STABILITY OF TCP-LIKE CONGESTION CONTROL IN WIRELESS NETWORKS

A Project Report

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THESIS CERTIFICATE

This is to certify that the thesis titled **Stability of TCP-like Congestion control in Wireless Networks**, submitted by **Mohana Kottu**, to the Indian Institute of Technology, Madras, for the award of the degree of **Master of Technology**, is a bonafide record of the research work done by her under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

KEYWORDS: Congestion control ; Hopf bifurcation ; TCP fluid flow model ; Stability ; Nonlinear delay differential equation ; Wireless Networks

In this current work, we aim at the bifurcation behavior of a TCP fluid flow model in wireless networks is investigated for internet congestion control. We study both wired and wireless networks where traditionally wired network supporting internet TCP. First, a nonlinear dynamic model for wireless networks is derived, where the study of modern control theory on delay differential systems been applied. Later, for this model we derive conditions to ensure local stability, then we perform Hopf bifurcation analysis using Poincaré normal forms and the theory of Center manifold. We show how these bifurcation behaviors may cause heavy oscillation of average queue length and induce network instability.

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NOTATION

TCP	Transmission Control Protocol
RTT	Round trip time
RED	Random Early Detection
$W_i(t)$	TCP window size at time t
$q(t)$	Queue length of the router buffer
M	Number of TCP sessions
$C(t)$	Queue capacity
$R_i(t)$	Round trip time of each flow i
$T_{p,i}$	Propagation delay of each flow i
$p(.)$	Packet marking probability
P_{ul}	Up link channel loss probability
P_{dl}	Down link channel loss probability

CHAPTER 1

Introduction

1.1 Internet Congestion

Main reason for internet congestion, is packets dropping and increasing delays, the upper formation application system performance drop, and can even break the whole system by causing congestion collapse. Network congestion already became a bottleneck that restricted the development and application of networks. If the congestion control scheme is not well designed, the sources will try to push even more packets through the network in response to packet drops, thus worsening the congestion.

Over the last decade, congestion control in the internet is an extremely important and challenging problem, which has been the main subject of intensive studies. The researchers in network states congestion control becomes a key issue. The stability of internet largely dependent on the congestion control and avoidance mechanisms implemented in its end to end transmission control protocol (TCP), developed by Jacobson in 1980s [Jacobson (1988)]. However, this implementing from the network edge control mechanism is extremely limited, it is not sufficient to provide good services with only the TCP congestion control on the internet in all circumstances. Therefore we are trying to stablize the TCP congestion problem in wireless networks for the increase in demand.

1.2 Wireless Networks

For a typical wireless network sources, we use access from a wired network, which adapts the internet TCP protocol to transfer data. We knew TCP while operating over wired networks, delivers good performance as the assumption made by TCP that packet loss means congestion is valid over wired network. As wired link have very low error rate which make very few packets get corrupted and dropped due to error introduce by channel. Thus packet drop over wired network usually occur due to congestion and TCP performs well as it is tuned for this. In case of wireless networks this assumption does not hold. Whenever TCP operating over these wireless network it detects packet losses caused by disconnection or error introduced by wireless channel, which causes unnecessary reduction in congestion window, results in degradation of TCP performance.

1.3 Linear Stability Analysis

In the study of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point of a system of nonlinear differential equations and it's first necessary condition for the bifurcation analysis. These equilibrium point help us to know more about the system's stability and unstability. In this paper, we examine nonlinear system for TCP fluid flow model to ensure local stability condition.

1.4 Hopf Bifurcation Analysis

In two dimensions a Hopf bifurcation occurs at a spiral point where stable switches to unstable (or vice versa) and a periodic solution appears. The fact that a critical point switches from stable to unstable (or vice versa) alone does not guarantee that a periodic solution will arise, though one almost always does but we need to check for the extra conditions have to be satisfied. We will use a second order delay differential equations to check the occurence of Hopf bifurcation, analytically we show type of the Hopf bifurcation either supercritical or subcritical and the periodic solution also.

1.5 Summary and Organisation of the Report

In this paper the bifurcation behavior of a TCP fluid flow model for internet congestion control in wireless networks is investigated. These bifurcation behaviors may cause heavy oscillation of average queue length and induce network instability. Simulation results show that the nonlinear behavior of the system. In section (3), modelling of nonlinear dynamic TCP fluid flow model for wireless systems is dervied. In section (4), for the system's stability existence we perform linearization. In section (5), first we focus on Hopf bifurcation occurence and then we study the direction and the stability of bifurcating solution using the Poincaré normal form and the theory of Center manifold. In section (6), we verify our theoretical analysis for the existence of Hopf bifurcation and bifurcating periodic solutions of system. Finally in section (7) conclusion will be drawn.

CHAPTER 2

Literature Review

Congestion is an inherent property of the current best effort internet, and hence congestion control plays a crucial role on the success of the modern internet. As we already stated about the literature Jacobson proposed his congestion avoidance and control algorithm [Jacobson (1988)]. Recently a significant attention to wireless access networks, especially to the internet has received. This introduces new challenge for network researchers. The main problem for the congestion control in wireless connections is that the packet loss caused by the fading of wireless channels may be mistaken as the packet loss caused by the network congestion. Hence the transmission rate is unnecessarily reduced. If the wireless connections coexist with other wired connections in the network, the well behaved fairness maintained by the original internet transmission control protocol (TCP) may be destroyed.

In the past decade, a lot of efforts been made for the congestion control problem for wired networks and a great progress has achieved. While the same problem for wireless networks has received relatively less attention, especially from the control theoretic viewpoint. Our attention will be focused on the RED (random early detection) congestion control algorithm. The reason is that the source transmission rate of wireless networks using RED will not be affected so severely due explicit feedback information for the packet marking probability [Zheng and Nelson (2009)] whereas the networks using other kinds of congestion control algorithms.

We study a nonlinear TCP fluid flow model for wireless network derived based on wired networks [Liu *et al.* (2012), Misra *et al.* (2000)]. We study the variations in system parameters can induce a Hopf bifurcation [Liu *et al.* (2012)], which would lead to the emergence of limit cycles. We also characterize the type of the Hopf bifurcation and verify the stability of the bifurcating limit cycles as in [Hassard *et al.* (1981), Raina (2005)]. Before all of these we focus on the necessary condition.

CHAPTER 3

Nonlinear TCP Fluid Flow Model in Wireless Networks

As it is stated above, we learn about the fluid flow model of TCP congestion avoidance algorithm for wired networks developed by [Misra *et al.* (2000)]. Then we derive a TCP fluid flow model for wireless access networks which is largely based on the wired networks fluid model. We focus more on the window size and queue length of the single congested router. Let the count of TCP flows be labeled as M where $i = 1, 2, 3, \dots M$ traverse the order. Let the round trip time (RTT) be $R_i(t)$ and TCP window size be $W_i(t)$ at time $t \geq 0$ of flow i, respectively, $i = 1, \dots M$. Let the queue length of the router at time t be $q(t)$ and the propagation delay of each flow, denoted as $T_{p,i}(i = 1, \dots M)$, which is fixed. Now, the round trip time of the flows is as the following form

$$R_i(t) = T_{p,i} + \frac{q(t)}{C(t)}, \qquad i = 1, \dots M \tag{3.1}$$

where $\frac{q(t)}{C(t)}$ is the queuing delay, $C(t)$ is available link bandwidth.

In the present scenario, the sources we use for the wireless network is to get access from a wired network, which adapts the internet TCP protocol to transfer data. We further assume that the network employs a RED algorithm to control the congestion. The dynamic model of window size is captured by the following equation [Misra *et al.* (2000)] for wired access network:

$$\dot{W}(t) = \frac{1}{R_i(t)} - \frac{W_i(t)}{2} \frac{W_i(t - R_i(t))}{R_i(t - R_i(t))} p(t - R_i(t)), \qquad i = 1, \dots M \tag{3.2}$$

where $p(t)$ is the packet's mark probability at time t.

In the above (3.2) the first term of the equation is a window's additive increase part i.e. $\frac{1}{R_i(t)}$ which adopts the phase of bandwidth probing from (3.1). This approach to increase the window size, probing for usable bandwidth, until loss occurs. The policy of additive increase, for every fixed amount round trip time increase the congestion window. Whereas, the second term of the equation is a window's multiplicative decrease part i.e. $\frac{W_i(t)}{2}$ in response to packet marking probability p. If once a congestion loss is detected in RED, the transmitter decreases the window size by a multiplicative factor i.e. the window size is halved after loss and the marking/dropping is implemented to distribute the losses in proportion to a flow's bandwidth share, which is $p(t)\frac{W_i(t)}{R_i(t)}$ at time t. The AIMD result is a saw tooth behavior that represents the probe for bandwidth [Zheng and Nelson (2009)].

Till now we discussed about wired network dynamical model for window size and AIMD congestion control parameters. Let us know more about the the wireless network model, transmission and its congestion properties. In wireless the connectivity happens through up link and down link transmission. As we already stated source for wireless we use wired network i.e. To transfer the data up link transmission is through wired network, when it comes to down link transmission the source, marking probability is fed back to sources. For down link communication two events can happen, One is that the source has correctly received the marking probability, and second is that the due to channel fading source has failed to receive the marking probability. In the down link transmission if feedback packet marking probability is lost the probability of the event at time t of flow i, respectively , $i = 1, \dots M$ be $P_{dl,i}, i = 1, \dots M$ (subscript dl represents down link). Whenever this event happens, the source will use the previous packet marking probability to reduce its window size, and the window size is decreased by one by convention. This is similar to the response of the source to a timeout loss in traditional wired networks. Thus for a wireless network dynamics for the transmission rate is governed by

$$\begin{aligned}\dot{W}(t) = {} & \frac{1}{R_i(t)} - \frac{W_i(t)}{2}\frac{W_i(t-R_i(t))}{R_i(t-R_i(t))}p(t-R_i(t)) - P_{dl,i}(t)(W_i(t)-1) \\ & \times \frac{W_i(t-R_i(t))}{R_i(t-R_i(t))}p(t-R_{ah,i}(t)), \qquad i = 1, \dots M \end{aligned} \tag{3.3}$$

where $R_{ah,i}$ denotes the time difference between the current time and the time at which the latest marking probability has been successfully received. In packet losses it clear that $R_{ah,i}(t) \geq R_i(t)$. In the congestion control algorithm we assume that the $R_{ah,i}(t) = \alpha R_i(t)$, where α is an integer larger than or equal to two.

Now we describe the dynamic model for the queue, in the case of wired networks, the queue length is represent by the following equation

$$\dot{q}(t) = \begin{cases} -C(t) + \sum_{i=1}^{M} \frac{W_i(t)}{R_i(t)}, & \text{when } q(t) > 0, \\ \max\Big(0, -C(t) + \sum_{i=1}^{M} \frac{W_i(t)}{R_i(t)}\Big) & \text{when } q(t) = 0. \end{cases} \quad (3.4)$$

In wireless access networks, due to channel fading some uplink transmitted packets will be lost. Let us denote the loss probability due to the fading in as $P_{ul,i}(t)$ (subscript ul represents uplink). Uplink channel loss probability at time t is $P_{ul,i}(t)$. Taking the channel loss into account, the actual queue length of wireless networks will be governed by the following equation

$$\dot{q}(t) = \begin{cases} -C(t) + \sum_{i=1}^{M} \frac{W_i(t)}{R_i(t)}(1 - P_{ul,i}(t)), & \text{when } q(t) > 0, \\ \max\Big(0, -C(t) + \sum_{i=1}^{M} \frac{W_i(t)}{R_i(t)}(1 - P_{ul,i}(t))\Big) & \text{when } q(t) = 0. \end{cases} \quad (3.5)$$

The system dynamic behavior is completely described by M + 1 differential equations consisting of (3.3) and (3.5). Based on these M + 1 dynamic equation, it would be very difficult to study the design method for the congestion control algorithm. To reduce this kind of difficulty, assumption are made that the channel fading of all the wireless connections has the same statistical property. We capture some basic characteristics of each individual flows, these same statistical property makes this kind of representation reasonable. Correspondingly, let the window size be W, the down and up link channel loss probabilities be P_{ul} and P_{dl}, respectively, and the round trip time of the generic flow be $R(t) = T_p + \frac{q(t)}{C(t)}$.

Then the system approximates a new set of ordinary differential equations for dynamic behavior described by

$$\begin{aligned} \dot{W}(t) &= \frac{1}{R(t)} - (1 - P_{dl})\frac{W(t)}{2}\frac{W(t-R(t))}{R(t-R(t))}p(t-R(t)) \\ &\quad - P_{dl}(t)(W(t)-1)\frac{W(t-R(t))}{R(t-R(t))}p(t-R_{ah}(t)) \\ \dot{q}(t) &= -C(t) + M\frac{W(t)}{R(t)}\left(1 - P_{ul}(t)\right), \qquad \text{when } q(t) > 0 \end{aligned} \tag{3.6}$$

CHAPTER 4

Linear Stability Analysis

A fluid based TCP dynamic model was developed (3.6). We simplify the model further, which ignores the timeout and slow start mechanism of TCP. The model relates the average value of key network variables and is described by the following set of nonlinear delay differential equations:

$$\begin{aligned}\dot{W}(t) &= \frac{1}{R(t)} - (1 - P_{dl})\frac{W(t)}{2}\frac{W(t-R(t))}{R(t-R(t))}p(t-R(t)) \\ &\quad - P_{dl}(t)(W(t)-1)\frac{W(t-R(t))}{R(t-R(t))}p(t-R_{ah}(t)), \\ \dot{q}(t) &= -C(t) + M\frac{W(t)}{R(t)}\Big(1 - P_{ul}(t)\Big), \qquad \text{when } q(t) > 0\end{aligned}$$

where $W(t)$ denotes the average of TCP windows size (packets), $q(t)$ is the average of queue length (packets), $M(t)$ is the number of TCP sessions, $\boldsymbol{C}$ is the queue capacity (packets/sec) and $R(t)$ is the round trip time which consists of the propagation delay T_p and queuing delay, $p(*)$ is the probability function of a packet mark. Assume that the round trip delay $R(t)(s)$ and the number of TCP connections $M(t)$ are constants, i.e, $M(t) = M$ and $R_{ah}(t) \approx R(t) = \tau$, when the queuing delay is much smaller than the propagation delay. Considering that the probability marking function $p(*)$ is proportional to the queue length, i.e. $p(t) = Kq(t)$. The down and up link channel loss probabilities P_{dl} and P_{ul} are assumed to be constants. The above system can be simplified as (4.1). We define

$$\begin{aligned}g(W,q) &= \frac{1}{\tau} - (1-P_{dl})\frac{W(t)}{2}\frac{W(t)}{\tau}Kq(t-\tau) - P_{dl}(W(t)-1)\frac{W(t)}{\tau}Kq(t-\tau), \\ h(W,q) &= -C + M\frac{W(t)}{\tau}(1-P_{ul})\end{aligned} \tag{4.1}$$

The equilibrium point (W^*, q^*) of system (3.6) is given by

$$W^* = \frac{\tau_0 C}{M(1-P_{dl})}, \quad p^* = \frac{2}{(1+P_{dl})W^{*2} - 2P_{dl}W^*} \tag{4.2}$$

We consider a small perturbation about the equilibrium point, i.e.,

$$W_d(t) = W(t) - W^*, \quad q_d(t) = q(t) - q^*, \quad p_d(t) = p(t) - p^* \tag{4.3}$$

Evaluating the following partials at the operating point (W^*, q^*) defined by (4.2) gives

$$\begin{aligned}
\frac{\partial g}{\partial W} &= \frac{P_{dl}P^* - 2(1+P_{dl})W^*P^*}{\tau} \\
\frac{\partial h}{\partial W} &= M\frac{(1-P_{ul})}{\tau} \\
\frac{\partial g}{\partial q} &= \frac{(2P_{dl}W^* - (1+P_{dl})W^{*2})K}{2\tau} \\
\frac{\partial h}{\partial q} &= 0
\end{aligned}$$

Then using the first order Taylor series expansion, we get the linearized expression and by substituting (4.3) into the these expression, we obtain the following linearized equations about the equilibrium points.

$$\begin{aligned}
\dot{W_d}(t) &= a_0 W_d(t) + b_0 q_d(t-\tau), \\
\dot{q_d}(t) &= c_0 W_d(t), \qquad \text{when } q(t) > 0
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
a_0 &= \frac{P_{dl}P^* - (1+P_{dl})W^*P^*}{\tau}, \\
b_0 &= \frac{2P_{dl}W^* - (1+P_{dl})W^{*2}}{2\tau}K \text{ and} \\
c_0 &= \frac{M(1-P_{ul})}{\tau}
\end{aligned}$$

Then the characteristic of (4.4) is

$$\lambda^2 + \zeta_1\lambda + \zeta_2 e^{-\lambda\tau} = 0. \tag{4.5}$$

where

$$\zeta_1 = a_0 = \frac{P_{dl}P^* - (1+P_{dl})W^*P^*}{\tau},$$
$$\zeta_2 = b_0c_0 = \frac{(2P_{dl}W^* - (1+P_{dl})W^{*2}K)(M(1-P_{ul}))}{2\tau^2}$$

We know the exponential series expansion, will consider first three terms in $e^{-\lambda\tau} \approx 1 - \lambda\tau + \frac{\lambda^2\tau^2}{2}$ and substitute in the characteristic equation, we obtain

$$(1+\frac{\zeta_2\tau^2}{2})\lambda^2 + (\zeta_1 - \zeta_2\tau)\lambda + \zeta_1 = 0 \tag{4.6}$$

To find out the system stability, necessary condition is to have all the roots of the characteristic equation are in the open left-half plane then only the system is stable. According to Routh-Hurwitz stability criterion states that the system is stable if and only if the the value of each determinant is greater than zero. For the above characteristic equation,

Determinant one: $\begin{vmatrix}(\zeta_1 - \zeta_2\tau)\end{vmatrix}$,

Determinant two: $\begin{vmatrix}(\zeta_1 - \zeta_2\tau) & 0 \\ (1+\frac{\zeta_2\tau^2}{2}) & \zeta_1\end{vmatrix}$

System will be stable if it satisfies the following coefficients conditions, then dynamic system (3.6) is considered to be stable.

$$(1+\frac{\zeta_2\tau^2}{2}) > 0, \quad (\zeta_1 - \zeta_2\tau) > 0, \quad \zeta_1 > 0. \tag{4.7}$$

CHAPTER 5

Hopf Bifurcation Analysis

5.1 Hopf Bifurcation Occurrence

In order to conduct a Hopf bifurcation analysis we choose a parameter which induces the bifurcation. We consider the bifurcation parameter to be K and rewrite the characteristic equation as

$$\lambda^2 + \zeta_1\lambda + \zeta_k K e^{-\lambda\tau} = 0. \tag{5.1}$$

where

$$\zeta_1 = \frac{P_{dl}P^* - (1+P_{dl})W^*P^*}{\tau},$$
$$\zeta_k = \zeta_2/K = \frac{[2P_{dl}W^* - (1+P_{dl})W^{*2}][M(1-P_{ul})]}{2\tau^2}$$

In this section, We investigate the bifurcation behavior of the TCP fluid model. We analyze Hopf bifurcation for (3.6) in the following steps

(i) First, we calculate parameter values such that the characteristic equation has pure imaginary roots, then

(ii) We determine these values and also the critical value for which characteristic equation has no positive real part roots and, then

(iii) We observe the parameter strictly upper bounded by the critical value.

As this parameter K varies, which induce a Hopf bifurcation.

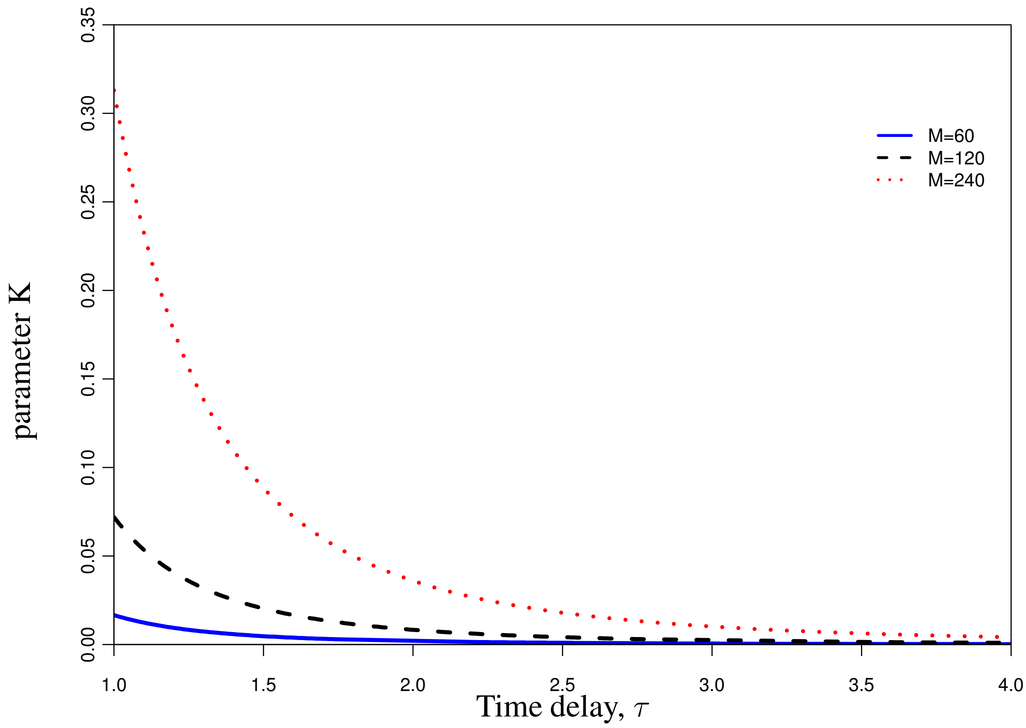


Figure 5.1: Stability chart for different TCP flows.

For $\tau > 0, K > 0$, let $\lambda = \pm i\omega, \omega > 0$, substituting $\lambda = i\omega$ and $e^{-\lambda\tau}$ in (5.1), which gives

$$
\begin{aligned}
\lambda^2 + \zeta_1\lambda + \zeta_k K e^{-\lambda\tau} =& (i\omega)^2 + \zeta_1(i\omega) + \zeta_k K e^{-i\omega\tau} \\
=& -\omega^2 + i\zeta_1\omega + \zeta_k K e^{-i\omega\tau} \\
=& -\omega^2 + i\zeta_1\omega + \zeta_k K(\cos\omega\tau - i\sin\omega\tau) \\
=& \Big(\zeta_k K\cos\omega\tau - \omega^2\Big) + i\Big(\zeta_1\omega - \zeta_k K\sin\omega\tau\Big)
\end{aligned} \tag{5.2}
$$

From (5.2) which gives

$$
\begin{aligned}
\zeta_k K\cos\omega\tau - \omega^2 = 0 \\
-\zeta_k K\sin\omega\tau + \zeta_1\omega = 0
\end{aligned} \tag{5.3}
$$

we obtain,

$$\omega_0 = \sqrt{\frac{-{\zeta_1}^2 + \sqrt{{\zeta_1}^4 \pm 4{\zeta_k}^2{K_c}^2}}{2}}$$
$$\tan(\omega_0\tau) = \frac{\zeta_1}{\omega_0} \tag{5.4}$$

where K_c denotes the critical value of K at $\omega = \omega_0$.

Next we prove that $\lambda = \pm i\omega_0$ are simple roots of (5.1) when $K = K_c$. Now defining

$$\Delta(\lambda, K) = \lambda^2 + \zeta_1\lambda + \zeta_k K e^{-\lambda\tau} \tag{5.5}$$

Now we need to prove the transversality condition for the occurrence of Hopf bifucartion. i.e. $\frac{d\Delta(\lambda,K)}{d\lambda}\Big|_{\lambda=i\omega_0} \neq 0$. we differentiate (5.5) and then we substitute $e^{-\lambda\tau}$ from (5.1)

$$\begin{aligned}
\frac{d\Delta(\lambda, K)}{d\lambda} &= 2\lambda + \zeta_1 - \zeta_k K\lambda e^{-\lambda\tau} \\
&= 2\lambda + \zeta_1 - \zeta_k K\tau\left(\frac{-\lambda(\lambda + \zeta_1)}{\zeta_k K}\right) \\
&= \tau\lambda^2 + (\tau\zeta_1 + 2)\lambda + \zeta_1 \\
\frac{d\Delta(\lambda, K)}{d\lambda}\bigg|_{\lambda=i\omega_0} &= \tau(i\omega_0)^2 + (\tau\zeta_1 + 2)i\omega_0 + \zeta_1 \\
&= \zeta_1 - \tau\omega_0^2 + i(\tau\zeta_1 + 2)\omega_0 \neq 0
\end{aligned} \tag{5.6}$$

Again we differentiate (5.5) and then we substitute $e^{-\lambda\tau}, \lambda = i\omega_0$ from (5.1),we get

$$\begin{aligned}
\frac{d\lambda}{dK} &= \frac{-\zeta_k e^{-\lambda\tau}}{2\lambda + \zeta_1 - \zeta_k K\tau e^{-\lambda\tau}} \\
&= \frac{1}{K}\frac{\lambda^2 + \zeta_1\lambda}{\lambda^2 + (\zeta_1\tau + 2)\lambda + \zeta_1} \\
&= \frac{-\omega_0^2 + i\zeta_1\omega_0}{K(\zeta_1 - \omega_0^2 + i(\zeta_1\tau + 2)\omega_0)} \\
Re\left(\frac{d\lambda}{dK}\right) &= \frac{1}{K}\frac{\tau\omega_0^4 + \tau\zeta_1^2\omega_0^2 + \zeta_1\omega_0^2}{(\zeta_1 - \omega_0^2)^2 + (\zeta_1\tau + 2)^2\omega_0^2} \\
Re\left(\frac{d\lambda}{dK}\right)\bigg|_{K=K_c} &= \frac{1}{K_c}\frac{\tau\omega_0^4 + \tau\zeta_1^2\omega_0^2 + \zeta_1\omega_0^2}{(\zeta_1 - \omega_0^2)^2 + (\zeta_1\tau + 2)^2\omega_0^2}
\end{aligned} \tag{5.7}$$

For $\omega_0 > 0$, we get $\quad Re\left(\frac{d\lambda}{dK}\right)\Bigg|_{K=K_c} > 0.$

5.2 Direction and Stability of Hopf bifurcation

In this section, we use the Poincaré normal forms and the theory of Center manifold helps to study the direction of Hopf bifurcation and the stability of bifurcating solution at the equilibrium when K passes through certain critical values. A set of nonlinear equations for TCP fluid flow model in wireless networks (3.6) are as follows:

$$\begin{aligned}\dot{W}(t) &= \frac{1}{R(t)} - (1-P_{dl})\frac{W(t)}{2}\frac{W(t-R(t))}{R(t-R(t))}p(t-R(t)) \\ &\quad - P_{dl}(t)(W(t)-1)\frac{W(t-R(t))}{R(t-R(t))}p(t-R_{ah}(t)) \\ \dot{q}(t) &= -C(t) + M\frac{W(t)}{R(t)}\Bigg(1-P_{ul}(t)\Bigg), \qquad \text{when } q(t) > 0 \end{aligned} \tag{5.8}$$

above set of equation are further simplified to

$$\begin{aligned}\dot{W}(t) &= \frac{1}{\tau} - \frac{(1+P_{dl})W(t)W(t) - 2P_{dl}W(t)}{\tau}Kq(t-\tau), \\ \dot{q}(t) &= -C + M\frac{W(t)}{\tau}\Bigg(1-P_{ul}\Bigg), \qquad \text{when } q(t) > 0 \end{aligned} \tag{5.9}$$

We know the linearized equation (4.4), Using Taylor series expansion we find quadratic and cubic terms of defined by (4.2) at equilibrium point (W^*, q^*)

$$\begin{aligned}\frac{\partial g}{\partial WW} &= -\frac{2(1+P_{dl})P^*}{\tau} \\ \frac{\partial h}{\partial WW} &= 0 \\ \frac{\partial g}{\partial Wq} &= \frac{P_{dl}K - (1+P_{dl})W^*K}{\tau} \\ \frac{\partial h}{\partial Wq} &= 0 \\ \frac{\partial g}{\partial qq} &= 0 \\ \frac{\partial h}{\partial qq} &= 0 \\ \frac{\partial g}{\partial qW} &= \frac{P_{dl}K - (1+P_{dl})W^*K}{\tau} \\ \frac{\partial h}{\partial qW} &= 0\end{aligned}$$

similarly we will find the cubic terms.

The Taylor series expansion of (3.6), about the equilibrium point, including the linear, quadratic, and cubic terms is

$$
\begin{aligned}
\dot{W}_d(t) &= \left(\frac{P_{dl}P^* - (1+P_{dl})W^*P^*}{\tau}\right)W_d(t) + \left(\frac{2P_{dl}W^* - (1+P_{dl})W^{*2}}{2\tau}K\right)q_d(t-\tau) \\
&\quad + \frac{1}{2!}\left[(-\frac{(1+P_{dl})P^*}{\tau})W_d^2(t) + (\frac{(2P_{dl} - 2(1+P_{dl})W^*)K}{\tau})W_d(t)q_d(t-\tau)\right] \\
&\quad + \frac{1}{3!}\left[\frac{-6K(1+P_{dl})}{\tau}W_d^2(t)q_d(t-\tau)\right] + \ldots\ldots \\
\dot{q}_d(t) &= \frac{M(1-P_{ul})}{\tau}W_d(t)
\end{aligned}
\tag{5.10}
$$

Let us now consider the following autonomous system

$$
\frac{d}{dt}\boldsymbol{u}(t) = \mathcal{L}_\mu \boldsymbol{u}_t + \mathcal{F}(\boldsymbol{u}_t, \mu), \tag{5.11}
$$

$t > 0$, $\mu \in \mathbb{R}$, where for $\tau > 0$

$$
\boldsymbol{u}_t(\theta) = \boldsymbol{u}(t+\theta) \qquad \boldsymbol{u} : \big[-\tau, 0\big] \to \mathbb{R}^2, \qquad \theta \in [-\tau, 0].
$$

Note that $\mathcal{L}_\mu$ is a one-parameter family of continuous (bounded) linear operators. The operator $\mathcal{F}(\boldsymbol{u}_t, \mu)$ contains the non-linear terms. Further assume that $\mathcal{F}$ is analytic and that $\mathcal{F}$ and $\mathcal{L}_\mu$ depend analytically on the bifurcation parameter μ for small $|\mu|$. Then equation (5.10) is of the form (5.11), with $\boldsymbol{u} = \begin{bmatrix} w & q \end{bmatrix}^T$ where

$$
\begin{aligned}
\mathcal{L}_\mu \boldsymbol{u}_t &= \begin{bmatrix} a_0 & 0 \\ c_0 & 0 \end{bmatrix} \boldsymbol{u}_t(0) + \begin{bmatrix} 0 & b_0 \\ 0 & 0 \end{bmatrix} \boldsymbol{u}_t(-\tau), \\
\mathcal{F}(\boldsymbol{u}_t, \mu) &= \begin{bmatrix} \varepsilon_{ww}W_d^2(t) + \varepsilon_{wq}W_d(t)q_d(t-\tau) + \varepsilon_{w^2q}W_d^2(t)q_d(t-\tau) \\ 0 \end{bmatrix}.
\end{aligned}
$$

where

$$a_0 = \frac{P_{dl}P^* - (1+P_{dl})W^*P^*}{\tau},$$
$$b_0 = \frac{2P_{dl}W^* - (1+P_{dl})W^{*2}}{2\tau}K,$$
$$c_0 = \frac{M(1-P_{ul})}{\tau},$$
$$\varepsilon_{ww} = -\frac{(1+P_{dl})P^*}{2\tau},$$
$$\varepsilon_{wq} = \frac{(P_{dl} - (1+P_{dl})W^*)K}{\tau},$$
$$\varepsilon_{w^2q} = \frac{-K(1+P_{dl})}{\tau}$$

The idea is to transform equation (5.11) into a form which contains only $\boldsymbol{u}_t$ instead of both $\boldsymbol{u}$ and $\boldsymbol{u}_t$, i.e.

$$\frac{d}{dt}\boldsymbol{u}_t = \mathcal{A}(\mu)\boldsymbol{u}_t + \mathcal{R}\boldsymbol{u}_t. \tag{5.12}$$

First, we transform the linear problem $d\boldsymbol{u}(t)/dt = \mathcal{L}_\mu \boldsymbol{u}_t$. For this we employ the Riesz representation theorem which states that there exists a 2×2 matrix function $\eta(\cdot, \mu) : [-\tau, 0] \to \mathbb{R}^{2\times 2}$, such that the components of η have bounded variation and for all $\boldsymbol{\phi} \in C[-\tau, 0]$

$$\mathcal{L}_\mu \boldsymbol{\phi} = \int_{-\tau}^{0} \mathrm{d}\eta(\theta, \mu)\boldsymbol{\phi}(\theta).$$

In particular,

$$\mathcal{L}_\mu \boldsymbol{u}_t = \int_{-\tau}^{0} \mathrm{d}\eta(\theta, \mu)\boldsymbol{u}(t+\theta). \tag{5.13}$$

Comparing equation (5.13) with the expression for $\mathcal{L}_\mu \boldsymbol{\phi}$, we obtain

$$\mathrm{d}\eta(\theta, \mu) = \begin{bmatrix} a_0\delta(\theta) & b_0\delta(\theta+\tau) \\ c_0\delta(\theta) & 0 \end{bmatrix} \mathrm{d}\theta,$$

where $\delta(\theta)$ is the Dirac-delta function.

We now define $\mathcal{A}(\mu)\boldsymbol{\phi}(\theta)$, for $\boldsymbol{\phi} \in C^1\big[-\tau, 0\big]$, as

$$\mathcal{A}(\mu)\boldsymbol{\phi}(\theta) = \begin{cases} \frac{\mathrm{d}\boldsymbol{\phi}(\theta)}{\mathrm{d}\theta}, & \theta \in [-\tau, 0) \\ \int_{-\tau}^{0} \mathrm{d}\eta(s,\mu)\boldsymbol{\phi}(s) \equiv \mathcal{L}_\mu \boldsymbol{\phi}, & \theta = 0, \end{cases} \tag{5.14}$$

and

$$\mathcal{R}\boldsymbol{\phi}(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0) \\ \mathcal{F}(\boldsymbol{\phi}, \mu), & \theta = 0. \end{cases}$$

As $\mathrm{d}\boldsymbol{u}_t/\mathrm{d}\theta = \mathrm{d}\boldsymbol{u}_t/\mathrm{d}t$, (5.11) becomes (5.12) as desired.

Let $\boldsymbol{q}(\theta)$ be the eigenfunction for $\mathcal{A}(0)$ corresponding to $\lambda(0) = i\omega$, namely

$$\mathcal{A}(0)\boldsymbol{q}(\theta) = i\omega\boldsymbol{q}(\theta).$$

To find $\boldsymbol{q}(\theta)$ let $\boldsymbol{q}(\theta) = \boldsymbol{q}_0 e^{i\omega\theta}$, where $\boldsymbol{q}_0 = [1 \;\; \phi_1]^T$. Substituting in the above equation and using the expression for $\mathcal{A}$ as in (5.14), we obtain

$$\begin{aligned} \mathcal{A}(0)\boldsymbol{q}(\theta) &= \int_{-\tau}^{0} \begin{bmatrix} a_0\delta(\theta) & b_0\delta(\theta+\tau) \\ c_0\delta(\theta) & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \phi_1 \end{bmatrix} e^{i\omega\theta}\mathrm{d}\theta \\ &= \int_{-\tau}^{0} \begin{bmatrix} a_0\delta(\theta) + b_0\phi_1\delta(\theta+\tau) \\ c_0\delta(\theta) \end{bmatrix} e^{i\omega\theta}\mathrm{d}\theta \\ &= \begin{bmatrix} a_0 + b_0\phi_1 e^{-i\omega\tau} \\ c_0 \end{bmatrix} \end{aligned}$$

we know $\mathcal{A}(0)\boldsymbol{q}(\theta) = i\omega\boldsymbol{q}(\theta),$ and $\boldsymbol{q}(\theta) = \begin{bmatrix} 1 \\ \phi_1 \end{bmatrix} e^{i\omega\theta}$ by equating, we get $\phi_1 = \frac{c_0}{i\omega}$.

From above solved ω_0 is

$$\omega_0 = \sqrt{\frac{-{\zeta_1}^2 + \sqrt{{\zeta_1}^4 \pm 4{\zeta_k}^2{K_c}^2}}{2}}.$$

Now define the adjoint operator $\mathcal{A}^*(0)$ for $\boldsymbol{\alpha} \in C^1[0, \tau]$ as

$$\mathcal{A}^*(0)\boldsymbol{\alpha}(s) = \begin{cases} -\frac{\mathrm{d}\boldsymbol{\alpha}(s)}{\mathrm{d}s}, & s \in (0, \tau] \\ \int_{-\tau}^{0} \mathrm{d}\eta^T(t, 0)\boldsymbol{\alpha}(-t), & s = 0. \end{cases} \tag{5.15}$$

As

$$\mathcal{A}(0)\boldsymbol{q}(\theta) = \lambda(0)\boldsymbol{q}(\theta),$$

$\overline{\lambda}(0) = -i\omega\big(\text{conjugate of } \lambda(0)\big)$ is an eigenvalue for $\mathcal{A}^*$, and

$$\mathcal{A}^*(0)\boldsymbol{q}^* = -i\omega\boldsymbol{q}^*,$$

for some non-zero vector $\boldsymbol{q}^*$. Let $\boldsymbol{q}^*(s) = \boldsymbol{B}e^{i\omega s}$ be an eigenvector of $\mathcal{A}^*$ corresponding to eigenvalue $-i\omega$, where $\boldsymbol{B} = B[\phi_2 \ \ 1]^T$. Substitute in (5.15) we obtain

$$\begin{aligned}\mathcal{A}^*(0)\boldsymbol{q}^*(\theta) &= \int_{-\tau}^{0} \mathrm{B}\begin{bmatrix} a_0\delta(\theta) & c_0\delta(\theta) \\ b_0\delta(\theta+\tau) & 0 \end{bmatrix}\begin{bmatrix}\phi_2 \\ 1\end{bmatrix} e^{-i\omega\theta}\mathrm{d}\theta \\ &= \int_{-\tau}^{0} \mathrm{B}\begin{bmatrix} a_0\phi_2\delta(\theta) + c_0\delta(\theta) \\ b_0\delta(\theta+\tau) \end{bmatrix} e^{-i\omega\theta}\mathrm{d}\theta \\ &= \mathrm{B}\begin{bmatrix} a_0\phi_2 + c_0 \\ b_0e^{i\omega\tau}\end{bmatrix}\end{aligned}$$

We Know $\mathcal{A}^*(0)\boldsymbol{q}^* = -i\omega\boldsymbol{q}^*$, and $\boldsymbol{q}^* = \mathrm{B}\begin{bmatrix}\phi_2 \\ 1\end{bmatrix} e^{-i\omega\theta}$ by equating, we get $\phi_2 = \frac{-c_0}{a_0+i\omega}$.

For $\boldsymbol{\phi} \in C[-\tau, 0]$ and $\boldsymbol{\psi} \in C[0, \tau]$, define an inner product

$$\langle \boldsymbol{\psi}, \boldsymbol{\phi}\rangle = \overline{\boldsymbol{\psi}}(0)\cdot\boldsymbol{\phi}(0) - \int_{\theta=-\tau}^{0}\int_{\zeta=0}^{\theta} \overline{\boldsymbol{\psi}}^T(\zeta-\theta)\mathrm{d}\eta(\theta)\boldsymbol{\phi}(\zeta)\mathrm{d}\zeta,$$

where $p\cdot q$ means $\sum_{i=1}^{n} p_iq_i$. Then, $\langle\boldsymbol{\psi}, \mathcal{A}\boldsymbol{\phi}\rangle = \langle\mathcal{A}^*\boldsymbol{\psi}, \boldsymbol{\phi}\rangle$ for $\boldsymbol{\phi} \in \mathrm{Dom}(\mathcal{A})$ and $\boldsymbol{\psi} \in \mathrm{Dom}(\mathcal{A}^*)$.

Let $\boldsymbol{q}$ and $\boldsymbol{q}^*$, with

$$\overline{B} = \left(\phi_1 + \overline{\phi}_2 + \tau b_0\phi_1\overline{\phi}_2e^{-i\omega\tau}\right)^{-1},$$

be the eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$ corresponding to the eignevalues $i\omega$ and $-i\omega$ respectively. These eigenvectors are required to satisfy $\langle\boldsymbol{q}^*, \boldsymbol{q}\rangle = 1$ and $\langle\boldsymbol{q}^*, \overline{\boldsymbol{q}}\rangle = 0$.

Now, we verify that $\langle \boldsymbol{q}^*, \boldsymbol{q}\rangle = 1$.

$$\begin{aligned}
\langle \boldsymbol{q}^*, \boldsymbol{q}\rangle =& \overline{B}\begin{bmatrix}\overline{\phi}_2 & 1\end{bmatrix}\cdot\begin{bmatrix}1\\ \phi_1\end{bmatrix} - \int_{\theta=-\tau}^{0}\int_{\zeta=0}^{\theta}\overline{B}e^{-i\omega(\zeta-\theta)}\begin{bmatrix}\overline{\phi}_2\\ 1\end{bmatrix}^T\\
&\times\begin{bmatrix}a_0\delta(\theta) & b_0\delta(\theta+\tau)\\ c_0\delta(\theta) & 0\end{bmatrix}d\theta\begin{bmatrix}1\\ \phi_1\end{bmatrix}e^{i\omega\zeta}d\zeta\\
=&\overline{B}\big(\overline{\phi}_2+\phi_1\big) - \overline{B}\int_{\theta=-\tau}^{0} e^{i\omega\theta}\begin{bmatrix}\overline{\phi}_2 & 1\end{bmatrix}\\
&\times\begin{bmatrix}a_0\delta(\theta)+\phi_1 b_0\alpha_l\delta(\theta+\tau)\\ c_0\delta(\theta)\end{bmatrix}\theta d\theta\\
=&\overline{B}\big(\overline{\phi}_2+\phi_1\big) - \overline{B}\int_{\theta=-\tau}^{0}\Big(\overline{\phi}_2\big(a_0\delta(\theta)\\
&+b_0\phi_1\alpha_l\delta(\theta+\tau)\big)+c_0\delta(\theta)\Big)\theta e^{i\omega\theta}d\theta\\
=&\overline{B}\big(\overline{\phi}_2+\phi_1\big)+\overline{B}\tau b_0\phi_1\overline{\phi}_2 e^{-i\omega\tau}\\
=&1.
\end{aligned}$$

Similarly, we may also verify that $\langle \boldsymbol{q}^*, \overline{\boldsymbol{q}}\rangle = 0$.

$$\begin{aligned}
\langle \boldsymbol{q}^*, \overline{\boldsymbol{q}}\rangle =& \overline{B}\begin{bmatrix}\overline{\phi}_2 & 1\end{bmatrix}\cdot\begin{bmatrix}1\\ \overline{\phi}_1\end{bmatrix} - \int_{\theta=-\tau}^{0}\int_{\zeta=0}^{\theta}\overline{B}e^{-i\omega(\zeta-\theta)}\begin{bmatrix}\overline{\phi}_2\\ 1\end{bmatrix}^T\\
&\times\begin{bmatrix}a_0\delta(\theta) & b_0\delta(\theta+\tau)\\ c_0\delta(\theta) & 0\end{bmatrix}d\theta\begin{bmatrix}1\\ \overline{\phi}_1\end{bmatrix}e^{-i\omega\zeta}d\zeta\\
=&\overline{B}\big(\overline{\phi}_2+\overline{\phi}_1\big) - \overline{B}\int_{\theta=-\tau}^{0} e^{i\omega\theta}\begin{bmatrix}\overline{\phi}_2 & 1\end{bmatrix}\\
&\times\begin{bmatrix}a_0\delta(\theta)+\overline{\phi}_1 b_0\delta(\theta+\tau)\\ c_0\delta(\theta)\end{bmatrix}\left(\frac{1-e^{-2i\omega\theta}}{2i\omega}\right)d\theta\\
=&\overline{B}\big(\overline{\phi}_2+\overline{\phi}_1\big) - \overline{B}\int_{\theta=-\tau}^{0}\Big(\overline{\phi}_2\big(a_0\delta(\theta)\\
&+\overline{\phi}_1 b_0\delta(\theta+\tau)\big)+c_0\delta(\theta)\Big)\left(\frac{e^{i\omega\theta}-e^{-i\omega\theta}}{2i\omega}\right)d\theta\\
=&\overline{B}\big(\overline{\phi}_2+\overline{\phi}_1\big) - \overline{B}\big(\overline{\phi}_2\overline{\phi}_1 b_0\big)\left(\frac{e^{-i\omega\tau}-e^{i\omega\tau}}{2i\omega}\right).
\end{aligned}\tag{5.16}$$

Using the expressions of ϕ_1, ϕ_2 and $e^{i\omega\tau}$ and their conjugates, we have

$$\overline{\phi}_1 + \overline{\phi}_2 = \frac{-c_0}{i\omega} + \frac{-c_0}{a_0 - i\omega} = -\frac{a_0 c_0}{i\omega(a_0 - i\omega)}, \tag{5.17}$$

and

$$\begin{aligned}
&(\overline{\phi}_2 \overline{\phi}_1 b_0)\left(\frac{e^{-i\omega\tau} - e^{i\omega\tau}}{2i\omega}\right) \\
&= \left(\frac{-c_0}{i\omega}\frac{-c_0}{a_0 - i\omega} b_0\right)\left(\frac{\dfrac{-\omega^2 + a_0 i\omega}{b_0 c_0} + \dfrac{\omega^2 + a_0 i\omega}{b_0 c_0}}{2i\omega}\right) \\
&= \frac{a_0 c_0}{i\omega(a_0 - i\omega)}.
\end{aligned} \tag{5.18}$$

Substituting (5.17) and (5.18) in (5.16), we get

$$\begin{aligned}
\langle \boldsymbol{q}^*, \overline{\boldsymbol{q}} \rangle &= \overline{B}\frac{a_0 c_0}{i\omega(a_0 - i\omega)} - \overline{B}\frac{a_0 c_0}{i\omega(a_0 - i\omega)} \\
&= 0.
\end{aligned}$$

For $\boldsymbol{u}_t$, a solution of (5.12) at $\mu = 0$, define

$$\begin{aligned}
z(t) &= \langle \boldsymbol{q}^*, \boldsymbol{u}_t \rangle, \\
\boldsymbol{w}(t, \theta) &= \boldsymbol{u}_t(\theta) - 2\text{Re}\big(z(t)\boldsymbol{q}(\theta)\big).
\end{aligned}$$

Then, on the manifold, C_0, $\boldsymbol{w}(t, \theta) = \boldsymbol{w}\big(z(t), \overline{z}(t), \theta\big)$, where

$$\boldsymbol{w}\big(z, \overline{z}, \theta\big) = \boldsymbol{w}_{20}(\theta)\frac{z^2}{2} + \boldsymbol{w}_{11}(\theta) z\overline{z} + \boldsymbol{w}_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots . \tag{5.19}$$

Effectively z and $\overline{z}$ are projections for C_0 in C in the directions of $\boldsymbol{q}^*$ and $\overline{\boldsymbol{q}}^*$, respectively. The existence of the center manifold enables the reduction of (5.12) to an ordinary differential equation for a single complex variable on C_0. At $\mu = 0$, this is

$$\begin{aligned}
z'(t) &= \big\langle \boldsymbol{q}^*, \mathcal{A}\boldsymbol{u}_t + \mathcal{R}\boldsymbol{u}_t \big\rangle \\
&= i\omega z(t) + \overline{\boldsymbol{q}}^*(0) \cdot \mathcal{F}\Big(\boldsymbol{w}(z, \overline{z}, \theta) + 2\text{Re}\big(z\boldsymbol{q}(\theta)\big)\Big) \\
&= i\omega z(t) + \overline{\boldsymbol{q}}^*(0) \cdot \mathcal{F}_0(z, \overline{z}),
\end{aligned} \tag{5.20}$$

which can be abbreviated as

$$z^{'}(t) = i\omega z(t) + g(z, \overline{z}). \tag{5.21}$$

Our next objective is to expand g in powers of z and $\overline{z}$ and to determine the coefficients $\boldsymbol{w}_{ij}(\theta)$ in (5.19). The differential equation (5.20) for z would be explicit when we determine $\boldsymbol{w}_{ij}$. Expanding $g(z, \overline{z})$ in powers of z and $\overline{z}$, we have

$$\begin{aligned} g(z, \overline{z}) &= \overline{\boldsymbol{q}}^{*}(0) \cdot \mathcal{F}_0\big(z, \overline{z}\big) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} \cdots . \end{aligned}$$

Following [Hassard *et al.* (1981)], we write

$$\boldsymbol{w}^{'} = \boldsymbol{u}_t^{'} - z^{'}\boldsymbol{q} - \overline{z}^{'}\overline{\boldsymbol{q}},$$

and using (5.19) and (5.20), we obtain

$$\boldsymbol{w}^{'} = \begin{cases} \mathcal{A}\boldsymbol{w} - 2\mathrm{Re}\big(\overline{\boldsymbol{q}}^{*}(0) \cdot \mathcal{F}_0\boldsymbol{q}(\theta)\big), & \theta \in [-\tau, 0) \\ \mathcal{A}\boldsymbol{w} - 2\mathrm{Re}\big(\overline{\boldsymbol{q}}^{*}(0) \cdot \mathcal{F}_0\boldsymbol{q}(0)\big) + \mathcal{F}_0, & \theta = 0, \end{cases}$$

which can be rewritten as

$$\boldsymbol{w}^{'} = \mathcal{A}\boldsymbol{w} + \boldsymbol{h}\big(z, \overline{z}, \theta\big) \tag{5.22}$$

using (5.19), where

$$\boldsymbol{h}(z, \overline{z}, \theta) = \boldsymbol{h}_{20}(\theta)\frac{z^2}{2} + \boldsymbol{h}_{11}(\theta)z\overline{z} + \boldsymbol{h}_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots . \tag{5.23}$$

We note that, on C_0, near the origin

$$\boldsymbol{w}^{'} = \boldsymbol{w}_z z^{'} + \boldsymbol{w}_{\overline{z}}\overline{z}^{'}.$$

Using (5.19) and (5.21) to replace $\boldsymbol{w}_z$, $z^{'}$ (and their conjugates by their power series expansion) and equating this with (5.22), we get

$$
\begin{aligned}
\big(2i\omega - \mathcal{A}\big)\boldsymbol{w}_{20}(\theta) &= \boldsymbol{h}_{20}(\theta), \\
-\mathcal{A}\boldsymbol{w}_{11}(\theta) &= \boldsymbol{h}_{11}(\theta), \\
-\big(2i\omega + \mathcal{A}\big)\boldsymbol{w}_{02}(\theta) &= \boldsymbol{h}_{02}(\theta).
\end{aligned} \tag{5.24}
$$

We note that

$$
\begin{aligned}
\boldsymbol{u}_t(\theta) &= \boldsymbol{w}\big(z, \overline{z}, \theta\big) + \boldsymbol{q}(\theta)z + \overline{\boldsymbol{q}}(\theta)\overline{z} \\
&= \boldsymbol{w}_{20}(\theta)\frac{z^2}{2} + \boldsymbol{w}_{11}(\theta)z\overline{z} + \boldsymbol{w}_{02}(\theta)\frac{\overline{z}^2}{2} + \boldsymbol{q}_0 e^{i\omega\theta} z \\
&\quad + \overline{\boldsymbol{q}}_0 e^{-i\omega\theta}\overline{z} + \cdots ,
\end{aligned} \tag{5.25}
$$

from which we obtain $\boldsymbol{u}_t(0)$ and $\boldsymbol{u}_t(-\tau)$. We only require the coefficients of z^2, $z\overline{z}$, $\overline{z}^2$, $z^2\overline{z}$ from (5.25). There are only two non-linear terms in (5.10) for which we can obtain the coefficients as given below

$$
\begin{aligned}
{u_{1,t}}^2(0) &= z^2 + \overline{z}^2 + 2z\overline{z} + z^2\overline{z}\Big(w_{201}(0) + 2w_{111}(0)\Big), \\
u_{1,t}(0)u_{2,t}(-\tau) &= \phi_1 e^{-i\omega\tau} z^2 + \overline{\phi}_1 e^{i\omega\tau}\overline{z}^2 + z\overline{z}(\phi_1 e^{-i\omega\tau} + \overline{\phi}_1 e^{i\omega\tau}) \\
&\quad + z^2\overline{z}\Big(\frac{w_{201}(0)}{2}\overline{\phi}_1 e^{i\omega\tau} + w_{111}(0)\phi_1 e^{-i\omega\tau} + \frac{w_{202}(-\tau)}{2} + w_{112}(-\tau)\Big), \\
{u_{1,t}}^2(0)u_{2,t}(-\tau) &= z^2\overline{z}\Big(2\phi_1 e^{-i\omega\tau} + \overline{\phi}_1 e^{i\omega\tau}\Big),
\end{aligned}
$$

where $\boldsymbol{w}_{ij} = [w_{ij1} \;\; w_{ij2}]^T$. Recall that

$$
g(z, \overline{z}) = \overline{\boldsymbol{q}}^*(0) \cdot \mathcal{F}_0(z, \overline{z}) \equiv \overline{\boldsymbol{B}} \cdot \mathcal{F}_0(z, \overline{z}),
$$

where $[\mathcal{F}_{01} \;\; \mathcal{F}_{02}]^T = \mathcal{F}_0$, and

$$
g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} \cdots .
$$

Comparing the two equations above, we get

$$\begin{aligned}
g_{20} &= \overline{B}\cdot\overline{\phi}_2\left(\varepsilon_{ww}+\varepsilon_{wq}\phi_1 e^{-i\omega\tau}\right),\\
g_{11} &= \frac{\overline{B}\cdot\overline{\phi}_2}{2}\left(2\varepsilon_{ww}+\varepsilon_{wq}(\phi_1 e^{-i\omega\tau}+\overline{\phi}_1 e^{i\omega\tau})\right),\\
g_{02} &= \overline{B}\cdot\overline{\phi}_2\left(2\varepsilon_{ww}+\varepsilon_{wq}\overline{\phi}_1 e^{i\omega\tau}\right),\\
g_{21} &= \overline{B}\cdot\overline{\phi}_2\Big(2\varepsilon_{ww}(w_{201}(0)+2w_{111}(0))\\
&\quad+\varepsilon_{wq}\big(\frac{w_{201}(0)}{2}\overline{\phi}_1 e^{i\omega\tau}+w_{111}(0)\phi_1 e^{-i\omega\tau}+\frac{w_{202}(-\tau)}{2}+w_{112}(-\tau)\big)\\
&\quad+\frac{\varepsilon_{w^2q}}{3}(2\phi_1 e^{-i\omega\tau}+\overline{\phi}_1 e^{i\omega\tau})\Big),
\end{aligned}$$

For the expression of g_{21}, we still need to evaluate $\boldsymbol{w}_{11}(-\tau)$ and $\boldsymbol{w}_{20}(-\tau)$. Now for $\theta \in [-\tau, 0)$

$$\begin{aligned}
\boldsymbol{h}(z,\overline{z},\theta) &= -\,2\mathrm{Re}\Big(\overline{\boldsymbol{q}}^*(0)\cdot\mathcal{F}_0\boldsymbol{q}(\theta)\Big)\\
&= -\,2\mathrm{Re}\Big(g(z,\overline{z})\boldsymbol{q}(\theta)\Big)\\
&= -\left(g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z}^2}{2}+\dots\right)\boldsymbol{q}(\theta)\\
&\quad-\left(\overline{g}_{20}\frac{\overline{z}^2}{2}+\overline{g}_{11}z\overline{z}+\overline{g}_{02}\frac{z^2}{2}+\dots\right)\overline{\boldsymbol{q}}(\theta),
\end{aligned}$$

which when compared with (5.23), yields

$$\begin{aligned}
\boldsymbol{h}_{20}(\theta) &= -g_{20}\boldsymbol{q}(\theta)-\overline{g}_{02}\overline{\boldsymbol{q}}(\theta)\\
\boldsymbol{h}_{11}(\theta) &= -g_{11}\boldsymbol{q}(\theta)-\overline{g}_{11}\overline{\boldsymbol{q}}(\theta)
\end{aligned}$$

From (5.14) and (5.24), we get

$$\boldsymbol{w}'_{20}(\theta) = 2i\omega\boldsymbol{w}_{20}(\theta)+g_{20}\boldsymbol{q}(\theta)+\overline{g}_{02}\overline{\boldsymbol{q}}(\theta), \tag{5.26}$$

$$\boldsymbol{w}'_{11}(\theta) = g_{11}\boldsymbol{q}(\theta)+\overline{g}_{11}\overline{\boldsymbol{q}}(\theta). \tag{5.27}$$

Solving the differential equations (5.26) and (5.27), we obtain

$$\boldsymbol{w}_{20}(\theta) = -\frac{g_{20}}{i\omega}\boldsymbol{q}_0 e^{i\omega\theta}-\frac{\overline{g}_{02}}{3i\omega}\overline{\boldsymbol{q}}_0 e^{-i\omega\theta}+\boldsymbol{e}e^{2i\omega\theta}, \tag{5.28}$$

$$\boldsymbol{w}_{11}(\theta) = \frac{g_{11}}{i\omega}\boldsymbol{q}_0 e^{i\omega\theta}-\frac{\overline{g}_{11}}{i\omega}\overline{\boldsymbol{q}}_0 e^{-i\omega\theta}+\boldsymbol{f}, \tag{5.29}$$

for some $\boldsymbol{e} = [e_1 \;\; e_2]^T$ and $\boldsymbol{f} = [f_1 \;\; f_2]^T$, which we will determine now. For $\boldsymbol{h}(z, \overline{z}, 0) = -2\text{Re}\big(\overline{\boldsymbol{q}}^*(0) \cdot \mathcal{F}_0 \boldsymbol{q}(0)\big) + \mathcal{F}_0$,

$$\begin{aligned}
\boldsymbol{h}_{20}(0) =& - g_{20}\boldsymbol{q}(0) - \overline{g}_{02}\overline{\boldsymbol{q}}(0) + \begin{bmatrix} \varepsilon_{ww} + \varepsilon_{wq}\phi_1 e^{-i\omega\tau} \\ 0 \end{bmatrix} \\
\boldsymbol{h}_{11}(0) =& - g_{11}\boldsymbol{q}(0) - \overline{g}_{11}\overline{\boldsymbol{q}}(0) + \begin{bmatrix} 2\varepsilon_{ww} + \varepsilon_{wq}(\phi_1 e^{-i\omega\tau} + \overline{\phi}_1 e^{i\omega\tau}) \\ 0 \end{bmatrix}
\end{aligned}$$

Again, from (5.14) and (5.24), we get

$$\begin{aligned}
\begin{bmatrix} (2i\omega - a_0)\boldsymbol{w}_{201}(0) - b_0\boldsymbol{w}_{202}(-\tau) \\ -c_0\boldsymbol{w}_{201}(0) + 2i\omega\boldsymbol{w}_{202}(0) \end{bmatrix} &= \boldsymbol{h}_{20}(0) \\
\begin{bmatrix} a_0\boldsymbol{w}_{111}(0) - b_0\boldsymbol{w}_{112}(-\tau) \\ -c_0\boldsymbol{w}_{111}(0) \end{bmatrix} &= \boldsymbol{h}_{11}(0)
\end{aligned} \tag{5.30}$$

from (5.30) and $\boldsymbol{h}_{20}(0), \boldsymbol{h}_{11}(0)$ gives

$$\begin{aligned}
\begin{bmatrix} (2i\omega - a_0)\boldsymbol{w}_{201}(0) - b_0\boldsymbol{w}_{202}(-\tau) \\ -c_0\boldsymbol{w}_{201}(0) + 2i\omega\boldsymbol{w}_{202}(0) \end{bmatrix} &= -g_{20}\boldsymbol{q}(0) - \overline{g}_{02}\overline{\boldsymbol{q}}(0) + \begin{bmatrix} \varepsilon_{ww} + \varepsilon_{wq}\phi_1 e^{-i\omega\tau} \\ 0 \end{bmatrix} \\
\begin{bmatrix} a_0\boldsymbol{w}_{111}(0) - b_0\boldsymbol{w}_{112}(-\tau) \\ -c_0\boldsymbol{w}_{111}(0) \end{bmatrix} &= -g_{11}\boldsymbol{q}(0) - \overline{g}_{11}\overline{\boldsymbol{q}}(0) + \begin{bmatrix} 2\varepsilon_{ww} + \varepsilon_{wq}(\phi_1 e^{-i\omega\tau} + \overline{\phi}_1 e^{i\omega\tau}) \\ 0 \end{bmatrix}
\end{aligned} \tag{5.31}$$

we further simplify and substitute the expression for $\boldsymbol{w}_{ij}(\theta), \;\; \theta \in \{-\tau, 0\}$ from (5.31) in (5.29) and finally solving for e_1, e_2, f_1 and f_2, we have

$$\begin{aligned}
\boldsymbol{e} &= \begin{bmatrix} 2i\omega\beta(\varepsilon_{ww} + \varepsilon_{wq}\phi_1 e^{-2i\omega\tau}) \\ c_0\beta(\varepsilon_{ww} + \varepsilon_{wq}\phi_1 e^{-2i\omega\tau}) \end{bmatrix} \\
\boldsymbol{f} &= \begin{bmatrix} 0 \\ \frac{1}{b_0 c_0}(2\varepsilon_{ww} + \varepsilon_{wq}(\phi_1 e^{-i\omega\tau} + \overline{\phi}_1 e^{i\omega\tau})) \end{bmatrix}
\end{aligned}$$

where

$$\beta = \frac{1}{2i\omega(2i\omega - a_0) - b_0 c_0 e^{-2i\omega\tau}}$$

Using the values of $\boldsymbol{e}$ and $\boldsymbol{f}$ in (5.28) and (5.29), followed by substituting $\theta = -\tau$, we can obtain the expressions for $\boldsymbol{w}_{11}(-\tau)$ and $\boldsymbol{w}_{20}(-\tau)$. Using these, we can evaluate g_{21}. We now, finally, have the expressions for g_{20}, g_{11}, g_{02} and g_{21}.

Recall that we motivated K as the bifurcation parameter. We denote $\alpha'(0) = \text{Re}\big(d\lambda/dK\big)\big|_{K=K_c}$. In order to determine the type of the Hopf bifurcation, and the stability of the limit cycles, we need to evaluate the following quantities [Hassard *et al.* (1981)]

$$\mu_2 = \frac{-\text{Re}\big(c_1(0)\big)}{\alpha'(0)}, \quad \mathcal{B}_2 = 2\text{Re}\big(c_1(0)\big), \tag{5.32}$$

where $c_1(0)$ is the first Lyapunov coefficient, and is given by

$$c_1(0) = \frac{i}{2\omega}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}.$$

Then

- the Hopf bifurcation is *supercritical* if $\mu_2 > 0$ and *subcritical* if $\mu_2 < 0$,
- the limit cycles are *asymptotically orbitally stable* if $\beta_2 < 0$ and *unstable* if $\beta_2 > 0$.

CHAPTER 6

Numerical Simulations

In this section, we verify our theoretical analysis for the existence of Hopf bifurcation in section (5) and determine the stability and direction of the bifurcating periodic solutions of system (3.6) with the parameters of the system as follows

Table 6.1: Nonlinear Model Results

Parameter	Values	Units
TCP connections M	60, 120, 240	–
Queue capacity C	100	packets/sec
Round trip time τ	2	sec
Uplink probability P_{ul}	0.001	–
Downlink probability P_{dl}	0.001	–
Bifurcation parameter K_c	0.034	–

Stability chart, for the system from Fig (5.1) we observe for $M = 240$ and $\tau = 2$ value using (5.4) equation, we get the bifurcating parameter value to be $K = 0.034$ and we can also visualize from the Fig (6.5).

In order to perform a Hopf bifurcation analysis, to verify and visualize the analytically result we observe the bifurcating parameter is $K = 0.034$ for $M = 240, \tau = 2$, and the critical values W^* and q^* are 0.834, and 82.47 respectively from (4.2) in section (4). Below are the waveform plot to study about the system's stability and phase portrait for $K < K_c$ and $K > K_c$.

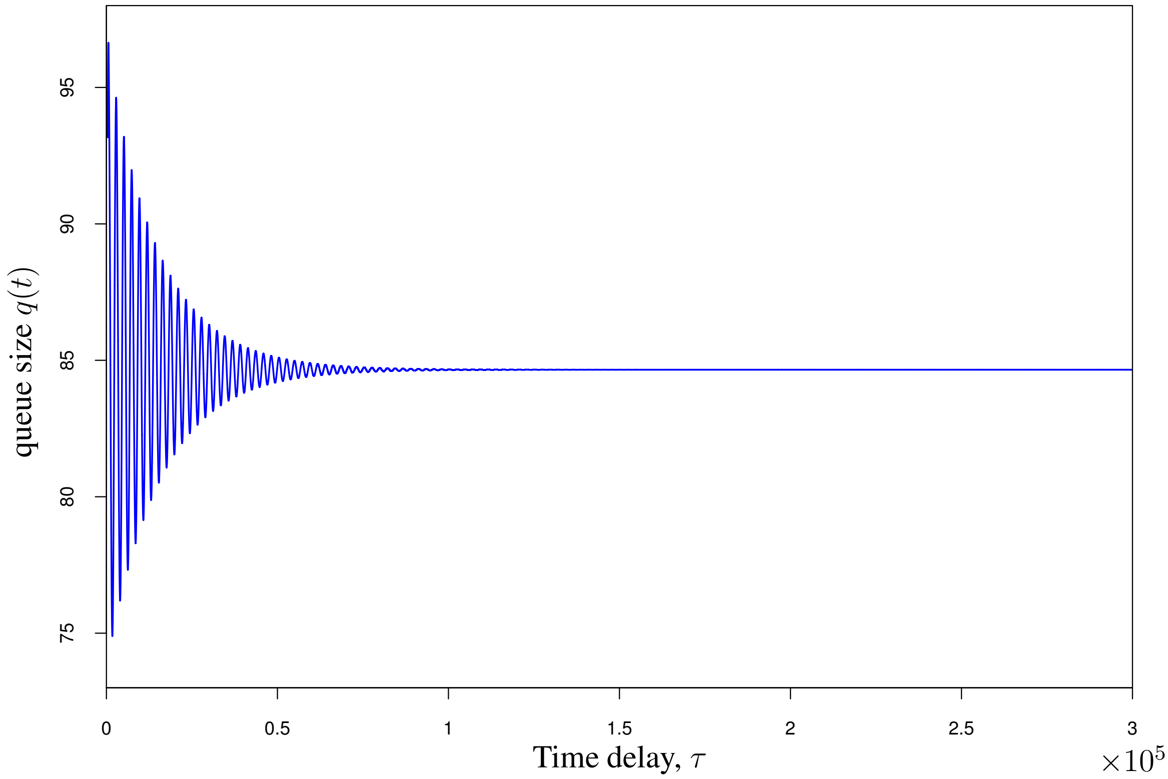


Figure 6.1: Waveform plot for $K < K_c$.

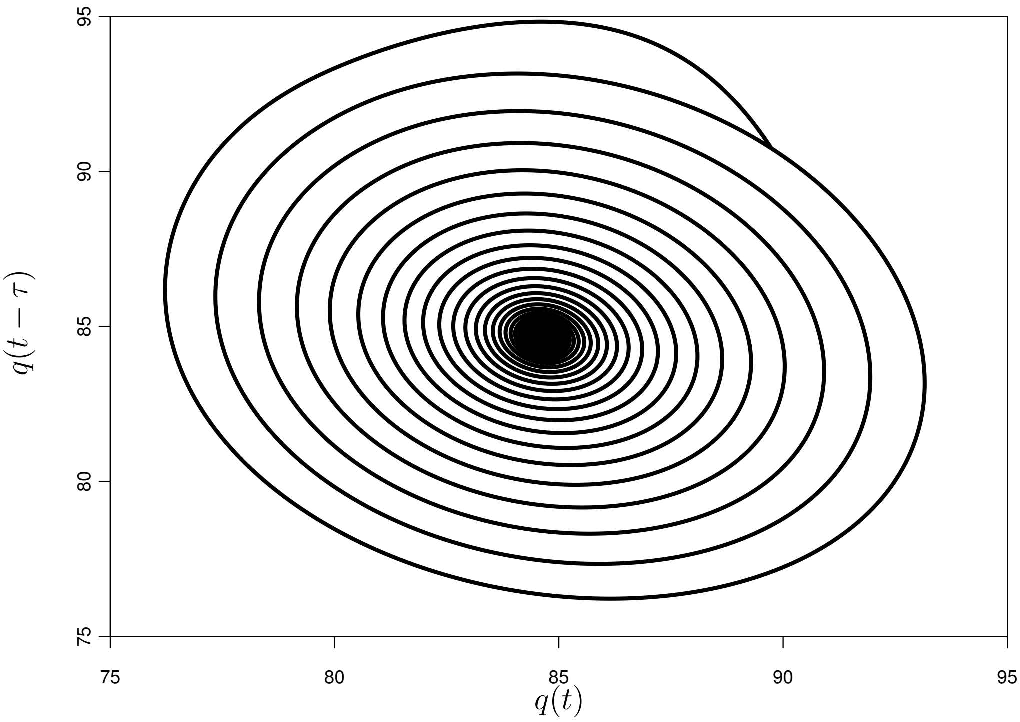


Figure 6.2: Phase portrait for $K < K_c$.

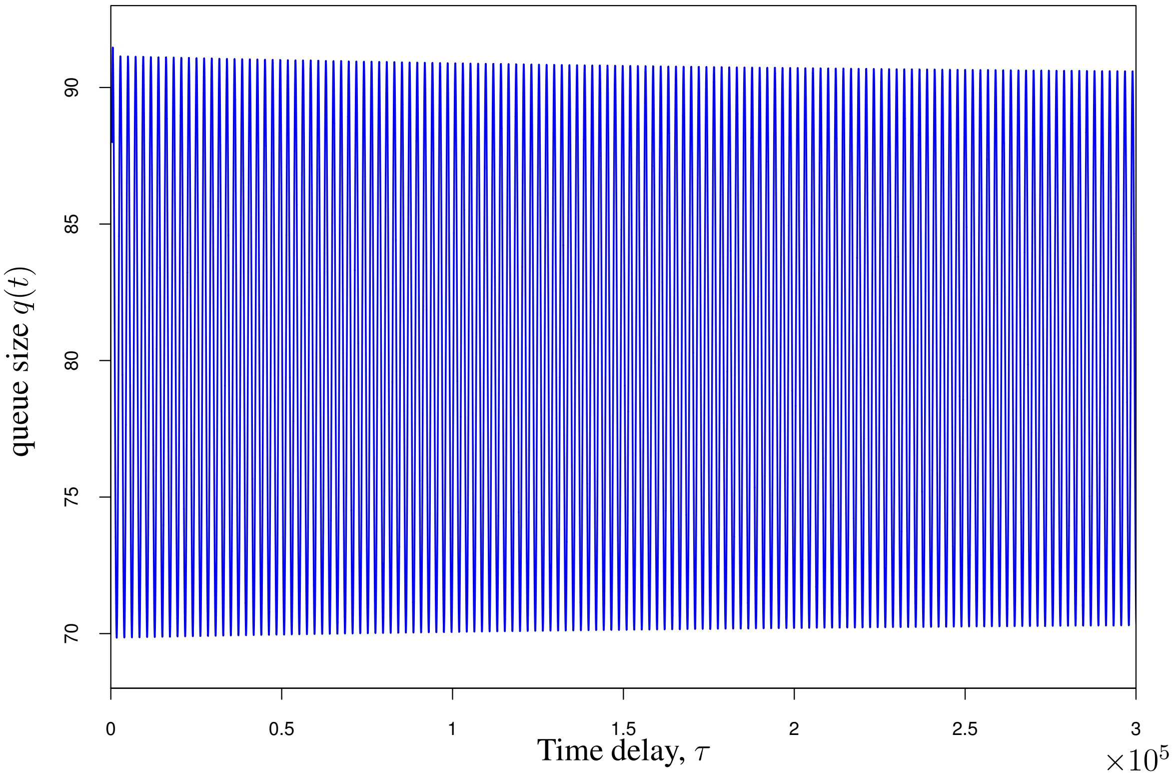


Figure 6.3: Waveform plot for $K > K_c$.

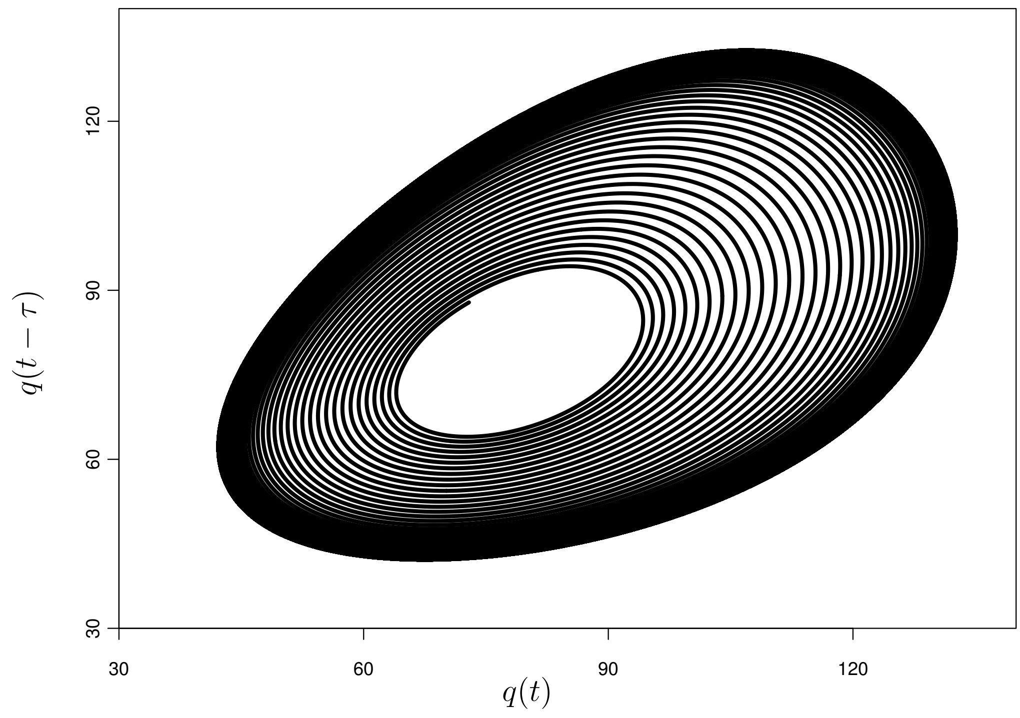


Figure 6.4: Phase portrait for $K > K_c$.

Hopf bifurcation occurs at $K = K_c$.

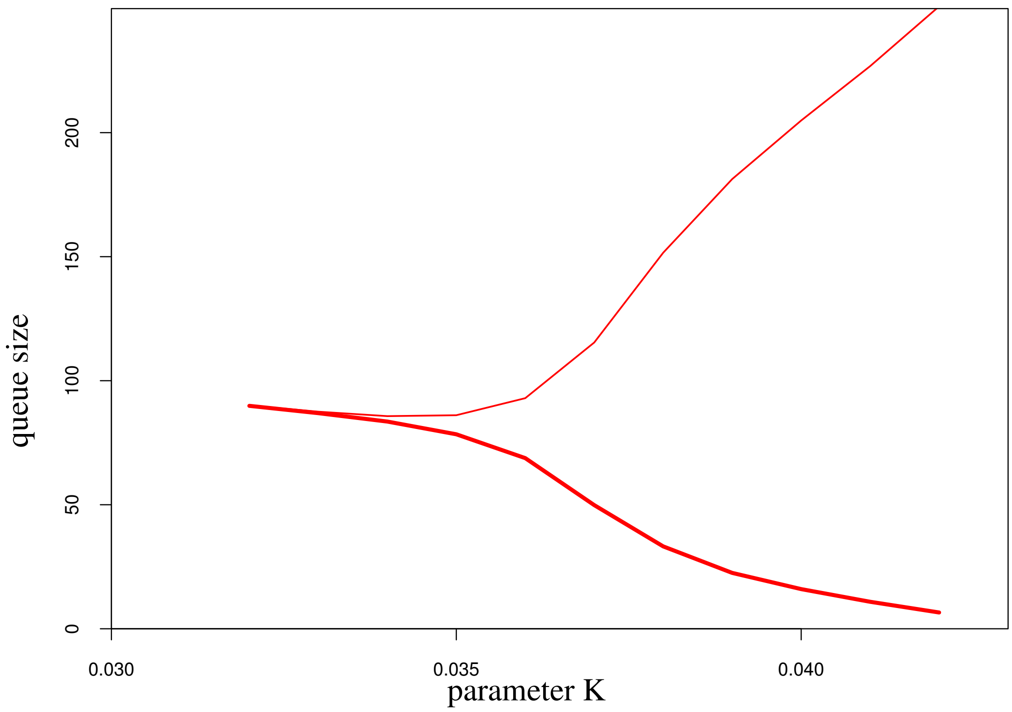
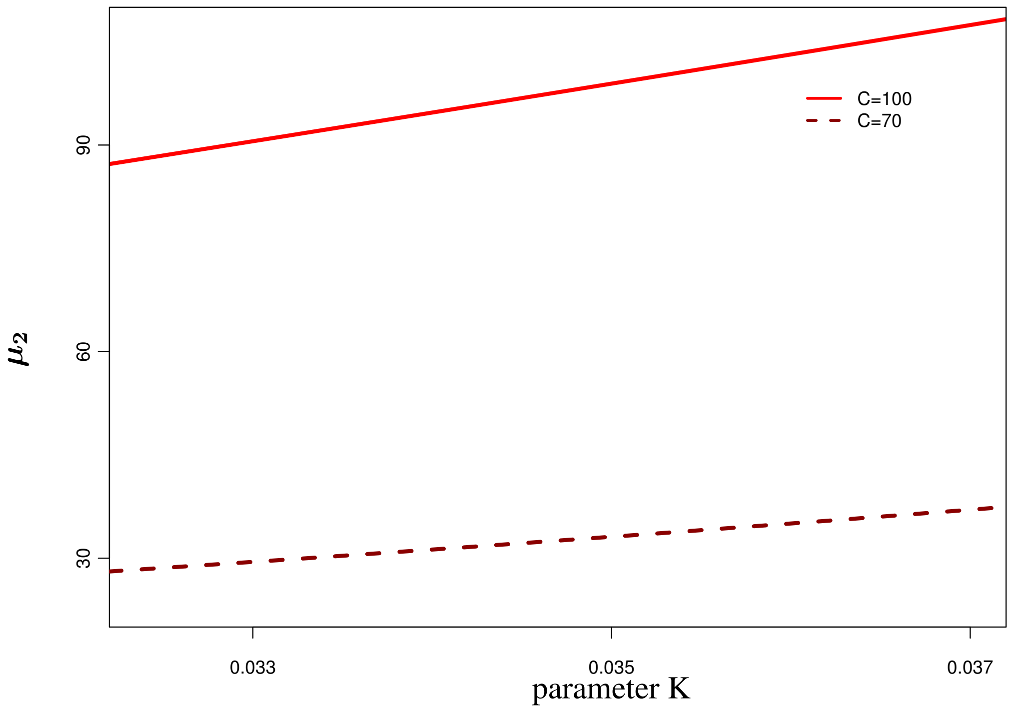


Figure 6.5: Bifurcation diagram for K.



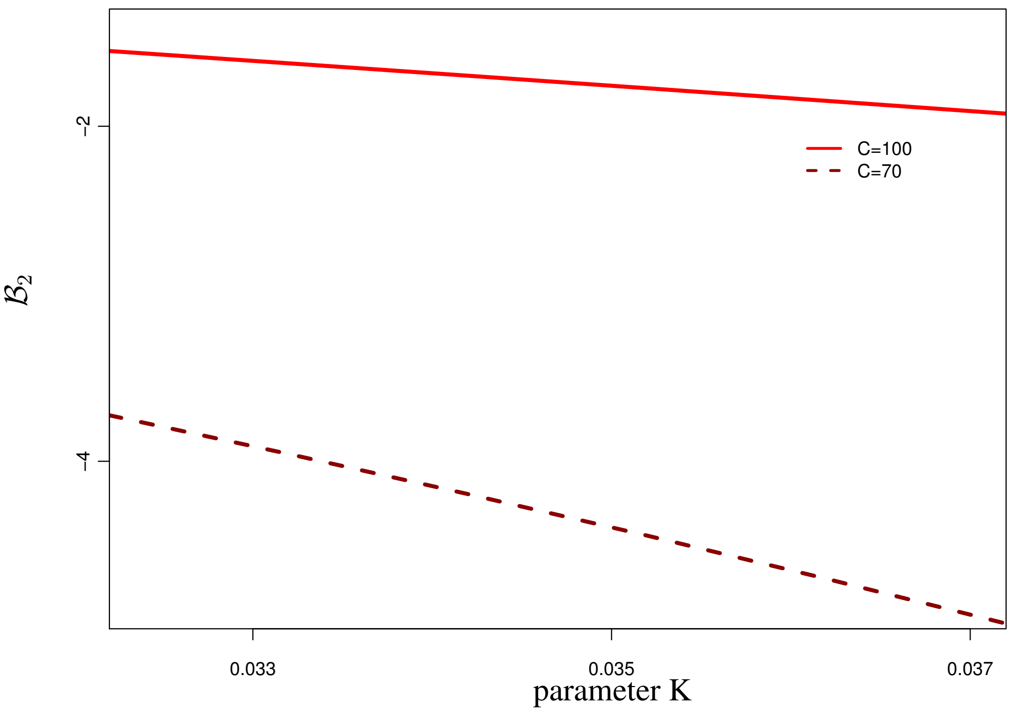


Figure 6.6: $\mu_2, \mathcal{B}_2$ for bifurcation parameter K.

CHAPTER 7

Conclusion

A nonlinear TCP fluid flow model was analyzed. From Fig (6.2) - Fig (6.5) we visualize behaviour of system's stability at $K < K_c, K > K_c$ and $K = K_c$. The local stability of the equilibrium was investigated from Fig (6.2) system is stable because bifurcation parameter K is lesser than the critical value. Occurence of Hopf bifurcation as K passes from left to right through the critical value $K = K_c$, which cause the system to sustain oscillations is where the system loses its stability.

We now knew that as K crosses a critical value, the fluid flow model (3.6) will lose stability, where Hopf bifurcation occurs. To stabilize the system's queueing length with a bifurcation parameter K, we further analysis the system by considering quadratic and cubic terms which plays a major role for system oscillations (5.10) and to determine the type of Hopf bifurcation, and stability of limit cycles (5.32) as in section (5). From Fig (6.6) $\mu 2 > 0$, implies the Hopf bifurcation is supercritical and $\mathcal{B}_2 < 0$, implies that the limit cycles are asymptotically orbitally stable.

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