} w'_k \frac{x'_k(1-\delta)}{\delta} \\
&= \frac{e}{\delta} x'(1-\delta)
\end{aligned}$$

□

Lemma 4.7. *Under the assumption that the incumbent auction is such that its top two position* *weights are the same, the third term of the error is bounded as*

$$|Z_y(1-\delta_N)(b(1-\delta_N)-\hat{b}_N)| \leq e\delta_N y'(1-\delta_N)$$

when the quantile $\hat{q}_N$ corresponding to the maximum sampled bid $\hat{b}_N$ lies in the range $[1-\delta_N, 1]$, where $\delta_N < 1/n$.

Proof. Consider the case when the quantile corresponding to $\hat{b}_N$, $\hat{q}$, lies in this extremal range i.e $\hat{q} \geq 1-\delta_N$.

$$\begin{aligned} Z_y(1-\delta_N)(\hat{b}_N - b(1-\delta_N))) &\leq Z_y(1-\delta_N)(b(1)-b(1-\delta_N)) \\ &= \delta_N \frac{y'(1-\delta_N)}{x'(1-\delta_N)}(b(1)-b(1-\delta_N)) \\ &= \delta_N \frac{y'(1-\delta_N)}{x'(1-\delta_N)} \int_{q=1-\delta_N}^{1} v(q)x'(q)dq \\ &= \delta_N \frac{y'(1-\delta_N)}{x'(1-\delta_N)} \int_{q=1-\delta_N}^{1} v(q)(1-q)\frac{x'(q)}{1-q}dq \\ &\leq \delta_N^2 \frac{y'(1-\delta_N)}{x'(1-\delta_N)} \frac{e}{\delta_N} x'(1-\delta_N) \\ &= e\delta_N y'(1-\delta_N) \end{aligned} \tag{4.8}$$

where the first inequality comes from the fact that the $b(1)$ is an upper bound on $\hat{b}_N$, the first equality from the definition of $Z_y(.)$, the second equality from the equilibrium bid definition of an all-pay incumbent auction and the second inequality follows from the assumption that $\max_q R(q) = 1$ and from Lemma 4.6.

□

Now we just need to look at the probability that out of N samples, at least one sample has a quantile in the extremal range. Let us denote this by event E.

$$Pr(\bar{E}) = (1-\delta_N)^N \leq e^{-\max\{25 \log \log N, n\}}$$

where the inequality comes from the fact that $\delta_N = \max\{25 \log\log N, n\}/N$.Thus,

$$Pr(E) \geq 1 - e^{-\max\{25 \log\log N, n\}}$$

Lemma 4.8. *Under the assumption that the incumbent all-pay auction is such that its top two position weights are the same, the third error term in the estimator is bounded as*

$$\mathbb{E}_{\hat{b}}[|Z_y(1-\delta_N)(b(1-\delta_N) - \hat{b}_N)|] \leq \frac{8}{N} y'(1-\delta_N) + e\delta_N y'(1-\delta_N)$$

Proof. Lemma 4.7 shows that when the incumbent all-pay auction is such that its top two position weights are the same and when the quantile corresponding to the maximum bid $\hat{q}$, i.e., with $b(\hat{q}) = \hat{b}_N$, satisfies $\hat{q} > 1 - \delta_N$ the third error term is bounded as

$$\mathbf{E}_{\hat{b}}[Z_y(1-\delta_N)(\hat{b}_N - b(1-\delta_N))|\hat{q} > 1 - \delta_N] \leq e\delta_N y'(1-\delta_N)$$

Now considering the case when, $\hat{q} < 1 - \delta_N$, we observe that

$$\begin{aligned}
\mathbf{E}_{\hat{q}}[b(1-\delta_N) - b(\hat{q})|\hat{q} < 1 - \delta_N] &= N \int_{q=0}^{1-\delta_N} (b(1-\delta_N) - b(q))q^{N-1} dq \\
&\leq \frac{N}{\delta_N} \int_{q=0}^{1-\delta_N} (x(1-\delta_N) - x(q))q^{N-1} dq \\
&\leq \frac{N}{\delta_N} \int_{q=0}^{1-\delta_N} (1 - x(q))q^{N-1} dq
\end{aligned}$$

where the first inequality comes from Lemma 4.3. Lemma A.8 in Chawla *et al.* (2017) tells us that for a k-unit auction $N \int_{q=0}^{1-\delta_N} (1 - x(q))q^{N-1} dq \leq 2(\frac{n}{N})^k$ for $N > 1.5n$. If $x = x_k$ for all $k \in \{1, 2, \ldots, n-1\}$, we have

$$
\begin{aligned}
\mathbf{E}_{\hat{\boldsymbol{b}}}[(|Z_y(1-\delta_N)(b(1-\delta_N)-\hat{b}_N)|)|\hat{q}<1-\delta_N] &\le \delta_N \frac{y'(1-\delta_N)}{x_k'(1-\delta_N)}\frac{2}{\delta_N}(\frac{n}{N})^k \\
&\le 2y'(1-\delta_N)\frac{1}{(n-1)\binom{n-2}{k-1}(1-\delta_N)^{n-1-k}\delta_N^{k-1}}(\frac{n}{N})^k \\
&\le \frac{2}{N}(\frac{n}{n-1})(\frac{n}{N\delta_N})^{k-1}y'(1-\delta_N)\frac{1}{\binom{n-2}{k-1}(1-\delta_N)^{n-1-k}} \\
&\le \frac{8}{N}y'(1-\delta_N)
\end{aligned}
$$

where the last inequality follows from $\binom{n-2}{k-1} \ge 1$, $(1-\delta_N)^n > \frac{1}{4}$, and $\frac{n}{N\delta_N} \le 1$. Thus for any general incumbent auction x,

$$
\begin{aligned}
\mathbf{E}_{\hat{\boldsymbol{b}}}[(|Z_y(1-\delta_N)(b(1-\delta_N)-\hat{b}_N)|)|\hat{q}<1-\delta_N] &\le 2\frac{y'(1-\delta_N)}{x'(1-\delta_N)}(\frac{n}{N})^k \\
&\le \max_k 2\frac{y'(1-\delta_N)}{x_k'(1-\delta_N)}(\frac{n}{N})^k \\
&\le \frac{8}{N}y'(1-\delta_N)
\end{aligned}
$$

Hence the third term in the error is bounded as

$$
\mathbb{E}_{\hat{b}}[|Z_y(1-\delta_N)(b(1-\delta_N)-\hat{b}_N)|] \le \frac{8}{N}y'(1-\delta_N) + e\delta_N y'(1-\delta_N)
$$

□

Now that we have obtained a bound on for each of the terms in the inference error, we are now ready to derive our multiplicative bound.

Lemma 4.9. *Under the assumption that the agents are regular with a monopoly revenue of one, the mean relative error in estimating the revenue of a k-unit auction using N samples from the bid distribution of an all-pay rank-based auction whose top two position weights are the same is*

$$\frac{\mathbf{E}_{\hat{b}}[|\hat{P}_k - P_k|]}{P_k} \le O(\frac{n^3 \log N}{\sqrt{N}})$$

where n is the number of positions and δ_N is set to $\max\{25 \log\log N, n\}/N$.

Proof. Using Lemma 3.11, Lemma 4.1, Lemma 4.2 and Lemma 4.8 we bound the error as

$$\begin{aligned}\mathbf{E}_{\hat{b}}[|\hat{P}_k - P_k|] \le \frac{4n^2 \log N}{\sqrt{N}}(1 + \frac{16n \log\log N}{\sqrt{N}}) + 2e\delta_N x'_k(\delta_N) + (e+1)\delta_N x'_k(1-\delta_N) \\ + x'_k(1-\delta_N)(\frac{8}{N} + e\delta_N)\end{aligned}$$

Lemma 3.5 tells us that

$$P_k \ge \frac{1}{4n} \quad \forall k \in \{1, \dots, n-1\}$$

The relative bound can be written as

$$\begin{aligned}\frac{\mathbf{E}_{\hat{b}}[|\hat{P}_k - P_k|]}{P_k} \le \frac{16n^3 \log N}{\sqrt{N}}(1 + \frac{16n \log\log N}{\sqrt{N}}) + 8e\delta_N n x'_k(\delta_N) + 4(e+1)\delta_N n x'_k(1-\delta_N) \\ + 4n x'_k(1-\delta_N)(\frac{8}{N} + e\delta_N)\end{aligned}$$

Assuming $64n^3 \log N < \sqrt{N}$, the first term simplifies to $\frac{32n^3 \log N}{\sqrt{N}}$. Furthermore, the error terms are no more than the error from the moderate quantiles. Hence the relative bound simplifies to

$$\frac{\mathbf{E}_{\hat{b}}[|\hat{P}_k - P_k|]}{P_k} \le O(\frac{n^3 \log N}{\sqrt{N}})$$

□

The result above obtains a multiplicative bound for estimating the revenue of a multi-unit auction. For a general position auction, we utilize the fact that it can be represented as a convex combination of multi unit auctions and obtain the multilpicative result of Lemma 3.3.

Proof. (Proof of Lemma 3.3)

$$\begin{aligned} \mathbf{E}_{\hat{b}}[|\hat{P}_y - P_y|] &\leq \sum_{k=1}^{n} w'_k \mathbf{E}_{\hat{b}}[|\hat{P}_k - P_k|] \\ &\leq O(\frac{n^3 \log N}{\sqrt{N}}) \sum_{k=1}^{n} w'_k P_k \end{aligned} \tag{4.9}$$

where the second inequality follows from Lemma 4.9. Hence the multiplicative bound we obtain is

$$\frac{\mathbf{E}_{\hat{b}}[|\hat{P}_y - P_y|]}{P_y} \leq O(\frac{n^3 \log N}{\sqrt{N}})$$

□

The bound obtained seems to be crude in the sense that it does not utilize the similarity between the incumbent and counterfactual auctions. If we observe the extreme case of the incumbent and counterfactual auctions being the same, we should expect the inference error to only have a $\sqrt{N}$ dependence corresponding to the statistical error of the bids (for a fixed n). We can obtain a better result when we account for the similarity of the auctions.

Lemma 4.10. *Let x and x_k denote the allocation rules for any all-pay rank based auction and the k-highest-bids wins auction over n positions respectively. Let $\hat{P}_k$ denote the estimated revenue for estimating revenue P_k of the latter auction by using N equilibrium bid samples of the former, with δ_N set to $\max\{25 \log\log N, n\}/N$. If $\delta_N < 1/n$ and the incumbent auction is such that its top two position weights are the same, the absolute relative bound when the agents are regular and have a monopoly revenue of one, is given as*

$$\frac{\mathbf{E}[|\hat{P}_k - P_k|]}{P_k} \leq O(\frac{n}{\sqrt{N}} \Phi_{x,x_k})$$

where $\Phi_{x,y} := \sup_q\{y'(q)\} \max\{1, \log \sup_{q:y'(q)\geq 1} \frac{x'(q)}{y'(q)}, \log \sup_q \frac{y'(q)}{x'(q)}\}$

Proof. We can bound the additive absolute error for any $\alpha > 0$ as

$$\mathbf{E}[|\hat{P}_k - P_k|] \leq \mathbf{E}[\frac{(\log(1+Z_k(q)))^\alpha}{Z_k(q)}|Z_k'(q)|] \sup_q |\frac{Z_k(q)}{(\log(1+Z_k(q)))^\alpha}(\hat{b}(q) - b(q))|$$

Lemma A.1 in Chawla *et al.* (2017) bounds the first term in the split as

$$\mathbf{E}[\frac{(\log(1+Z_k(q)))^\alpha}{Z_k(q)}|Z_k'(q)|] \leq \frac{2}{\alpha} + \frac{2}{1+\alpha}(\log Z_k^* + 1)^{1+\alpha}$$

where $Z_k^* = \sup_q Z_k(q)$. The first term is at most $2(1+e)/\alpha$ for $\alpha < 1/\log Z_k^*$

We can bound the second term as

$$\begin{aligned}
&\mathbf{E}_{\hat{b}}[\sup_q |\frac{Z_k(q)}{(\log(1+Z_k(q)))^\alpha}(\hat{b}(q) - b(q))|] \\
&\leq \sup_q |\frac{Z_k(q)}{(\log(1+Z_k(q)))^\alpha} b'(q)| \mathbf{E}[\sup_q |\frac{\hat{b}(q) - b(q)}{b'(q)}|] \\
&\leq 2^\alpha \sup_q (x_k'(q)) \max\{1, \sup_{q: x_k'(q) \geq 1} \frac{x'(q)}{x_k'(q)})\}^\alpha \frac{1}{\sqrt{N}}(1 + 16\frac{\log\log N}{\sqrt{N}} \sup_q q(1-q)b'(q)) \\
&\leq 2^\alpha \sup_q (x_k'(q)) \max\{1, \sup_{q: x_k'(q) \geq 1} \frac{x'(q)}{x_k'(q)})\}^\alpha \frac{1}{\sqrt{N}}(1 + 16\frac{\log\log N}{\sqrt{N}} \sup_q q(1-q)v(q)x'(q)) \\
&\leq 2^\alpha \sup_q (x_k'(q)) \max\{1, \sup_{q: x_k'(q) \geq 1} \frac{x'(q)}{x_k'(q)})\}^\alpha \frac{1}{\sqrt{N}}(1 + 16n\frac{\log\log N}{\sqrt{N}})
\end{aligned}$$

where the last inequality comes from the fact that $\sup_q v(q)(1-q) = 1$, $\sup_q x'(q) \leq n$ and from Lemma 3.9 which gives a lower bound on the bid in a general all-pay auction. We define A as $A := \max\{1, \sup_{q: x_k'(q) \geq 1} \frac{x'(q)}{x_k'(q)})\}$. Choosing $\alpha = \min\{1, 1/\log A, 1/\log Z_k^*\}$, we obtain

$$\begin{aligned}
\mathbf{E}[|\hat{P}_k - P_k|] &\leq \frac{2(1+\epsilon)}{\alpha} 2^\alpha A^\alpha \frac{1}{\sqrt{N}} \sup_q (x_k'(q))(1 + 16n\frac{\log\log N}{\sqrt{N}}) \\
&\leq \frac{40}{\sqrt{N}} \sup_q (x_k'(q)) \max\{1, \log A, \log \sup_q \{\frac{x_k'(q)}{x'(q)}\}\}(1 + 16n\frac{\log\log N}{\sqrt{N}})
\end{aligned}$$

Using the lower bound on the per-agent revenue derived and performing a similar analysis for the relative bound in Lemma 4.8, we obtain the relative bound

$$\frac{\mathbf{E}[|\hat{P}_k - P_k|]}{P_k} \leq O(\frac{n}{\sqrt{N}}\Phi_{x,x_k})$$

□

4.2 Sample Complexity

In this section we look at the sample complexity for a multiplicative revenue objective with unbounded value distributions. As far as the revenue is concerned, let ψ^k be the k^{th} expected order statistic of the virtual value function. This is given by $\psi^k = n(P_k - P_{k-1})$, where P_k is the per-agent revenue of a k-unit auction in an environment of n-agents. It is shown that errors in estimating the surrogate values flow in a well behaved fashion in the surrogate-ranking-mechanism for an additive error.

Theorem 5. *(Hartline and Taggart (2019)) For all i and j, let ψ_i^j be the expected j^{th} order statistic of agent i's virtual value distribution, and let $\hat{\psi}_i^j$ be an estimate of ψ_i^j satisfying $|\hat{\psi}_i^j - \psi_i^j| < \epsilon_i$, where ϵ_i is an agent specific error bound. The difference between the expected virtual surplus of the surrogate ranking mechanisms with the true expected order statistics is at most $2\sum_i \epsilon_i$.*

To obtain an additive error of ϵ for each ψ^k, it suffices to estimate each P_k to an additive error of $\frac{\epsilon}{n}$. We can show that if the optimal surrogate values were estimated to a multiplicative error of ϵ, the multiplicative error flows in an efficient manner as well with some constraints on the way the surrogate ranking mechanism allocates to maximize surplus.

Claim 5.1. *For all i and j, let ψ_i^j be the expected j^{th} order statistic of agent i's virtual value distribution, let $\hat{\psi}_i^j$ be an estimate of $\psi_i^j > 0$ satisfying $\frac{|\hat{\psi}_i^j - \psi_i^j|}{\psi_i^j} \leq \epsilon_i$, where ϵ_i is an agent-specific relative error bound. The relative error between the expected virtual surplus of the surrogate ranking mechanism which only allocates if the surrogate value is positive with the true expected order statistics is at most $2\max_i \epsilon_i$*

Proof. Let $\mathbf{x}$ and $\hat{\mathbf{x}}$ denote the allocation rule of the surrogate-ranking mechanism as a function of the agent's rank $\mathbf{r}$ among their run-time samples with optimal surrogate values ψ and estimated and ironed surrogate values $\hat{\psi}$, respectively.

Based on the multiplicative error in estimating the expected order statistic of the virtual value for agent i, we get the inequality

$$(1-\epsilon_i)\psi_i^j \leq \hat{\psi}_i^j \leq (1+\epsilon_i)\psi_i^j$$

$$\begin{aligned}
\mathbf{E_r}[\sum_i \mathbf{E}_{q_i}[\psi_i(q_i)|r_i]\hat{x}_i(\mathbf{r})] &= \mathbf{E_r}[\sum_i \psi_i^{r_i}\hat{x}_i(\mathbf{r})] \\
&\geq \mathbf{E_r}[\sum_i \frac{\hat{\psi}_i^{r_i}}{1+\epsilon_i}\hat{x}_i(\mathbf{r})] \\
&\geq \frac{1}{1+\max_i \epsilon_i}\mathbf{E_r}[\sum_i \hat{\psi}_i^{r_i}x_i(\mathbf{r})] \\
&\geq \frac{1}{1+\max_i \epsilon_i}\mathbf{E_r}[\sum_i (1-\epsilon_i)\psi_i^{r_i}x_i(\mathbf{r})]
\end{aligned}$$

where the first inequality comes from the multiplicative error bound, the second inequality comes from the fact that $\hat{\mathbf{x}}(\mathbf{r})$ is the allocation rule that maximizes $\mathbf{E_r}[\sum_i \hat{\psi}_i^{r_i}\hat{x}_i(\mathbf{r})]$ and the last inequality comes from the multiplicative error on the expected order statistic of the virtual value.

Hence the error between the expected virtual between the expected virtual surplus of the surrogate ranking mechanisms with the true expected order statistics is

$$\begin{aligned}
\mathbf{E_r}[\sum_i \psi_i^{r_i}x_i(\mathbf{r})] - \mathbf{E_r}[\sum_i \psi_i^{r_i}\hat{x}_i(\mathbf{r})] &\leq \mathbf{E_r}[\sum_i \frac{2\max_i \epsilon_i}{1+\max_i \epsilon_i}\psi_i^{r_i}x_i(\mathbf{r})] \\
&\leq \mathbf{E_r}[\sum_i \psi_i^{r_i}x_i(\mathbf{r})]2\max_i \epsilon_i
\end{aligned}$$

the first inequality comes from the fact that $\epsilon_i > 0$ for all i and hence the claim on the relative error follows.

□

It seems unlikely that a multiplicative bound for the optimal surrogate value would exist as for any i and j, ψ_i^j can take the value zero while the estimate $\hat{\psi}_i^j$ is nothing but the difference of the estimate of the per-agent revenues of consecutive multi-unit auctions which need not be zero. The assumption that the surrogate ranking mechanism allocates only when the surrogate value is positive also induces a constraint on the allocation set, something like downward-closure. Thus the above proved result is not of much use to us. However, note that the allocation of the surrogate ranking mechanism is such that it maximizes surrogate surplus and hence in the case the surrogate value were zero, the mechanism would indifferent to allocating or not allocating that particular agent. Thus if we only estimate surrogate values which are strictly positive, we might be able to use this multiplicative estimation error.

For our current result, we only estimate the optimal surrogate values to an additive error of ϵ. The additive bound derived as a by product of the multiplicative bound leads to the following corollary.

Corollary 5.1. *Consider a T-agent all-pay or winner-pays-bid i.i.d position auction such that its agents are regular with possibly unbounded valuations, with a monopoly revenue of one. There exists an estimator $\hat{\psi}_k$ for the expected k^{th} order statistic of the virtual value distribution ψ_k such that with $N \geq \tilde{O}(T^4\epsilon^{-2})$ sampled bids from the unique BNE, $|\hat{\psi}_k - \psi_k| \leq \epsilon$.*

We can now arrive at the number of samples required to obtain a multiplicative approximation to the revenue of the optimal mechanism.

Lemma 5.1. *For agents with regularly distributed values having a monopoly revenue of one leading to potentially unbounded values, there are families of winner-pays-bid and all-pay mechanisms that satisfy conditions C1, C2 and C3 with $p_{run}(n, \epsilon^{-1}) = O(n\epsilon^{-3})$ and $p_{design}(n, \epsilon^{-1}) = \tilde{O}(n^6\epsilon^{-14})$ for multiplicative loss and the revenue objective.*

Proof. $T = n\epsilon^{-3}$ surrogate values suffice per agent to obtain a multiplicative revenue approximation of ϵ. Now note that for each agent the monopoly revenue is one. Estimating each surrogate value to an error of $\frac{\epsilon}{n}$ gives an $O(\epsilon)$ multiplicative approximation to the optimal revenue. Thus the number of design time samples that suffice to obtain a multiplicative loss of ϵ is $O((n\epsilon^{-3})^4(\epsilon/n)^{-2}) = O(n^6\epsilon^{-14})$. □

CHAPTER 5

CONCLUSION

We study the sample complexity for nun-truthful mechanisms for a multiplicative revenue objective when the value distributions are regular and potentially unbounded. We show a polynomial sample complexity in this setting but the bounds obtained are impractical.

We also perform a thorough empirical analysis of the revenue estimator and verify its dependence on various parameters such as the number of agents, number of bid samples, value distribution, etc.

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