Prediction of Surface Currents on a Grounded Sphere placed Near a Circular Loop Antenna

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1 Aim of the project

To find the surface currents on a grounded sphere placed near a circular loop antenna from measuring the magnetic fields very close to the surface of the grounded plane.

2 Apparatus

A thin metallic sphere of radius 'a' is placed with its centre at origin. The sphere is then grounded. A circular loop/arc antenna with radius 'r_0' with its centre also located at origin is placed in x-y plane. Assume the feed in point for the antenna is on x-axis at the point $(r_0, 0, 0)$. Take circumference of the loop antenna as one wavelength, i.e $2\pi r_0 = \lambda_0$. And the difference between the radii of antenna and sphere is taken as '$\lambda_0/20$'.

$$r_0 - a = \lambda_0/20$$

$$a = r_0 - \lambda_0/20$$

$$a = \lambda_0 \left(\frac{1}{2\pi} - \frac{1}{20} \right)$$

3 Motivation

The idea behind estimating surface currents from the magnetic fields close to the surface of the grounded sphere comes from Ampere's law. For the tangential magnetic field to be continuous at a material discontinuty, there should not be any surface currents at the interface.If there is a surface current, then it should give rise to its own magnetic field making the magnetic fields on both sides discontinuous and the discontinuty exactly equals to the surface current. Now the discontinuty is at/on the sphere because there will be no fields inside the sphere and there are fields just outside the sphere. As there is current on the antenna, there appears to be an image current inside the sphere. The discontinuity at the surface can be assumed to be cause by the image current. Since every current element on the antenna is equidistant from centre of the sphere, the image current will also be a circular loop. Let us take the current on antenna as **Note:Here we omit**$e^{j\omega t}$**in the equations and assume it will always be a multiplier to the expressions from now**

$$I_0\left(r,\theta,\phi\right)=I_0\left(\phi\right)\delta\left(r-r_0\right)\delta\left(\theta-\frac{\pi}{2}\right)\hat{\phi}$$

From The Method of Images we can find image current as

$$I_1\left(r,\theta,\phi\right)=-\left(\frac{a}{r_0}\right)I_0\left(\phi\right)\delta\left(r-\frac{a^2}{r_0}\right)\delta\left(\theta-\frac{\pi}{2}\right)\hat{\phi}$$

and the radius of image current is $=\frac{a^2}{r_0}$. This is just is an approximate solution. The **Green's Function** solution of the wave equation is the right one to use. The method we use for magnetostatics cannot be applicable for radiation because it cannot satisfy the boundary condition, potential on the surface of the sphere should be **constant.**

Now this is just a superposition problem. Find the radiated magnetic fields due to both the currents at any point$(r>a)$ and add them vectorially. We can now evaluate magnetic vector potential$\left(\vec{A}\right)$ due to a circular loop in spherical coordinates as (**Note: source coordinates are primed**)[**Reference: Pg 235,Balanis, Antenna Theory, Analysis, and Design 3rd edition**].

$$A_0\left(r,\phi,\theta\right)=\frac{\mu_0}{4\pi}\left(\iiint_V J\left(r',\theta',\phi'\right)\frac{e^{-j\beta R}}{R}dv\right)\hat{\phi}$$

$$A_0\left(r,\phi,\theta\right)=\frac{\mu_0 r_0}{4\pi}\left(\int_0^{2\pi} I_0\left(\phi'\right)\left(\hat{\phi}\cdot\hat{\phi}'\right)\frac{e^{-j\beta R}}{R}d\phi'\right)\hat{\phi}$$

$$A_0\left(r,\phi,\theta\right)=\frac{\mu_0 r_0}{4\pi}\left(\int_0^{2\pi} I_0\left(\phi'\right)cos\left(\phi-\phi'\right)\frac{e^{-j\beta R}}{R}d\phi'\right)\hat{\phi}$$

where $R=\sqrt{r^2+r_0^2-2rr_0 sin\theta cos(\phi-\phi')}$

Similarly for image current,

$$A_1\left(r,\phi,\theta\right)=-\left(\frac{a^3}{r_0^2}\right)\left(\frac{\mu_0}{4\pi}\right)\left(\int_0^{2\pi} I_0\left(\phi'\right)cos\left(\phi-\phi'\right)\frac{e^{-j.\beta.R_1}}{R_1}d\phi'\right)\hat{\phi}$$

where $R_1=\sqrt{r^2+\left(\frac{a^2}{r_0}\right)2-2r\left(\frac{a^2}{r_0}\right)sin\theta cos(\phi-\phi')}$

So, the net magnetic vector potential at a point$(r>a)$ is

$$\overrightarrow{A}_{net} = \overrightarrow{A_0} + \overrightarrow{A_1}$$

Now Magnetic field is given by, $B = \nabla \times A$. Since $\overrightarrow{A}$ has only $\widehat{\phi}$ component,

$$\nabla \times A = \frac{1}{rsin\theta} \left(\frac{\partial (A_\phi sin\theta)}{\partial \theta} \right) \widehat{r} - \frac{1}{r} \left(\frac{\partial (r.A_\phi)}{\partial r} \right) \widehat{\theta}$$

Since only $\widehat{\theta}$ component of the magnetic field is tangential to the sphere and involved in calculating the currents, we only use it

Magnetic field for first loop is given by

$$H_\theta = -\frac{1}{\mu_0 r} \left(\frac{\partial (r.A_\phi)}{\partial r} \right) \widehat{\theta}$$

$$H_\theta = -\frac{1}{r} \left(\frac{\partial (r.\frac{r_0}{4\pi} \left(\int_0^{2\pi} I_0\left(\phi'\right) cos\left(\phi - \phi'\right) \frac{e^{-j\beta R}}{R} d\phi' \right))}{\partial r} \right) \widehat{\theta}$$

$$H_\theta = -\frac{1}{r} \left(\frac{r_0}{4\pi} \int_0^{2\pi} I_0\left(\phi'\right) cos\left(\phi - \phi'\right) \frac{\partial (r.\frac{e^{-j\beta R}}{R})}{\partial r} d\phi' \right) \widehat{\theta}$$

First evaluate $\frac{\partial (r.\frac{e^{-j\beta R}}{R})}{\partial r}$

$$\frac{\partial (r.\frac{e^{-j\beta R}}{R})}{\partial r} = \frac{e^{-j\beta R}}{R} + r. \left(\frac{Re^{-j\beta R}\left(-j\beta\right) - e^{-j\beta R}}{R^2} \right) \left(\frac{r - r_0 cos\left(\phi - \phi'\right) sin\theta}{R} \right)$$

$$\Rightarrow \frac{\partial (r.\frac{e^{-j\beta R}}{R})}{\partial r} = \frac{e^{-j\beta R}}{R} \left(1 - r \left(j\beta + \frac{1}{R} \right) \left(\frac{r - r_0 cos\left(\phi - \phi'\right) sin\theta}{R} \right) \right)$$

$$\Rightarrow -\frac{1}{r}. \frac{\partial (r.\frac{e^{-j\beta R}}{R})}{\partial r} = \frac{e^{-j\beta R}}{R} \left(-\frac{1}{r} + \left(j\beta + \frac{1}{R} \right) \left(\frac{r - r_0 cos\left(\phi - \phi'\right) sin\theta}{R} \right) \right)$$

That gives

$$H_\theta\left(r, \phi, \theta\right) = \frac{r_0}{4\pi} \left(\int_0^{2\pi} I_0\left(\phi'\right) cos\left(\phi - \phi'\right) \frac{e^{-j\beta R}}{R} \left(-\frac{1}{r} + \left(j\beta + \frac{1}{R} \right) \left(\frac{r - r_0 cos\left(\phi - \phi'\right) sin\theta}{R} \right) \right) d\phi' \right) \hat{\theta}$$

Field for image current is given by

$$H_{\theta 1}\left(r, \phi, \theta\right) = -\frac{1}{4\pi} \left(\frac{a^3}{r_0^2} \right) \left(\int_0^{2\pi} I_0\left(\phi'\right) cos\left(\phi - \phi'\right) \frac{e^{-j\beta R_1}}{R_1} \left(-\frac{1}{r} + \left(j\beta + \frac{1}{R_1} \right) \left(\frac{r - \left(\frac{a^2}{r_0} \right) cos\left(\phi - \phi'\right) sin\theta}{R_1} \right) \right) \right)$$

Since there are no fields inside the sphere, the surface currents($\overrightarrow{K}$) on the sphere is given by

$$\overrightarrow{K}(\phi,\theta) = \left[\overrightarrow{H_\theta} - \overrightarrow{H_{\theta 1}}\right]_{r=a}$$

for this arrangement, $\overrightarrow{K}$ is given by $\left[\overrightarrow{H_\theta}(r+dr,\phi,\pi/2) - \overrightarrow{H_\theta}(r-dr,\phi,\pi/2)\right]_{r=r_0}$

Let's validate this with a cirular loop antenna without the grounded sphere, so there is no image current. The antenna is placed in x-y plane. Take the current distribution on the antenna as

$$I_0(r,\theta,\phi) = cos(\phi)\,\delta(r-r_0)\,\delta\left(\theta-\frac{\pi}{2}\right)\hat{\phi}$$

$\overrightarrow{H_\theta}\left(\theta=\frac{\pi}{2}\right) = \overrightarrow{H_{\theta_1}}\left(\theta=\frac{\pi}{2}\right) = 0$ and To apply this(Ampere's Law) to the loop we also need to calculate the Displacement current$\left(\overrightarrow{D}\right)$.

$$\oint_C H.dl = \iint_S \left(J + \frac{\partial D}{\partial t}\right)\cdot ds$$

$$\oint_C H.dl = \iint_S \left(J + \varepsilon_0\frac{\partial E}{\partial t}\right)\cdot ds$$

$$\oint_C H.dl = \iint_S \left(J - j\omega\varepsilon_0\frac{\partial A}{\partial t}\right)\cdot ds$$

That gives

$$\iint_S J\cdot ds = I_0(r,\theta,\phi) = \oint_C H_\theta.dl - \iint_S \left(j\omega\varepsilon_0\frac{\partial A_\phi}{\partial t}\right)\cdot ds$$

We descretize this problem by taking $I_0(\phi)$ as vector of 360 elements. Therfore ϕ is divided into 360 divisions. For calculating Magnetic field at every grid point, it becomes a matrix problem

$$H_\theta(r,\phi_i,\theta) = \frac{r_0}{4\pi}\sum_j I_0(\phi_j)\,cos(\phi_i-\phi_j)\,\frac{e^{-j\beta R_{ij}}}{R_{ij}}\left(-\frac{1}{r} + \left(j\beta + \frac{1}{R_{ij}}\right)\left(\frac{r - r_0 cos(\phi_i-\phi_j)\,sin\theta}{R_{ij}}\right)\right)$$

where $R_{ij} = \sqrt{r^2 + r_0^2 - 2rr_0 sin\theta cos(\phi_i-\phi_j)}$

Take $k_{ij} = \frac{r_0}{4\pi}cos(\phi_i-\phi_j)\frac{e^{-j\beta R_{ij}}}{R_{ij}}\left(-\frac{1}{r} + \left(j\beta + \frac{1}{R_{ij}}\right)\left(\frac{r-r_0cos(\phi_i-\phi_j)sin\theta}{R_{ij}}\right)\right)$
and H_θ vector(dimensions 360x1) is given by, $K = [k_{ij}]$

$$H_\theta(r,\phi_i,\theta) = KI_i$$

For Displacement Current

$$I_D = \iint_S \left(-j\omega\varepsilon_0\frac{\partial A_\phi}{\partial t}\right)\cdot ds$$

$$I_D = \iint_S \left(-\omega^2 \varepsilon_0 A_\phi\right) \cdot ds$$

Since calculation of Displacement current exactly is cumbersome, we approximate it by taking average of two values at at distance $\triangle r$ on either side of the loop. Take area of cross-section for discretizing as $\triangle s = 2 \cdot r_0 \cdot \triangle\theta \cdot \triangle r$

$$I_{D_j} = -\omega^2 \varepsilon_0 A_{\phi_i} \cdot \triangle s$$

$$I_{D_j} = -\omega^2 \varepsilon_0 \left(\frac{A_{\phi_i}\left(r_0 + \triangle r, \phi_i, \theta\right) + A_{\phi_i}\left(r_0 - \triangle r, \phi_i, \theta\right)}{2} \right) (2r_0) \cdot \left(\frac{2\triangle\theta}{2} \right) \cdot \triangle r$$

$$I_{D_j} \simeq -\omega^2 \varepsilon_0 \left(A_{\phi_i}\left(r_0 + \triangle r, \phi_i, \theta\right) + A_{\phi_i}\left(r_0 - \triangle r, \phi_i, \theta\right) \right) (r_0) \cdot (\triangle\theta) \cdot \triangle r$$

$$I_{D_i} = d_{ij} I_j$$

where $D = [d_{ij}]$ is **Displacement current Matrix.**

We can neglect the contribution due to displacement current as it is order of $10^{-3}$of the original current. We will prove this after the simulation. So the Ampere's law around the loop becomes

$$I(\phi_i) = H_\theta \left(r_0 + \triangle r, \phi_i, \frac{\pi}{2}\right)(r_0 + \triangle r) - H_\theta \left(r_0 - \triangle r, \phi_i, \frac{\pi}{2}\right)(r_0 - \triangle r)$$

Let

$$b_{ij} = k_{ij}\left(r + \triangle r, \phi_j, \theta\right)(r + \triangle r) - k_{ij}\left(r - \triangle r, \phi_j, \theta\right)(r - \triangle r)$$

where $B = [b_{ij}]$. Call 'B' as **Estimation Matrix**.

Therefore current inside the circular loop antenna can be estimated as

$$I_i = b_{ij} I_j$$

We simulate this using the following **python code**.

NOTE: There are two problems with my current code.

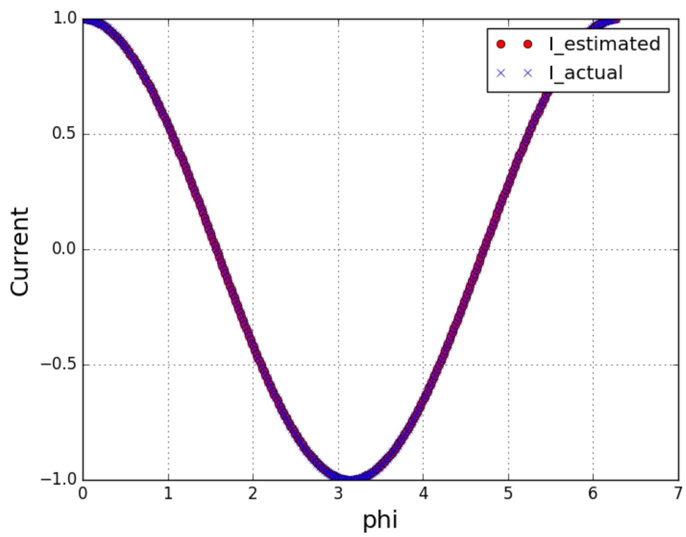
1. There is a spatial phase shift of 2π in the fields due to the indexing which I cannot debug. But I corrected it with rotating the obtained Estimation matrix by a spatial shift of 2π. It is named as 'Rot_matrix' in the code. It is an identity matrix of order $n \times n$ rotated $\frac{n}{2}$ times row wise.
2. There also seen to be a need for a scaling factor of **1.4983731750083773** to make atleast one of the eigen values go on to the unit circle and for the Estimated and actual currents to match.

```
import numpy as np
import pylab as pl
import matplotlib.pyplot as plt
scaling_factor = 1.4989436575087889
```

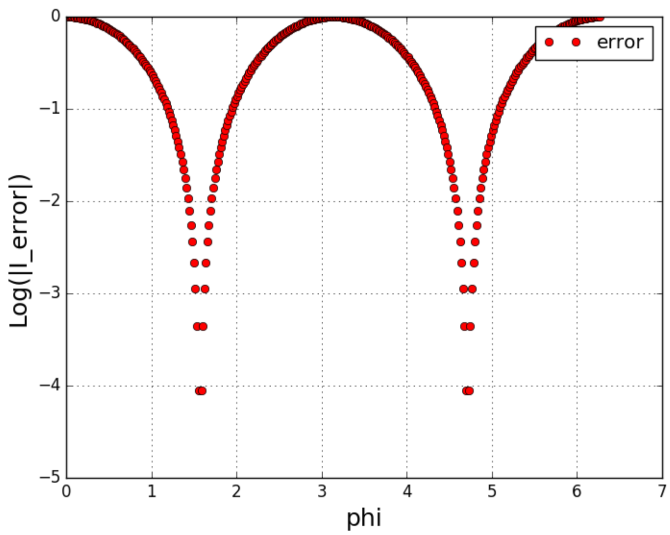
```
n = 360         # n = size(phis), number of phi (or) theta divisions
mu = 4*pl.pi*10**-7
eps = 8.85*10**-12
c = 3*10**8
phis = pl.linspace(0,2*pl.pi,n+1)  # 'phi' array
phis = phis[:-1]
dphis = phis[1]-phis[0];
thetas = pl.linspace(0,pl.pi,n+1)  # 'theta' array
thetas = thetas[:-1]
dthetas = thetas[1]-thetas[0]
r0 = 1.0    # radius of antenna
N = 1#0.61*2*pl.pi  # dimension of the antenna , 2*pi*r0 = N * wavelength
w0 = N*c/r0
k = 2                              # 1- full wave, 2- half wave, 4-quarter wave
I = pl.zeros(pl.size(phis))
# current array initialization for arbitary currents
iphis = phis[0 : n/(k*N)]
# 'iphis' are the values of phi where current is zero
I[ 0:n/(k*N) ] = pl.cos(N*iphis)
#I = pl.ones(pl.size(phis))
# current array initialization for constant current
H = pl.zeros((n,n),dtype=np.complex)# H_theta = H*I
E = pl.zeros((n,n),dtype=np.complex)# E_phi = E*I
theta = pl.pi/2
r = (1+2*pl.pi/n)*r0
r1 = (1-2*pl.pi/n)*r0
for i in range(pl.size(phis)):

    for j in range(pl.size(phis)):

        R = (r**2+r0**2+2.0*r*r0*(pl.cos(phis[i]-phis[j]))*(pl.sin(theta)))**(0.5)
        R1= (r1**2+r0**2+2.0*r1*r0*(pl.cos(phis[i]-phis[j]))*(pl.sin(theta)))**(0.5)
        E[i,j] = (np.cos(phis[i]-phis[j]))*(((np.exp(1j*N*R))/R)+ \
           ((np.exp(1j*N*R1))/R1))/2
        H[i,j] = (np.cos(phis[i]-phis[j]))*((np.exp(1j*N*R))/R)*(-1/r + (1/R+1j*N)* \

           ((r-r0*(np.cos(phis[i]-phis[j]))*(np.sin(theta)))/R))*r*dthetas*2- \
           (np.cos(phis[i]-phis[j]))*((np.exp(1j*N*R1))/R1)*(-1/r1 + (1/R1+1j*N)* \
           ((r1-r0*(np.cos(phis[i]-phis[j]))*(np.sin(theta)))/R1))*r1*dthetas*2

H = scaling_factor*(pl.pi/2)*(r0/(4*pl.pi))*H*dphis    #  H_theta = H*I
E = -1j*w0*mu*(r0/(4*pl.pi))*E*dphis # Displacement    # E_phi = E*I
D = -scaling_factor*1j*w0*eps*E*r0*(r-r1)*dthetas*2    # I_D_j = D*j
ID_n = np.identity(n)
Rot_matrix = -np.roll(ID_n,n/2,axis=0)
Est_matrix = scaling_factor*np.matmul(Rot_matrix,H-D)
```

```
Iest = np.matmul(Est_matrix,I)
# H_discontinuity at every theta or Iest
#Iest = np.matmul(H,I)-np.matmul(D,I)
# H_discontinuity at every theta or Iest
EV,EV_matrix = np.linalg.eigvals(Est_matrix) #eigen values
eigen_real = EV.real
eigen_img = EV.imag
#plt.plot(eigen_real,eigen_img,'o',color='red')
plt.plot(phis,Iest,'o',color='red',label='I_estimated')
plt.xlabel("phi ")
plt.ylabel("Current")
plt.plot(phis,I,'x',color='blue',label='I_actual')
plt.legend(loc='upper right')
plt.grid()
plt.show()
```

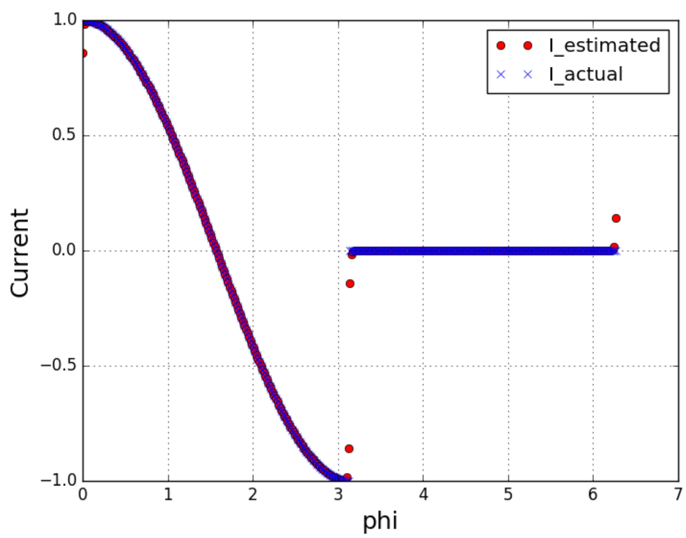
*Use k =1 for fullwave current, k=2 for halfwave and k=4 for quarter wave current in line no.13 of the code



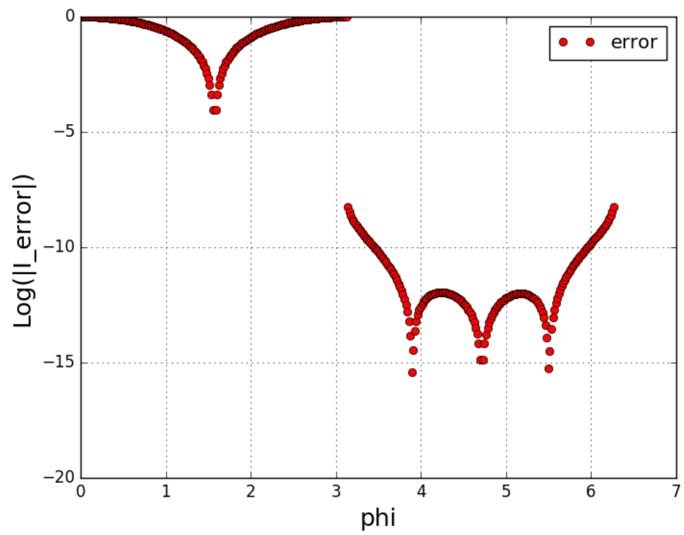
(a) Full wave current



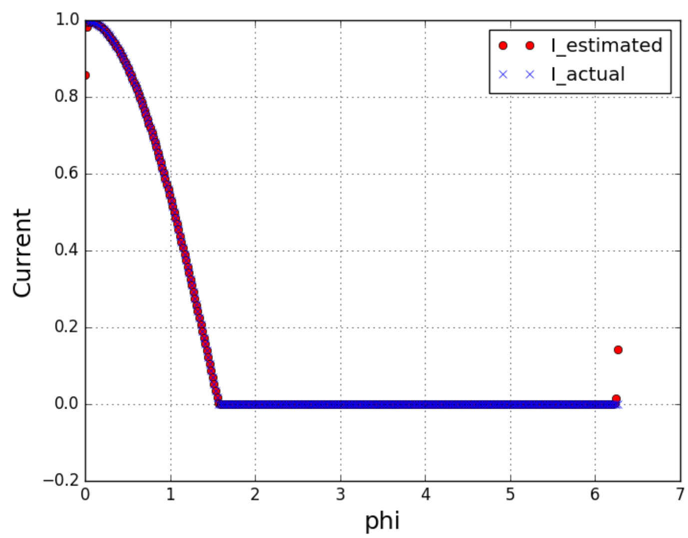
(b) Full wave current Error



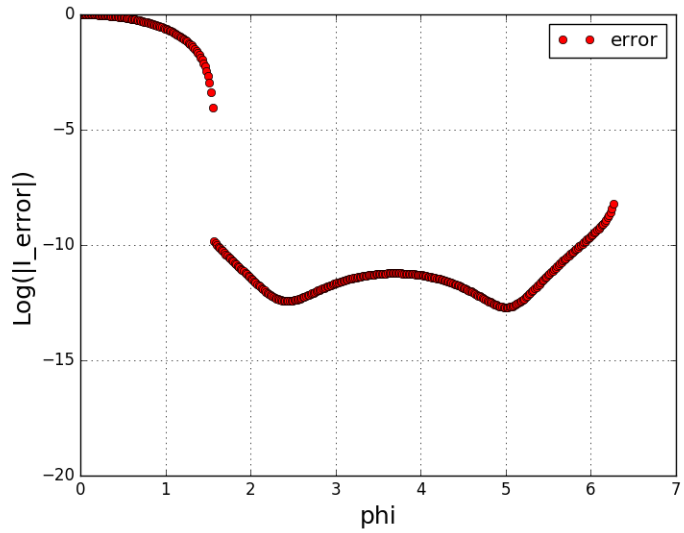
(c) Halfwave current



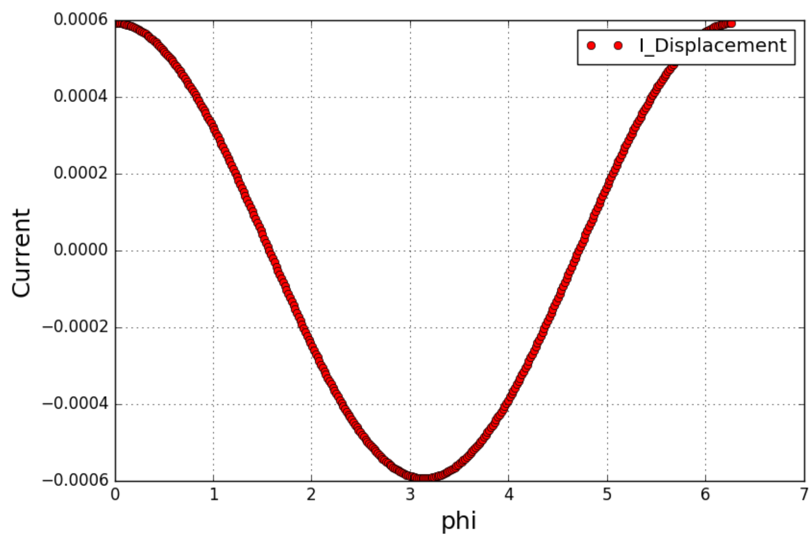
(d) Half wave current Error



(e) Quarter wave current



(f) Quarter wave current Error



(g) Displacement Current for a Full wave current Input

Inferences

- The estimated current agrees with the actual current.

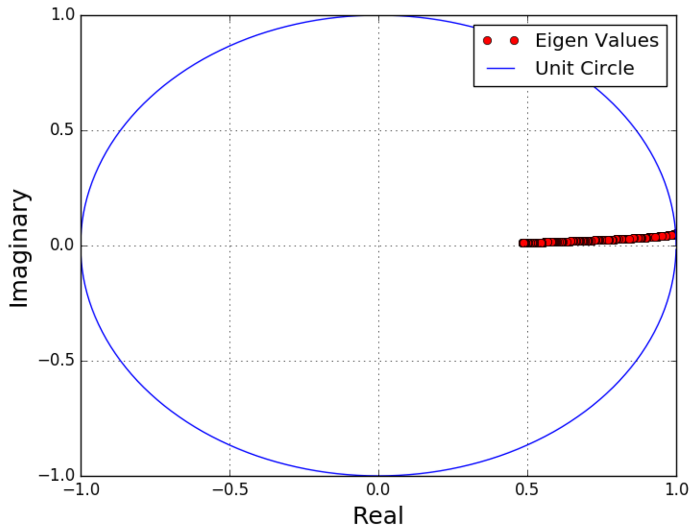
4 Estimation Matrix(B)

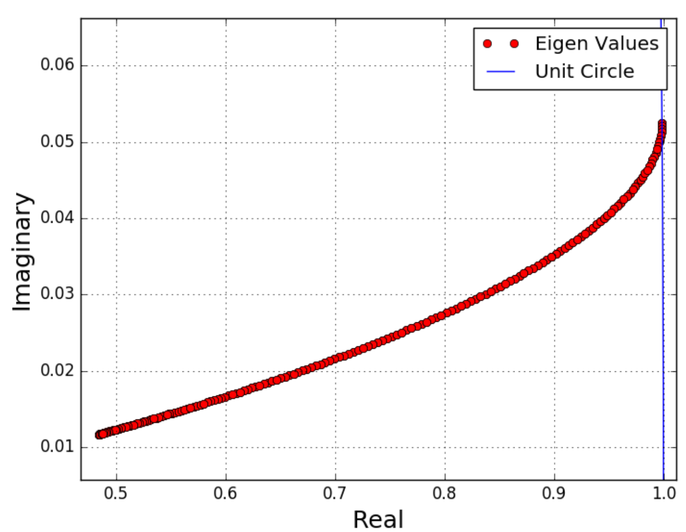
Solving for eigen values of B

$$B - \lambda I = 0$$

where I is identity matrix of order $n \times n$.

Eigen values of B when plotted along with unit circle, look like

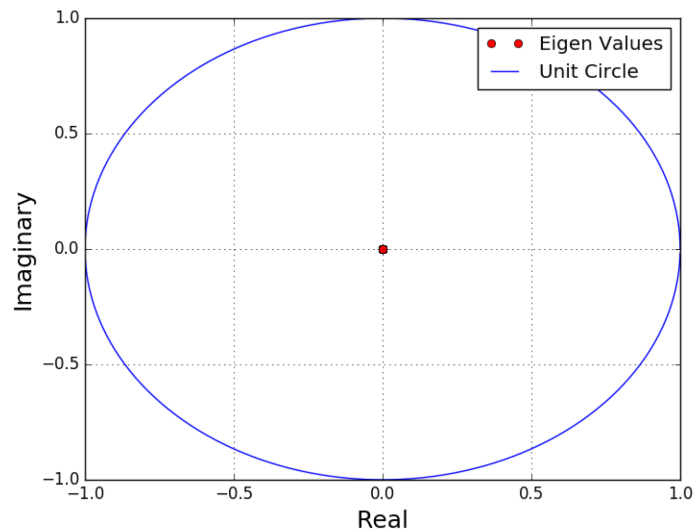


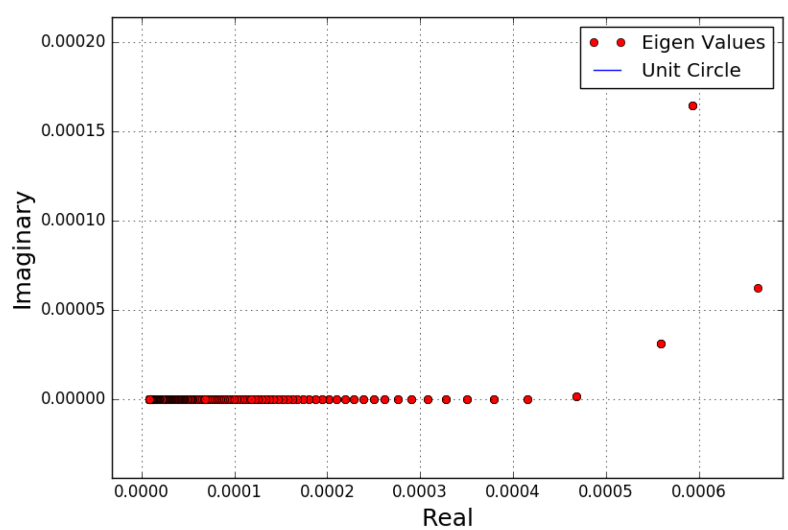


Rank of matrix **B** is 'n'**.** It is a **row full rank** matrix.All eigen values lie inside the unit circle.So, the system is stable.Maximum of the eigen values is

$\lambda_{max} = (0.99862427481785354 + 0.052436225498374532j) \approx 1$. We expect all the harmonics of $cos(\phi)$ to be the solutions. It implies that **B matrix** should have a low rank than n. That implies most of the eigen values to be either zero or unity. But that is not the case, we do not know why it is happening.

Now we will see the eigen values corresponding to the **Displacement matrix.**





The eigen values of Displacement current matrix seem to be concentrated near origin, they have a very less contribution to the displacement current. The very less value of the this may be because of the small size of the cross section around the loop we're calculating displacement current. We have to verify it varying the cross sectional area of the loop integral for displacement current.

The actual estimation matrix should be **(B-D)**. Since the eigen values of **Displacement Matrix (D) are very close to zero, they have a very little effect on the eigen values of (B-D) matrix and most of the eigen values of (B-D) matrix lie very close the eigen values of B matrix.** The eigen modes supported by this system are

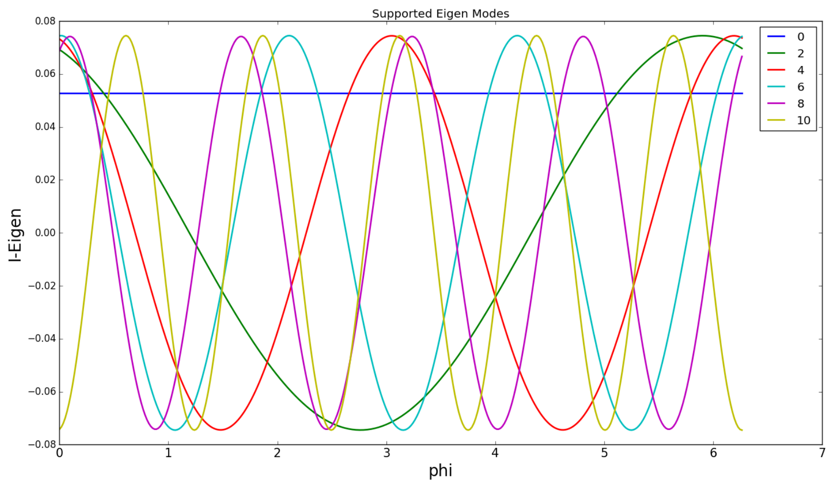


Figure: Eigen Modes supported by this system

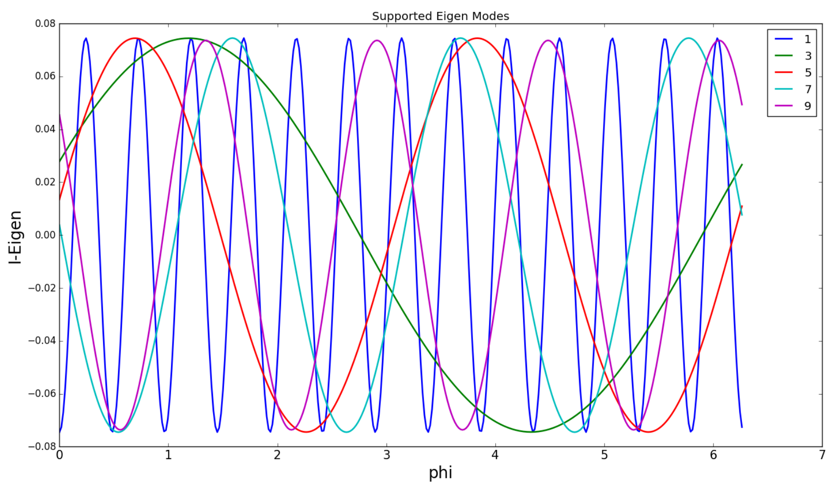


Figure: Eigen Modes supported by this system

5 Directivity

The fields calculated are agreeing with literature

[Reference: Pg 235,Balanis, Antenna Theory, Analysis, and Design 3rd edition]

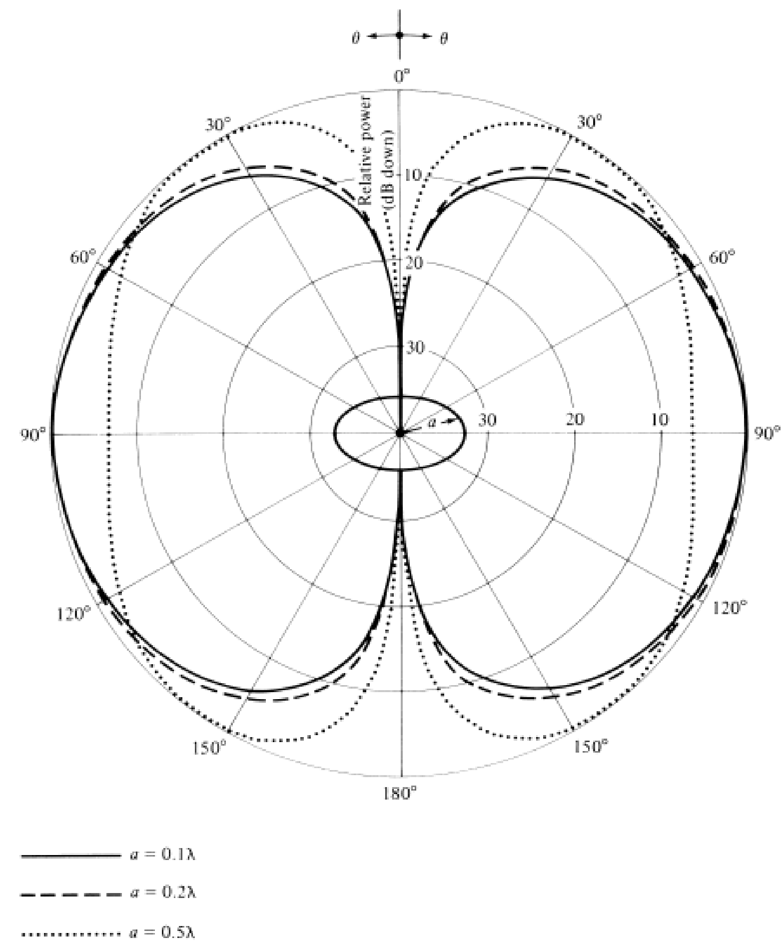


Figure 5.7 Elevation plane amplitude patterns for a circular loop of constant current ($a = 0.1\lambda$, 0.2λ, and 0.5λ).

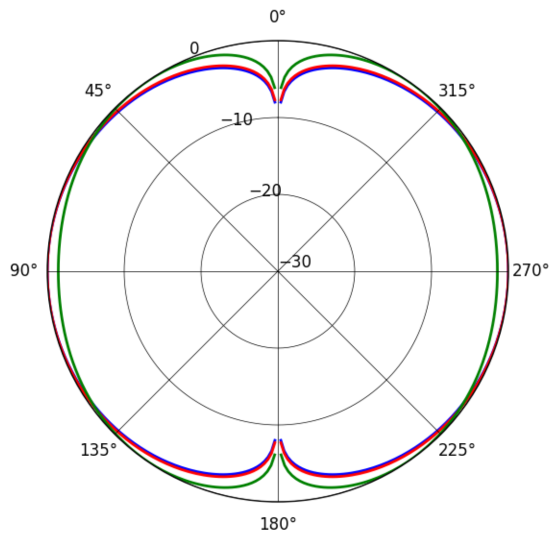


Figure: Normalized Elevation plane patterns(dB) for constant current from 0.1λ to 0.5λ (from left to right, blue to green)

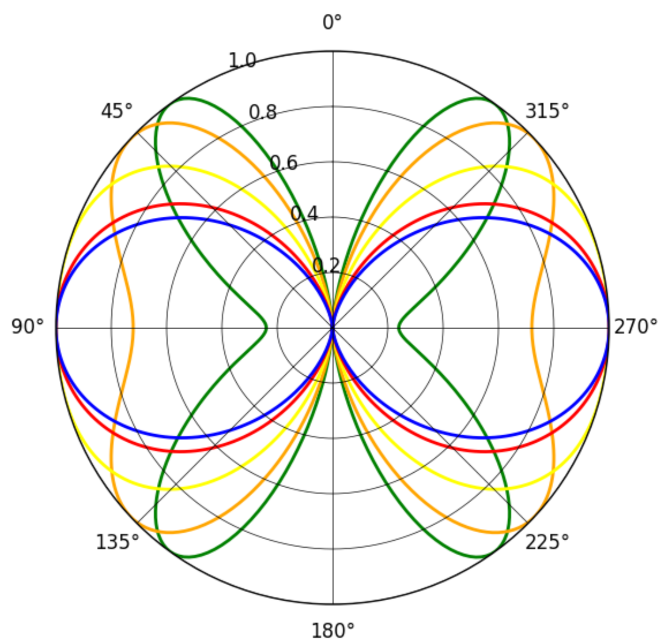
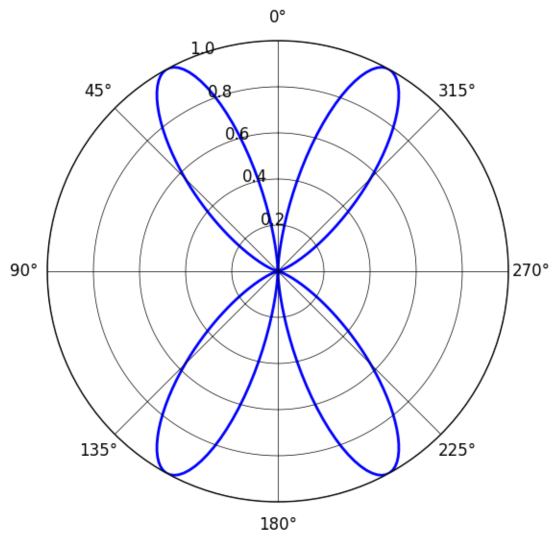


Figure: Normalized Elevation plane patterns(linear) for constant current from 0.1λ to 0.5λ (from left to right, blue to green)



$Figure : r_0 = 0.61\lambda$

As the radius goes on increasing from 0.1λ, the field intensity along the plane of the loop($\theta = 90^o$) diminishes and eventually forms a null when $r_0 \simeq 0.61\lambda$. Beyond $r_0 \simeq 0.61\lambda$, the radiation along the plane of the loop begins to intensify and the pattern attains a multilobe form.

[Reference: Pg 249,Balanis, Antenna Theory, Analysis, and Design 3rd edition]

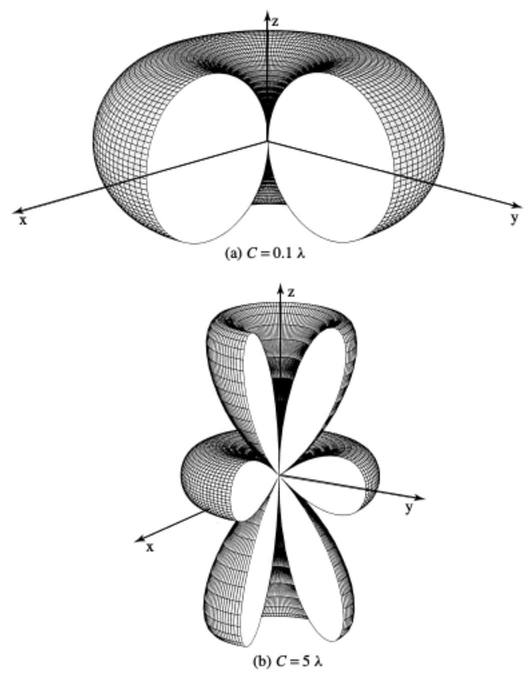
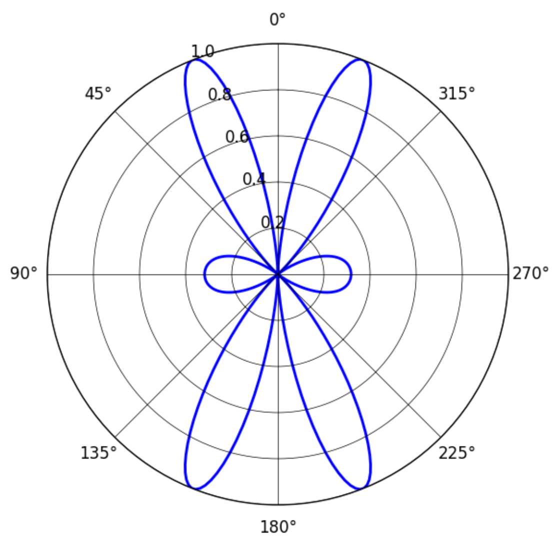


Figure 5.8 Three-dimensional amplitude patterns of a circular loop with constant current distribution.



$$C = 5\lambda$$

The above plot of normalized fields agrees with Figure 5.8

6 PEC Sphere

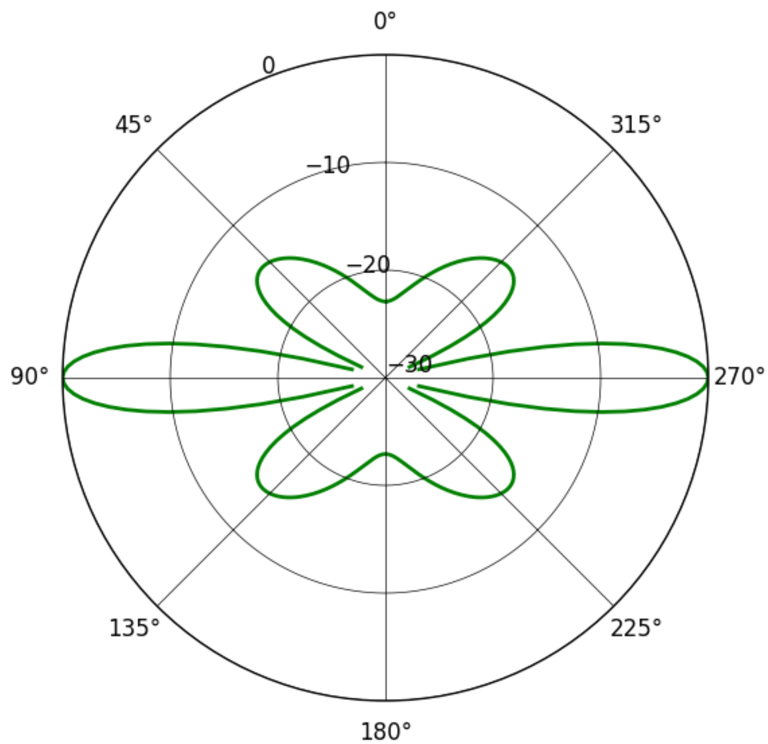


Figure: Elevation plane pattern(in dB) for PEC sphere surrounded by a circular loop antenna($2\pi r_0 = \lambda$)