Asymptotic properties of Minimax risk for estimating the transition probabilities of Markov chains

A Project Report

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THESIS CERTIFICATE

This is to certify that the thesis titled **Asymptotic properties of Minimax risk for estimating the transition probabilities of Markov chains**, submitted by **Sukhdeep Singh Kahlon**, to the Indian Institute of Technology, Madras, for the award of the degree of **Dual degree(B.Tech + M.Tech)**, is a bona fide record of the research work done by him under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

KEYWORDS: Two state Markov chain ; $Minimax$ risk.

Markov chains are used in a broad variety of academic fields, ranging from biology to economics. Markov chains are commonly used for modeling complex systems due to its simplicity and effectiveness. The objective is to bound the $minimax$ risk of estimating the parameters of a two state Markov chain. We propose various risk metrics and bounded the $minimax$ risk for these metrics. Firstly, the most popular quadratic risk metric is consider for all parameters of the Markov chain distribution and its minimax risk is shown to be lower bounded by a constant making it less insightful. We then define two other risk metrics and present both lower and upper bounds. We modify the Le Cam's method into a more generalized version in order to determine lower bounds.

TABLE OF CONTENTS

NOTATION

α	Transition probability from state(0) to state(1)
β	Transition probability from state(1) to state(0)
$\mathcal{P}$	Set of all distributions of n observation form 2 state ergodic Markov chain
P	A particular distributions for n observation form 2 state ergodic Markov chain. $P \in \mathcal{P}$
$\hat{\alpha}$	Estimator of α
$\hat{\beta}$	Estimator of β
$\hat{P}$	Estimator for $\{\alpha, \beta\}$
R_n^*	$Minimax$ risk
π_0	Steady state probability of state(0)
π_1	Steady state probability of state(1)

CHAPTER 1

INTRODUCTION

This study focus on the rate at which Markov chain can be learned. We define the rate in terms of number of samples required for the worst case error to be less than some value.

1.1 Problem setting

Consider the following ergodic two state Markov chain $(\alpha \neq 0, \beta \neq 0)$.

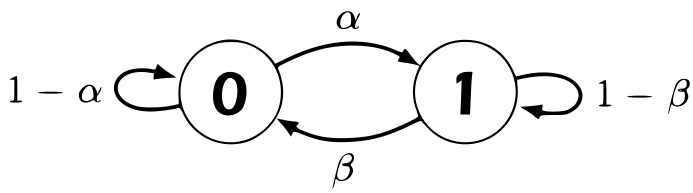

Figure 1.1: Two state Markov chain

We are interested in finding the $minimax$ risk associated with the estimation of parameters α and β given n observations.

Let $\hat{\alpha} = \hat{\alpha}(X_1, ..., X_n)$ denotes an estimator for α where $X_1, ..., X_n$ are the n continuous observations from the two state Markov chain. Similarly $\hat{\beta} = \beta(X_1, ..., X_n)$ denotes an estimator for β and $\hat{P}$ be the set of all estimators for $\{\alpha, \beta\}$

Let $R_n(P, \hat{P})$ be the risk function for n observations, then the $minimax$ risk is

$$R_n^* = \inf_{\hat{P}} \sup_{P \in \mathcal{P}} R_n \tag{1.1}$$

1.2 Various Risk functions used

Firstly, quadratic loss function is considered. The risk is defined as follows:

$$R_n = E\Big[\, \big|\hat{\alpha} - \alpha\big|^2 + \big|\hat{\beta} - \beta\big|^2 \Big] \tag{1.2}$$

The $minimax$ risk is

$$R_n^* = \inf_{\hat{p}} \sup_{P \in \mathcal{P}} E\Big[\big|\hat{\alpha} - \alpha\big|^2 + \big|\hat{\beta} - \beta\big|^2 \Big] \tag{1.3}$$

The bounds on R_n^* are found and shown below

$$0.005 \leq R_n^* \leq 0.5 \tag{1.4}$$

The lower bound is found using Le Cam method and the upper bound is found by taking a constant estimator i.e. $\hat{\alpha} = \hat{\beta} = \frac{1}{2}$

The $minimax$ risk does not approach to zero as n approach to infinity. This suggest that worst case performance of every estimator is poor. The distributions which have very small α or β are the one which are very difficult to estimate, this is because if α is very small then probability of entering state(1) is very small which makes estimating β very difficult. Similarly, if β is very small then probability of entering state(0) is very small which makes estimating α very difficult.

Next a new risk metric is proposed defined as follows:

$$R_n = E\Big[\beta\big|\hat{\alpha} - \alpha\big| + \alpha\big|\hat{\beta} - \beta\big| \Big] \tag{1.5}$$

This risk metric take care of the problem faced previously, i.e. when β is very less then estimating α is difficult which gives high error. So we have scaled the error term of α by β.

$Minimax$ risk corresponding to this error metric is defined as follows:

$$R_n^* = \inf_{\hat{p}} \sup_{P \in \mathcal{P}} E\Big[\beta\big|\hat{\alpha} - \alpha\big| + \alpha\big|\hat{\beta} - \beta\big| \Big] \tag{1.6}$$

The bounds found on R_n^* are

$$\frac{1}{16\sqrt{6n}} - o\Big(\frac{1}{\sqrt{n}}\Big) \leq \; R_n^* \; \leq \frac{4}{\sqrt{n}} + o\Big(\frac{1}{\sqrt{n}}\Big) \tag{1.7}$$

For lower bound the Lecam method cannot be used directly due to α and β terms present outside the mod terms in the risk metric $E\Big[\beta\big|\hat{\alpha} - \alpha\big| + \alpha\big|\hat{\beta} - \beta\big| \Big]$. Le Cam's method is modified to accommodate this change. For upper bound, empirical estimator is used.

The last risk metric is defined as

$$R_n = E\left[\sum_{x_{n+1}=0}^{1} \left| p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n) \right|^2 \right] \tag{1.8}$$

where $p(X_{n+1}|X^n)$ is the conditional probability of X_{n+1} given X^n the n observations. Note that for a two state Markov chain $p(X_{n+1}|X^n) = p(X_{n+1}|X_n)$

The $minimax$ risk for this risk metric is defined as

$$R_n^* = \inf_{\hat{p}} \sup_{P \in \mathcal{P}} E\left[\sum_{x_{n+1}=0}^{1} \left| p(x_{n+1}|X^n) - \hat{p}(x_{n+1}|X^n) \right|^2 \right] \tag{1.9}$$

and the bounds for the above risk are

$$\frac{2}{81n} - o\Big(\frac{1}{n}\Big) \leq R^* \leq \frac{\log\big(\log(n)\big)}{n} + o\bigg(\frac{\log\log(n)}{n}\bigg) \tag{1.10}$$

The upper bound and lower bound are not of the same order. Hence there is scope of improvement.

CHAPTER 2

Le Cam method modification

Le Cam method is a popular method used for finding lower bounds on $minimax$ risks. Le Cam method cannot be applied directly on the risk functions we have considered. A more generalized version of Le cam method is found and used to find the lower bounds on the risk functions we have considered. First we will describe the Le Cam method.

2.1 Le Cam's method

Let $\mathcal{P}$ be a set of distributions and let $X_1, ..., X_n$ be a sample from some distribution $P \in \mathcal{P}$. Let $\theta = \theta(P)$ be some function of P. Let $\hat{\theta} = \hat{\theta}(X_1, ..., X_n)$ denotes an estimator and d is some metric distance satisfying triangular inequality and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non decreasing function with $\Phi(0) = 0$, then the $minimax\ risk$ is

$$R_n^* = \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} E_P \Big[\Phi(\, d\big(\hat{\theta}, \theta)\,) \Big] \tag{2.1}$$

Le Cam method states that:
For any pair $P_0, P_1 \in \mathcal{P}$, Let $\Delta = \frac{d\big(\theta(P_0), \theta(P_1)\big)}{2}$ then

$$R_n^* \geq \frac{1}{2}\,\Phi(\Delta) \Big[1 - ||P_0 - P_1||_{TV} \Big] \tag{2.2}$$

We cannot directly use Le Cam method to calculate the lower bounds for the risk functions which we have defined as they are not of the form (2.1). Therefore we need to modify the Le Cam method.

2.2 Modified Le Cam's method

Let $\mathcal{P}$ be a set of distributions and let $X_1, ..., X_n$ be a sample from some distribution $P \in \mathcal{P}$. Let $\theta_1 = \theta_1(P)$, $\theta_2 = \theta_2(P)$, $\gamma_1 = \gamma_1(P)$ and $\gamma_2 = \gamma_2(P)$ be some functions of P. Let $\hat{\theta}_1 = \hat{\theta}_1(X_1, ..., X_n)$ and $\hat{\theta}_2 = \hat{\theta}_2(X_1, ..., X_n)$ are the estimators for

θ_1 and θ_2 respectively. d is some metric distance satisfying triangular inequality and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non decreasing function with $\Phi(0) = 0$, then the $minimax\ risk$ is

$$R_n^* = \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} E_P \Big[\gamma_1 \, \Phi(\, d\big(\hat{\theta}_1, \theta_1\big) \,) + \gamma_2 \, \Phi(\, d\big(\hat{\theta}_2, \theta_2\big) \,) \Big] \tag{2.3}$$

The lower bound on R_n^* is give by

Theorem 1: For any pair $P_0, P_1 \in \mathcal{P}$, Let $\Delta_1 = \frac{d\big(\theta_1(P_0), \theta_1(P_1)\big)}{2}$ and $\Delta_2 = \frac{d\big(\theta_2(P_0), \theta_2(P_1)\big)}{2}$ then

$$\begin{aligned} R_n^* \geq & \frac{1}{2} \min \Big(\gamma_1(P_0), \gamma_1(P_1) \Big) \, \Phi(\Delta_1) \left[1 - \sqrt{\frac{1}{2} D_{KL}(P_0 || P_1)} \right] \\ & + \frac{1}{2} \min \Big(\gamma_2(P_0), \gamma_2(P_1) \Big) \, \Phi(\Delta_2) \left[1 - \sqrt{\frac{1}{2} D_{KL}(P_0 || P_1)} \right] \end{aligned} \tag{2.4}$$

Proof: Most of the steps are taken directly from Wasserman (2010) and John.
An estimator $\hat{\theta}_1$ defines a test static ψ_1, namely,

$$\psi_1(X_1, ..., X_n) = \begin{cases} 1, & \text{if } d(\hat{\theta}, \theta_1(P_0)) \geq d(\hat{\theta}, \theta_1(P_1)) \\ 0, & \text{if } d(\hat{\theta}, \theta_1(P_0)) < d(\hat{\theta}, \theta_1(P_1)) \end{cases} \tag{2.5}$$

Similarly, estimator $\hat{\theta}_2$ defines a test static ψ_2,

$$\psi_2(X_1, ..., X_n) = \begin{cases} 1, & \text{if } d(\hat{\theta}, \theta_2(P_0)) \geq d(\hat{\theta}, \theta_2(P_1)) \\ 0, & \text{if } d(\hat{\theta}, \theta_2(P_0)) < d(\hat{\theta}, \theta_2(P_1)) \end{cases} \tag{2.6}$$

If $P = P_0$ and $\Psi_1 = 1$ then

$$2\Delta_1 = d(\theta_1(P_0), \theta_1(P_1)) \leq d(\theta_1(P_0), \hat{\theta}_1(P_1)) + d(\theta_1, \hat{\theta}) \leq 2d(\theta_0, \hat{\theta}) \implies d(\theta_1(P_0)), \hat{\theta}) \geq \Delta_1$$

and so $\Phi\big(d(\theta_1(P_0),\hat{\theta})\big) \geq \Phi(\Delta)$. Hence

$$\begin{aligned} E_{P_0}\Big[\gamma_1(P_0)\,\Phi\big(\,d(\hat{\theta},\theta_1(P_0))\,\big)\Big] &\geq E_{P_0}\Big[\gamma_1(P_0)\,\Phi\big(d(\hat{\theta},\theta_1(P_0))\big)I(\Psi_1=1)\Big] \\ &\geq \gamma_1(P_0)\,\Phi(\Delta_1)\,E_{P_0}\big[I(\Psi_1=1)\big] \\ &= \gamma_1(P_0)\,\Phi(\Delta_1)\,P_0(\Psi_1=1) \end{aligned} \tag{2.7}$$

similarly,

$$E_{P_1}\Big[\gamma_1(P_1)\,\Phi\big(\,d(\hat{\theta},\theta_1(P_1))\big)\Big] \geq \gamma_1(P_1)\Phi(\Delta_1)P_1(\Psi_1=0) \tag{2.8}$$

$$E_{P_0}\Big[\gamma_2(P_0)\,\Phi\big(\,d(\hat{\theta},\theta_2(P_0))\big)\Big] \geq \gamma_2(P_0)\Phi(\Delta_2)P_0(\Psi_2=1) \tag{2.9}$$

$$E_{P_1}\Big[\gamma_2(P_1)\,\Phi\big(\,d(\hat{\theta},\theta_2(P_1))\big)\Big] \geq \gamma_2(P_1)\Phi(\Delta_2)P_1(\Psi_2=0) \tag{2.10}$$

From (2.7) and (2.9)

$$\begin{aligned} R_{P_1} = E_{P_0}\Big[\gamma_1(P_0)\,\Phi\big(\,d(\hat{\theta},\theta_1(P_0))\,\big) + \gamma_2(P_0)\,\Phi\big(\,d(\hat{\theta},\theta_2(P_0))\big)\Big] &\geq \gamma_1(P_0)\,\Phi(\Delta_1)\,P_0(\Psi_1=1) \\ &+ \gamma_2(P_0)\Phi(\Delta_2)P_0(\Psi_2=1) \end{aligned} \tag{2.11}$$

From (2.8) and (2.10)

$$\begin{aligned} R_{P_2} = E_{P_1}\Big[\gamma_1(P_1)\,\Phi\big(\,d(\hat{\theta},\theta_1(P_1))\big) + \gamma_2(P_1)\,\Phi\big(\,d(\hat{\theta},\theta_2(P_1))\big)\Big] &\geq \gamma_1(P_1)\Phi(\Delta_1)P_1(\Psi_1=0) \\ &+ \gamma_2(P_1)\Phi(\Delta_2)P_1(\Psi_2=0) \end{aligned} \tag{2.12}$$

Taking the maximum of (2.11) and (2.12), we have

$$\begin{aligned}\sup_{P\in\mathcal{P}} R_P &\geq \max_{P\in P_0,P_1} R_P \\ &\geq \frac{R_{P_1}+R_{P_2}}{2} \\ &\geq \frac{1}{2}\Big(\gamma_1(P_0)\,\Phi(\Delta_1)\,P_0(\Psi_1=1)+\gamma_2(P_0)\Phi(\Delta_2)P_0(\Psi_2=1)\Big)+ \\ &\quad \frac{1}{2}\Big(\gamma_1(P_1)\Phi(\Delta_1)P_1(\Psi_1=0)+\gamma_2(P_1)\Phi(\Delta_2)P_1(\Psi_2=0)\Big) \\ &\geq \min(\gamma_1(P_0),\gamma_1(P_1))\Phi(\Delta_1)\Big[\frac{P_0(\Psi_1=1)+P_1(\Psi_1=0)}{2}\Big]+ \\ &\quad \min(\gamma_2(P_0),\gamma_2(P_1))\Phi(\Delta_2)\Big[\frac{P_0(\Psi_2=1)+P_1(\Psi_2=0)}{2}\Big]\end{aligned} \tag{2.13}$$

Taking the infimum over all estimators, we have

$$\begin{aligned}R_n^* = \inf_{\hat\theta}\sup_{P\in\mathcal{P}} R_P \geq \min(\gamma_1(P_0),\gamma_1(P_1))\Phi(\Delta_1)\inf_{\Psi_1}\left[\frac{P_0(\Psi_1=1)+P_1(\Psi_1=0)}{2}\right]+ \\ \min(\gamma_2(P_0),\gamma_2(P_1))\Phi(\Delta_2)\inf_{\Psi_2}\left[\frac{P_0(\Psi_2=1)+P_1(\Psi_2=0)}{2}\right]\end{aligned} \tag{2.14}$$

Using the result from John

$$\inf_{\Psi}\Big(P_0(\Psi=1)+P_1(\Psi=0)\Big) = 1-||P_0-P_1||_{TV} \tag{2.15}$$

We get

$$\begin{aligned}R_n^* &\geq \frac{1}{2}\min\Big(\gamma_1(P_0),\gamma_1(P_1)\Big)\,\Phi(\Delta_1)\,\Big[1-||P_0-P_1||_{TV}\Big] \\ &\quad+\frac{1}{2}\min\Big(\gamma_2(P_0),\gamma_2(P_1)\Big)\,\Phi(\Delta_2)\,\Big[1-||P_0-P_1||_{TV}\Big]\end{aligned} \tag{2.16}$$

We know

$$||P_0-P_1||_{TV}^2 \leq \frac{1}{2}D_{KL}(P_0||P_1) \tag{2.17}$$

Therefore

$$\begin{aligned}R_n^* &\geq \frac{1}{2}\min\Big(\gamma_1(P_0),\gamma_1(P_1)\Big)\,\Phi(\Delta_1)\,\left[1-\sqrt{\frac{1}{2}D_{KL}(P_0||P_1)}\right] \\ &\quad+\frac{1}{2}\min\Big(\gamma_2(P_0),\gamma_2(P_1)\Big)\,\Phi(\Delta_2)\,\left[1-\sqrt{\frac{1}{2}D_{KL}(P_0||P_1)}\right]\end{aligned} \tag{2.18}$$

CHAPTER 3

Minimax risk for the first risk metric

The first risk metric which we have used is quadratic loss. $Minimax$ risk is defined as follows:

$$R_n^* = \inf_{\hat{p}} \sup_{P} \; E\Big[\, \big|\hat{\alpha} - \alpha\big|^2 + \big|\hat{\beta} - \beta\big|^2 \Big] \tag{3.1}$$

We will first find the lower bound using modified Le Cam method and then derived the upper bound.

For applying (2.4), KL distance between two distributions is required. We will now find KL distance between two general distributions for two state Markov chain.

3.1 KL distance for two state Markov chain distribution

Let $P_0, P_1 \in \mathcal{P}$

P_0 : is defined by $\alpha_0, \beta_0, q_{00}$ and q_{01}

Where α_0, β_0 are the transition probabilities as shown in Figure 1.1, q_{00} is the initial probability of state(0) and q_{01} is the initial probability of state(1).

Similarly, P_1 : is defined by $\alpha_0, \beta_0, q_{10}$ and q_{11}

$$D_{KL}(P_0||P_1) = \sum_{x^n \in \{0,1\}^n} p_0(x^n) \log\left(\frac{p_0(x^n)}{p_1(x^n)}\right) \tag{3.2}$$

where $p_0(x^n) = p_0(x_1) \prod_{k=2}^{n} p_0(x_k|x_{k-1})$, similarly for $p_1(x^n)$

$$\begin{aligned} D_{KL}(P_0||P_1) &= \sum_{x^n \in \{0,1\}^n} p_0(x_1) \prod_{k=2}^{n} p_0(x_k|x_{k-1}) \log\left(\frac{p_0(x_1) \prod_{k=2}^{n} p_0(x_k|x_{k-1})}{p_1(x_1) \prod_{k=2}^{n} p_1(x_k|x_{k-1})}\right) \\ &= \sum_{x_1 \in \{0,1\}} p_0(x_1) \log\left(\frac{p_0(x_1)}{p_1(x_1)}\right) + \\ &\quad \sum_{k=2}^{n} \sum_{x_{k-1}, x_k \in \{0,1\}} p_0(x_{k-1}) p_0(x_k|x_{k-1}) \log\left(\frac{p_0(x_k|x_{k-1})}{p_1(x_k|x_{k-1})}\right) \end{aligned} \tag{3.3}$$

We get the second step in (3.3) by writing the product inside $\log$ function as sum and by summing over the variables which are not inside the $\log$ function.
The first term in the above expression is the KL distance between $P_0(x_1)$ and $P_1(x_1)$. Since, $p_0(x_k|x_{k-1})$ can be written in terms of α and β depending on the value of x_{k-1} and x_k, the second term on R.H.S in 3.3 can be written as $\sum_{k=2}^{n} p_0(x_{k-1} = 0)D_{KL}(\alpha_0||\alpha_1) +$ $p_0(x_{k-1} D_{KL}(\beta_0||\beta_1))$
Therefore

$$\begin{aligned} D_{KL}(P_0||P_1) = D_{KL}(p_0(X_1)||p_1(X_1)) \; + &\sum_{k=2}^{n} p(X_{k-1} = 0)D_{KL}(\alpha_0||\alpha_1) + \\ &\sum_{k=2}^{n} p(X_{k-1} = 1)D_{KL}(\beta_0||\beta_1) \end{aligned} \tag{3.4}$$

Let π_{00} and π_{01} be the steady state probabilities for the distribution P_0 and λ_0 be the second eigen value of state probability transition matrix of P_0, i.e $\pi_{00} = \frac{\beta_0}{\alpha_0+\beta_0}, \pi_{01} = \frac{\alpha_0}{\alpha_0+\beta_0}$ and $\lambda_0 = 1 - \alpha_0 - \beta_0$, then we can show that

$$p_0(X_{k-1} = 0) = p_0(X_1 = 0)(\pi_{00} + \pi_{01}\lambda_0^{k-2}) + p_0(X_1 = 1)(\pi_{00} - \pi_{00}(\lambda_0)^{k-2}) \tag{3.5}$$

$$p_0(X_{k-1} = 1) = p_0(X_1 = 0)(\pi_{01} - \pi_{01}\lambda_0^{k-2}) + p(X_1 = 1)(\pi_{01} + \pi_{00}(\lambda_0)^{k-2}) \tag{3.6}$$

From (3.4), (3.5) and (3.6) we get

$$\begin{aligned} D_{KL}(P_0||P_1) &= D_{KL}(p_0(X_1)||p_1(X_1)) + \\ &\sum_{k=2}^{n} \Big(p_0(X_1 = 0)(\pi_{00} + \pi_{01}\lambda_0^{k-2}) + p_0(X_1 = 1)(\pi_{00} - \pi_{00}(\lambda_0)^{k-2})\Big) D_{KL}(\alpha_0||\alpha_1) \\ &+ \sum_{k=2}^{n} \Big(p(X_1 = 0)(\pi_{01} - \pi_{01}\lambda^{k-2}) + p(X_1 = 1)(\pi_{01} + \pi_{00}(\lambda)^{k-2})\Big) D_{KL}(\beta_0||\beta_1) \end{aligned} \tag{3.7}$$

Here $D_{KL}(\alpha_0||\alpha_1) = \alpha_0 \log(\frac{\alpha_0}{\alpha_1}) + (1 - \alpha_0)\log(\frac{1-\alpha_0}{1-\alpha_1})$, similarly for $D_{KL}(\beta_0||\beta_1)$

If the initial distribution of P_0 is steady state distribution, i.e. $p_0(X_1 = 0) = \pi_{00}$ and $p_0(X_1 = 1) = \pi_{01}$ then the KL distance becomes

$$D_{KL}(P_0||P_1) = D_{KL}(p_0(X_1)||p_1(X_1)) + \sum_{k=2}^{n} \pi_{00} D_{KL}(\alpha_0||\alpha_1) + \sum_{k=2}^{n} \pi_{01} D_{KL}(\beta_0||\beta_1) \tag{3.8}$$

Now we will use these results to find the lower bound.

3.2 Lower bound

Now we consider 2 distributions P_0 and P_1 required for Le Cam lower bound.
The initial distribution of P_0 is chosen as steady state distribution, i.e. $p_0(X_1 = 0) = \pi_{00}$ and $p_0(X_1 = 1) = \pi_{01}$. Taking $\alpha_0 = 0.9, \beta_0 = \frac{1}{n-1}$
Also for distribution P_1 we take $\alpha_1 = 0.1$, $\beta_1 = \frac{1}{n-1}$, $p_1(X_1 = 0) = p_0(X_1 = 0)$ and $p_1(X_1 = 1) = p_0(X_1 = 1)$.
With these choices we get
$D_{KL}(p_0(X_1)||p_1(X_1)) = 0$, $D_{KL}(\beta_0||\beta_1) = 0$ and $D_{KL}(\alpha_0||\alpha_1) = 1.7578$ Using the above values and (3.8), we get

$$D_{KL}(P_0||P_1) = \frac{1.7578}{0.9 + \frac{1}{n-1}} < 1.9531$$

Comparing (3.1) with (2.1), we get
$\gamma_1(P) = \gamma_2(P) = 1,\ d(x, y) = |x - y|,\ \Phi(x) = x^2$
Here $\delta_1 = \frac{|\alpha_0 - \alpha_1|}{2} = 0.4$ and $\delta_2 = \frac{|\beta_0 - \beta_1|}{2} = 0$
Now using (2.4), we get

$$R_n^* \geq \frac{1}{2}(0.4)^2(1 - \sqrt{\frac{1}{2}1.94}) = 0.005 \tag{3.9}$$

3.3 Upper Bound

Let $R(P, \hat{\theta})$ be the risk when the estimator is $\hat{\theta}$ and the distribution is P, then

$$\inf_{\hat{\theta}} \sup_{P} R(P, \hat{\theta}) \leq \sup_{P} R(P, \hat{\theta}) \tag{3.10}$$

The above equation means that the *minimax* risk is less than the worst case risk of any given estimator. We will use this result to find the upper bound.
We define estimator $\hat{\alpha} = \frac{1}{2}$ and $\hat{\beta} = \frac{1}{2}$. Notice these estimator always estimate the value

of α and β as $\frac{1}{2}$. Now we will find the worst case Risk for this estimator.

$$\begin{aligned}
\inf_{\hat{p}} \sup_{P\in\mathcal{P}} E\Big[\, \big|\hat{\alpha}-\alpha\big|^2 + \big|\hat{\beta}-\beta\big|^2\Big] &\le \sup_{P\in\mathcal{P}} E\Big[\, \big|\hat{\alpha}-\alpha\big|^2 + \big|\hat{\beta}-\beta\big|^2\Big] \\
&= \sup_{P\in\mathcal{P}} E\Big[\, \big|\frac{1}{2}-\alpha\big|^2 + \big|\frac{1}{2}-\beta\big|^2\Big] \\
&\le \sup_{P\in\mathcal{P}} \big(\frac{1}{2}\big)^2 + \big(\frac{1}{2}\big)^2 \\
&= \big(\frac{1}{2}\big)^2 + \big(\frac{1}{2}\big)^2 \\
&= \frac{1}{2}
\end{aligned} \tag{3.11}$$

$|\frac{1}{2}-\alpha| \le \frac{1}{2}$ is used in the third step of the above equation. This is true because $\alpha \in (0,1]$. Similarly $|\frac{1}{2}-\beta| \le \frac{1}{2}$ for $\beta \in (0,1]$.
Using (3.9) and (3.11)

$$0.005 \ \le \inf_{\hat{p}} \sup_{P\in\mathcal{P}} E\Big[\, \big|\hat{\alpha}-\alpha\big|^2 + \big|\hat{\beta}-\beta\big|^2\Big] \le \ 0.5 \tag{3.12}$$

The above result shows that the $minimax$ risk is $\Theta(1)$ for the quadratic risk. This suggest that worst case performance of every estimator is poor. The distributions which have very small α or β are the one which are very difficult to estimate, this is because if α is very small then probability of entering state(1) is very small which makes estimating β very difficult. Similarly, if β is very small then probability of entering state(0) is very small which makes estimating α very difficult.

CHAPTER 4

Minimax risk for the second risk metric

The next risk function considered is

$$R_n = E\Big[\ \beta\big|\hat{\alpha} - \alpha\big| + \alpha\big|\hat{\beta} - \beta\big|\Big] \tag{4.1}$$

and the $minimax$ risk is defined as

$$R_n^* = \inf_{\hat{P}} \sup_{P \in \mathcal{P}} E\Big[\ \beta\big|\hat{\alpha} - \alpha\big| + \alpha\big|\hat{\beta} - \beta\big|\Big] \tag{4.2}$$

The motivation behind this risk function is to adjust the problem faced by previous risk function, i.e. if β is very less then probability staying in state(0) is also less which estimation of α very difficult. Now the idea is: when β is very less then estimating α is difficult which gives high error, so we have scaled the error term of α by β.

The risk function is of the form (2.3) so we can use (2.4) to find the lower bound.

4.1 Lower Bound

For using Modified Le Cam method, we need to convert (4.1) into (2.1) by choosing appropriate γ, d, θ and Φ functions. By comparing (4.1) with (2.1) we get

$\gamma_1(P) = \beta,\ \gamma_2(P) = \alpha,\ \theta_1(P) = \alpha,\ \theta_2(P) = \beta,\ d(x, y) = |x - y|$ and $\Phi(x) = x$

The 2 distributions required for Le Cam method are as follows:

Let P_0 be the distribution of $X_1, ..., X_n$ when $\alpha_0 = 1 - \beta_0 = \frac{1+\delta}{2}$ and P_1 be the distribution when $\alpha_1 = 1 - \beta_1 = \frac{1-\delta}{2}$.

$\alpha = 1 - \beta \implies X_1, ..., X_n$ are i.i.d samples from Bernoulli distribution with $P(X = 0) = 1 - \alpha$ and $P(X = 1) = \alpha$. Therefore, P_0 and P_1 are Binomial distributions with $p_0 = \frac{1+\delta}{2}$ and $p_0 = \frac{1-\delta}{2}$ respectively. This choice of α and β simplifies the calculation of $D_{kl}(P_0||P_1)$.

$$
\begin{aligned}
D_{kl}(P_0||P_1) &= n\left[\frac{1+\delta}{2}\log(\frac{1+\delta}{1-\delta}) + \frac{1-\delta}{2}\log(\frac{1-\delta}{1+\delta})\right] \\
&= n\delta\log\left(\frac{1+\delta}{1-\delta}\right)
\end{aligned} \tag{4.3}
$$

Noting that $\delta\log(\frac{1+\delta}{1-\delta}) \leq 3\delta^2$ for $\delta \in [0, \frac{1}{2}]$, we obtain

$$
D_{kl}(P_0||P_1) \leq 3n\delta^2
$$

Therefore, $\gamma_1(P_0) = \frac{1-\delta}{2},\ \gamma_1(P_1) = \frac{1-\delta}{2},\ \gamma_2(P_0) = \frac{1+\delta}{2},\ \gamma_2(P_1) = \frac{1-\delta}{2},\ \theta_1(P_1) = \frac{1+\delta}{2},\ \theta_2(P_2) = \frac{1-\delta}{2},\ \Delta_1 = \frac{|\theta_1(P_1)-\theta_1(P_2)|}{2} = \delta$ and $\Delta_2 = \frac{|\theta_2(P_1)-\theta_2(P_2)|}{2} = \delta$

Using (2.2) we get

$$
\begin{aligned}
R_n^* &\geq \frac{1}{2}\min\left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right)\frac{\delta}{2}\left[1-\sqrt{\frac{3n\delta^2}{2}}\right] + \frac{1}{2}\min\left(\frac{1+\delta}{2}, \frac{1-\delta}{2}\right)\frac{\delta}{2}\left[1-\sqrt{\frac{3n\delta^2}{2}}\right] \\
&= \frac{(1-\delta)}{2}\frac{\delta}{2}\left[1-\sqrt{\frac{3n\delta^2}{2}}\right]
\end{aligned} \tag{4.4}
$$

For $\delta = \sqrt{\frac{1}{6n}}$ we get

$$
R_n^* \geq \frac{1}{8\sqrt{6n}} - o\left(\frac{1}{\sqrt{n}}\right) \tag{4.5}
$$

4.2 Upper Bound

Let $X_1, ..., X_n$ be the observations form the two state markov chain. Let N_{00} represents the number of transitions form state(0) to state(0), N_{01} represents the number of transitions from state(0) to state(1) and k_0 represents the number of times state(0) was visited in n observations. Then we have

$$
N_{00} + N_{01} \leq k_0 \leq N_{00} + N_{01} + 1 \tag{4.6}
$$

So k_0 is either $N_{00} + N_{01}$ or $N_{00} + N_{01} + 1$ depending on the transitions. For e.g. if $X_0 = state(0)$ and $X_n = state(1)$ then $k_0 = N_{00} + N_{01}$ and if $X_0 = state(0)$ and $X_n = state(0)$ then $k_0 = N_{00} + N_{01} + 1$

Similarly

$$N_{10} + N_{11} \leq k_0 \leq N_{10} + N_{11} + 1 \tag{4.7}$$

Let S_0 be the steady state probability of state(0) and S_1 be the steady state probability of state(1), then

$$S_0 = \frac{\beta}{\alpha + \beta} \tag{4.8}$$

$$S_1 = \frac{\alpha}{\alpha + \beta} \tag{4.9}$$

Consider an estimator $\hat{S}_0 = \frac{k_0}{n}$ for estimating S_0 and $\hat{S}_1 = \frac{k_1}{n}$ for estimating S_1

The following results are taken form Xue and Roy (2011), it will be used to prove the upper bound on R_n^*

$$E(\hat{S}_0) = S_0 \tag{4.10}$$

$$E(\hat{S}_1) = S_1 \tag{4.11}$$

$$var(\hat{S}_0) \leq \frac{1}{2n(1-\lambda_2)} - \frac{1}{4n} + \frac{1}{n^2(1-\lambda_2)^2} \tag{4.12}$$

$$var(\hat{S}_1) \leq \frac{1}{2n(1-\lambda_2)} - \frac{1}{4n} + \frac{1}{n^2(1-\lambda_2)^2} \tag{4.13}$$

where $\lambda_1 = 1$ and λ_2 are the eigen values of the transition matrix of the Markov chain in Fig. 1.1

Using $\lambda_2 = 1 - \beta - \alpha$

and (4.12)

$$var(\hat{S}_0) \leq \frac{1}{2n(\alpha+\beta)} - \frac{1}{4n} + \frac{1}{n^2(\alpha+\beta)^2} \tag{4.14}$$

$$var(\hat{S}_1) \leq \frac{1}{2n(\alpha+\beta)} - \frac{1}{4n} + \frac{1}{n^2(\alpha+\beta)^2} \tag{4.15}$$

Now we will upper bound the worst case risk for the estimators $\hat{\alpha}$ and $\hat{\beta}$ defined as follows

$$\hat{\alpha} = \begin{cases} \frac{N_{01}}{N_{00}+N_{01}}, & \text{if } N_{00}+N_{01} \neq 0 \\ 0, & \text{if } N_{00}+N_{01} = 0 \end{cases} \tag{4.16}$$

$$\hat{\beta} = \begin{cases} \frac{N_{10}}{N_{10}+N_{11}}, & \text{if } N_{10}+N_{11} \neq 0 \\ 0, & \text{if } N_{10}+N_{11} = 0 \end{cases} \tag{4.17}$$

Now we will upper bound the risk for these estimators

$$E\Big[\ \beta|\alpha-\hat{\alpha}|\ +\alpha|\beta-\hat{\beta}|\ \Big] \leq \sqrt{E\Big[\ \beta^2|\alpha-\hat{\alpha}|^2\ \Big]} + \sqrt{E\Big[\ \alpha^2|\beta-\hat{\beta}|^2\ \Big]} \tag{4.18}$$

Now we will find the upper bound on $E\Big[\ \beta^2|\alpha-\hat{\alpha}|^2\ \Big]$

Using (4.8) we get

$$E\big[\ \beta^2(\hat{\alpha}-\alpha)^2\big] = E\big[(\alpha+\beta)^2 S_0^2(\hat{\alpha}-\alpha)^2\big] \tag{4.19}$$

By manipulation we can write

$$\begin{aligned} E\big[(\alpha+\beta)^2 S_0^2(\hat{\alpha}-\alpha)^2\big] &= (\alpha+\beta)^2\Bigg[\ E\big[\ \big(\ S_0-\frac{k_0}{n}\big)^2\ (\hat{\alpha}-\alpha)^2\big]\ +E\big[\ 2\ S_0\frac{k_0}{n}(\hat{\alpha}-\alpha)^2\big] \\ &\qquad -E\big[\ \big(\ \frac{k_0}{n}\big)^2\ (\hat{\alpha}-\alpha)^2\big]\ \Bigg] \\ &\leq E\Bigg[\ (\alpha+\beta)^2\big(\ S_0-\frac{k_0}{n}\big)^2\ (\hat{\alpha}-\alpha)^2\Bigg]\ +E\Bigg[\ 2(\alpha+\beta)^2\ S_0\frac{k_0}{n}(\hat{\alpha}-\alpha)^2\Bigg] \end{aligned} \tag{4.20}$$

Consider the first term in R.H.S of (4.20)

Using $(\hat{\alpha}-\alpha)^2 \leq 1$ and ((4.14) we can write

$$\begin{aligned} E\Big[\ (\alpha+\beta)^2\big(\ S_0-\frac{k_0}{n}\big)^2\ (\hat{\alpha}-\alpha)^2\Big]\ &\leq E\Big[(\alpha+\beta)^2\big(\ S_0-\frac{k_0}{n}\big)^{\ 2}\Big] \\ &= (\alpha+\beta)^2 E\Big[\big(\ S_0-\frac{k_0}{n}\big)^{\ 2}\Big] \\ &= (\alpha+\beta)^2 var(\hat{S}_0) \\ &\leq \frac{(\alpha+\beta)}{2n}-\frac{(\alpha+\beta)^2}{4n}+\frac{1}{n^2} \end{aligned} \tag{4.21}$$

For a given value of $N_{00}+N_{01}=n_{00}+n_{01}\neq 0$ we can write

$$E\Big[(\hat{\alpha}-\alpha)^2\Big|n_{00}+n_{01}\Big]=\frac{\alpha(1-\alpha)}{n_{00}+n_{01}} \tag{4.22}$$

Now Consider the second term in R.H.S of (4.20)

Using (4.8) we can write

$$\begin{aligned} E\Big[\ 2(\alpha+\beta)^2\ S_0\frac{k_0}{n}(\hat{\alpha}-\alpha)^2\Big]\ &= E\Big[\ 2(\alpha+\beta)^2\ \frac{\beta}{\alpha+\beta}\frac{k_0}{n}(\hat{\alpha}-\alpha)^2\Big] \\ &= 2\beta(\alpha+\beta)\ E\Big[\ \frac{k_0}{n}(\hat{\alpha}-\alpha)^2\Big] \end{aligned} \tag{4.23}$$

To get the upper bound on (4.23), we will now find the upper bound on $E\Big[\frac{k_0}{n}(\hat{\alpha}-\alpha)^2\Big]$

$$\begin{aligned} E\Big[\ \frac{k_0}{n}(\hat{\alpha}-\alpha)^2\Big]\ &= E\Big[\ \frac{k_0}{n}(\hat{\alpha}-\alpha)^2\ \Big|\ n_{00}+n_{01}=0\Big]\ p(n_{00}+n_{01}=0)+ \\ &\quad E\Big[\ \frac{k_0}{n}(\hat{\alpha}-\alpha)^2\ \Big|\ n_{00}+n_{01}\neq 0\Big]\ p(n_{00}+n_{01}\neq 0) \end{aligned} \tag{4.24}$$

The RHS of the above equation has 2 terms, we will prove that the both terms decays are order n

Consider the first term on R.H.S of (4.24)

$$E\Big[\ \frac{k_0}{n}(\hat{\alpha}-\alpha)^2\ \Big|\ n_{00}+n_{01}\neq 0\Big]\ p(n_{00}+n_{01}\neq 0)\leq E\Big[\ \frac{k_0}{n}(\hat{\alpha}-\alpha)^2\ \Big|\ n_{00}+n_{01}\neq 0\Big] \tag{4.25}$$

$$
\begin{aligned}
E\Big[\, \frac{k_0}{n}(\hat{\alpha}-\alpha)^2 \,\Big|\, n_{00}+n_{01} \neq 0\Big] &\leq E\Bigg[\, E\Big[\, \frac{k_0}{n}(\hat{\alpha}-\alpha)^2 \mid n_{00}+n_{01} \neq 0\Big]\,\Bigg] \\
&\leq E\Bigg[\, E\Big[\, \frac{(n_{00}+n_{01}+1)}{n}(\hat{\alpha}-\alpha)^2 \mid n_{00}+n_{01} \neq 0\Big]\,\Bigg] \\
&\leq E\Big[\frac{2(n_{00}+n_{01})}{n}\frac{\alpha(1-\alpha)}{n_{00}+n_{01}} \mid n_{00}+n_{01} \neq 0\Big] \\
&= 2\frac{\alpha(1-\alpha)}{n}
\end{aligned}
\tag{4.26}
$$

The second step in the above equation uses (4.6), third step uses (4.22) and $n_{00}+n_{01}+1 \leq 2(n_{00}+n_{01})$ Consider the second term on R.H.S of (4.24)

Using (4.6) we know $k_0 \leq n_{00}+n_{01}+1 \implies k_0 \leq 1$ when $n_{00}+n_{01}=0$

$$
\begin{aligned}
E\Big[\, \frac{k_0}{n}(\hat{\alpha}-\alpha)^2 \,\Big|\, n_{00}+n_{01}=0\Big]\, p(n_{00}+n_{01}=0) &\leq E\Big[\, \frac{k_0}{n}(\hat{\alpha}-\alpha)^2 \,\Big|\, n_{00}+n_{01}=0\Big] \\
&\leq E\Big[\, \frac{1}{n}(0-\alpha)^2 \,\Big|\, n_{00}+n_{01}=0\Big] \\
&= \frac{\alpha^2}{n}
\end{aligned}
\tag{4.27}
$$

From (4.24), (4.26) and (4.27) we get

$$
\begin{aligned}
E\Big[\, \frac{k_0}{n}(\hat{\alpha}-\alpha)^2\Big] \; &\leq 2\frac{\alpha(1-\alpha)}{n} + \frac{\alpha^2}{n} \\
&\leq \frac{2\alpha-\alpha^2}{n}
\end{aligned}
\tag{4.28}
$$

From (4.23) and (4.28) we get

$$
E\Big[\, 2(\alpha+\beta)^2\, S_0\frac{k_0}{n}(\hat{\alpha}-\alpha)^2\Big] \; \leq 2\beta(\alpha+\beta)\frac{(2\alpha-\alpha^2)}{n} \tag{4.29}
$$

Using (4.20), (4.21) and (4.29) we get

$$
E\big[\, \beta^2(\hat{\alpha}-\alpha)^2\big] \; \leq \frac{(\alpha+\beta)}{2n} - \frac{(\alpha+\beta)^2}{4n} + 2\beta(\alpha+\beta)\frac{(2\alpha-\alpha^2)}{n} + \frac{1}{n^2} \tag{4.30}
$$

Maximizing the above equation with respect to alpha and beta, we get

$$
E\big[\, \beta^2(\hat{\alpha}-\alpha)^2\big] \; \leq \frac{4}{n} + \frac{1}{n^2} \tag{4.31}
$$

The maximum is attained when alpha is 1 and beta is 1.

From (4.31) we get

$$\begin{aligned} E\big[\,\beta|\hat{\alpha}-\alpha|\big] &\le \sqrt{\frac{4}{n}+\frac{1}{n^2}} \\ &\le \frac{2}{\sqrt{n}} + o\Big(\frac{1}{\sqrt{n}}\Big) \end{aligned} \tag{4.32}$$

Similarly we can show that

$$\begin{aligned} E\big[\,\alpha|\hat{\beta}-\beta|\big] &\le \sqrt{\frac{4}{n}+\frac{1}{n^2}} \\ &\le \frac{2}{\sqrt{n}} + o\Big(\frac{1}{\sqrt{n}}\Big) \end{aligned} \tag{4.33}$$

From (4.18), (4.32) and (4.33) we get

$$E\Big[\,\beta|\alpha-\hat{\alpha}|\;+\alpha|\beta-\hat{\beta}|\,\Big] \le \frac{4}{\sqrt{n}} + o\Big(\frac{1}{\sqrt{n}}\Big) \tag{4.34}$$

The above equation is true for all $\alpha, \beta \in (0,1]$, Hence

$$\sup_{P\in\mathcal{P}} E\Big[\,\beta|\alpha-\hat{\alpha}|\;+\alpha|\beta-\hat{\beta}|\,\Big] \le \frac{4}{\sqrt{n}} + o\Big(\frac{1}{\sqrt{n}}\Big) \tag{4.35}$$

Since

$$\inf_{\hat{P}}\sup_{P\in\mathcal{P}} E\Big[\,\beta|\alpha-\hat{\alpha}|\;+\alpha|\beta-\hat{\beta}|\,\Big] \le \sup_{P\in\mathcal{P}} E\Big[\,\beta|\alpha-\hat{\alpha}|\;+\alpha|\beta-\hat{\beta}|\,\Big] \tag{4.36}$$

From (4.35) and (4.36) we get

$$\inf_{\hat{P}}\sup_{P\in\mathcal{P}} E\Big[\,\beta|\alpha-\hat{\alpha}|\;+\alpha|\beta-\hat{\beta}|\,\Big] \le \frac{4}{\sqrt{n}} + o\Big(\frac{1}{\sqrt{n}}\Big) \tag{4.37}$$

Therefore the bounds on R_n^* are

$$\frac{1}{8\sqrt{6n}} - o\bigg(\frac{1}{\sqrt{n}}\bigg) \le \inf_{\hat{P}}\sup_{P\in\mathcal{P}} E\Big[\,\beta|\alpha-\hat{\alpha}|\;+\alpha|\beta-\hat{\beta}|\,\Big] \le \frac{4}{\sqrt{n}} + o\Big(\frac{1}{\sqrt{n}}\Big) \tag{4.38}$$

The above equation shows that the $minimax$ risk is $\Theta(\frac{1}{\sqrt{n}})$

CHAPTER 5

Minimax risk for the third risk metric

Let x^n be a sample form this Markov chain, $\hat{p}$ is an estimator consisting of $\hat{p}(0|x^n)$ and $\hat{p}(1|x^n)$. $\hat{p}(0|x^n)$ is an estimator for $p(X_{n+1} = 0|x^n)$ and $\hat{p}(1|x^n)$ is an estimator for $p(X_{n+1} = 1|x^n)$

For n samples of Markov chain the $minimax\ risk$ is defined as

$$R_n^* = \inf_{\hat{p}} \sup_{P \in \mathcal{P}} E\left[\sum_{x_{n+1}=0}^{1} \left|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\right|^2\right] \tag{5.1}$$

The motivation behind this risk function is that the following risk is considered in Falahatgar *et al.* (2016)

$$R_n^* = \inf_{\hat{p}} \sup_{P \in \mathcal{P}} E\left[D_{KL}\Big(P(X_{n+1}|X_n)\Big|\Big|\hat{P}(X_{n+1}|X_n)\Big)\right] \tag{5.2}$$

Where $\hat{P}(X_{n+1}|X_n)$ is an estimator for $P(X_{n+1}|X_n)$.

We have replaced the KL distance in the above equation by quadratic distance to get the new risk function.

5.1 Lower Bound

In the risk function

$$R_n = E\left[\sum_{x_{n+1}=0}^{1} \left|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\right|^2\right] \tag{5.3}$$

Notice that the parameter which we are trying to estimate $p(X_{n+1}|X_n)$ is a function of distribution as well as observations.

If $x_n = 0$ then $p(0|x^n) = 1 - \alpha$ and $p(1|x^n) = \alpha$.

Similarly, if $x_n = 1$ then $p(0|x^n) = 1 - \beta$ and $p(1|x^n) = \beta$.

The Le Cam method works only when the parameter is just a function of distribution.

To overcome this problem we will condition the risk for given x_n.

Let p_{X^n} be the distribution of x^n

We are interested in the estimators for which $\hat{p}(0|x^n) + \hat{p}(1|x^n) = 1$ holds.

Let's define $\hat{p}(1|x^n) = \hat{\alpha}$ and $\hat{p}(0|x^n) = 1 - \hat{\alpha}$ when $x_n = 0$

Similarly $\hat{p}(1|x^n) = \hat{\beta}$ and $\hat{p}(0|x^n) = 1 - \hat{\beta}$ when $x_n = 1$

$$
\begin{aligned}
R &= E_{p_{X^n}}\left[\sum_{x_{n+1}=0}^{1} \left|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\right|^2\right] \\
&= p(x_n = 0) E_{p_{X^n|x_n=0}}\left[\sum_{x_{n+1}=0}^{1} \left|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\right|^2\right] + \\
&\qquad p(x_n = 1) E_{p_{X^n|x_n=1}}\left[\sum_{x_{n+1}=0}^{1} \left|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\right|^2\right] \\
&= 2p(x_n = 0) E_{p_{X^n|x_n=0}} |\alpha - \hat{\alpha}|^2 \; + \; 2p(x_n = 1) E_{p_{X^n|x_n=1}} |\beta - \hat{\beta}|^2
\end{aligned} \tag{5.4}
$$

Now the risk is almost of the form (2.1), the only difference is that the expectation here is conditional.

Following the steps of proof of theorem 1, we can show that

$$
\begin{aligned}
R_n^* \geq\; &\min\Big(p_0(x_n = 0), p_1(x_n = 0)\Big) \left(\frac{|\alpha_0 - \alpha_1|}{2}\right)^2 \left[1 - \sqrt{\frac{1}{2} D_{KL}(P_{0|X_n=0} || P_{1|X_n=0})}\right] \\
&+ \min\Big(p_0(x_n = 1), p_1(x_n = 1)\Big) \left(\frac{|\beta_0 - \beta_1|}{2}\right)^2 \left[1 - \sqrt{\frac{1}{2} D_{KL}(P_{0|X_n=1} || P_{1|X_n=1})}\right]
\end{aligned} \tag{5.5}
$$

Here, $P_{0|X_n=0}$ is the conditional distribution of X^n given $X_n = 0$ under P_0, $P_{0|X_n=0}$ is the conditional distribution of X^n given $X_n = 1$ under P_0, $P_{1|X_n=0}$ is the conditional distribution of X^n given $X_n = 0$ under P_1 and $P_{1|X_n=1}$ is the conditional distribution of X^n given $X_n = 1$ under P_1

For finding the lower bound the following two distributions are considered

$P_0 : \alpha_0 = 1 - \beta_0 = \frac{1+\delta}{2}$

$P_1 : \alpha_1 = 1 - \beta_1 = \frac{1-\delta}{2}$

P_0 be the distribution of $X_1, ..., X_{n-1}$ when $\alpha_0 = 1 - \beta_0 = \frac{1+\delta}{2}$ and P_1 be the distribution when $\alpha_1 = 1 - \beta_1 = \frac{1-\delta}{2}$.

$\alpha = 1 - \beta \implies X_1, ..., X_{n-1}$ are iid samples from Bernoulli distribution with

$P(X=0) = 1-\alpha$ and $P(X=1) = \alpha$

Therefore, P_0 and P_1 are Binomial distributions with $p = \frac{1+\delta}{2}$ and $p = \frac{1-\delta}{2}$ respectively. Now since both the distributions considered are Bernoulli, therefore $P(X^n|X_n) = P(X^{n-1})$. Using this we can write the KL distribution as

$$D_{KL}\Big(P_0(X^n|X_n=0)||P_1(X^n|X_n=0)\Big) = D_{KL}\Big(P_0(X^{n-1})||P_1(X^{n-1})\Big) \quad (5.6)$$

Similarly

$$D_{KL}\Big(P_0(X^n|X_n=1)||P_1(X^n|X_n=1)\Big) = D_{KL}\Big(P_0(X^{n-1})||P_1(X^{n-1})\Big) \quad (5.7)$$

Now the KL distance is given by

$$D_{KL}\Big(P_0(X^{n-1})||P_1(X^{n-1})\Big) = (n-1)\delta \log\Big(\frac{1+\delta}{1-\delta}\Big) \quad (5.8)$$

Noting that $\delta \log(\frac{1+\delta}{1-\delta}) \leq 3\delta^2$ for $\delta \in [0, \frac{1}{2}]$, we obtain

$$D_{KL}\Big(P_0(X^{n-1})||P_1(X^{n-1})\Big) \leq 3(n-1)\delta^2 \quad (5.9)$$

Using (5.5), (5.6), (5.7) and (5.9) we get

$$\begin{aligned} R_n^* \geq\ & \min\Big(\frac{1-\delta}{2}, \frac{1+\delta}{2}\Big)\Big(\frac{\delta}{2}\Big)^2\left[1-\sqrt{\frac{3(n-1)\delta^2}{2}}\right]+ \\ & \min\Big(\frac{1+\delta}{2}, \frac{1-\delta}{2}\Big)\Big(\frac{\delta}{2}\Big)^2\left[1-\sqrt{\frac{3(n-1)\delta^2}{2}}\right] \\ \geq\ & \frac{1}{4}(1-\delta)\delta^2\Big(1-\delta\sqrt{\frac{3(n-1)}{2}}\Big) \end{aligned} \quad (5.10)$$

For $\delta = \frac{2}{3}\sqrt{\frac{2}{3(n-1)}}$

$$R^* \geq \frac{2}{81n} - o(\frac{1}{n}) \quad (5.11)$$

5.2 Upper Bound

We know that, for any two distributions P_0 and P_1

$$||P_0 - P_1||_{TV}^2 \leq \frac{1}{2} D_{kl}(P_0||P_1) \tag{5.12}$$

Consider $P_0 = P(X_{n+1}|X_n)$ and $P_1 = \hat{P}(X_{n+1}|X_n)$, then

$$\begin{aligned} ||P_0 - P_1||_{TV}^2 &= \Big[\, \big|p(0|x_n) - \hat{p}(0|x_n)\big| + \big|p(1|x_n) - \hat{p}(1|x_n)\big| \, \Big]^2 \\ &\geq \big|p(0|x_n) - \hat{p}(0|x_n)\big|^2 + \big|p(1|x_n) - \hat{p}(1|x_n)\big|^2 \\ &= \sum_{x_{n+1}=0}^{1} \big|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\big|^2 \end{aligned} \tag{5.13}$$

Using (5.12) and (5.13) we get

$$\sum_{x_{n+1}=0}^{1} \big|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\big|^2 \; \leq \frac{1}{2} D(p_{x_{n+1}|x^n}||\hat{p}_{x_{n+1}|x^n}) \tag{5.14}$$

Therefore

$$\inf_{\hat{p}} \sup_{\alpha,\beta} \; E\left[\sum_{x_{n+1}=0}^{1} \big|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\big|^2 \right] \leq \frac{1}{2} \inf_{\hat{p}} \sup_{\alpha,\beta} \; E\left[D(p_{x_{n+1}|x^n}||\hat{p}_{x_{n+1}|x^n}) \right] \tag{5.15}$$

Taking this result from Falahatgar *et al.* (2016)

$$\inf_{\hat{p}} \sup_{\alpha,\beta} \; E\left[D(p_{x_{n+1}|x^n}||\hat{p}_{x_{n+1}|x^n}) \right] \leq 2 \frac{\log\big(\log(n)\big)}{n} + O(\frac{1}{n}) \tag{5.16}$$

Using (5.15) and (5.16) we get

$$R_n^* \leq \frac{\log\big(\log(n)\big)}{n} + O(\frac{1}{n}) \tag{5.17}$$

Hence we can conclude

$$\frac{2}{81n} - o(\frac{1}{n}) \leq R^* \leq \frac{\log\big(\log(n)\big)}{n} + O(\frac{1}{n}) \tag{5.18}$$

Notice, the upper bound and the lower bound are not of the same order however

$\log(\log(n))$ changes very slowly with n. There is a scope of improvement in bounds.

CHAPTER 6

Conclusions and Scope for Future Work

We have proposed various risk metrics for estimating the parameters of a two state Markov chain and bounded the $minimax$ risk for these metrics. We have modified Le Cam's method into more generalized version which was used to find lower bounds. The generalized KL distance for two state Markov chain was derived which helped in finding a constant lower bound for quadratic risk function. For the risk function $R_n = E\left[\sum_{x_{n+1}=0}^{1} \left|p(x_{n+1}|x^n) - \hat{p}(x_{n+1}|x^n)\right|^2\right]$, the upper and lower bound found are not of the same order, and hence can be improved. Future work includes extension of this theory on k state Markov chain.

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