ANALYSIS OF A SPECTRUM SHARING MODEL USING LOSS NETWORKS

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THESIS CERTIFICATE

This is to certify that the thesis titled **ANALYSIS OF A SPECTRUM SHARING MODEL USING LOSS NETWORKS**, submitted by **Nithin Seyon Ramesan**, to the Indian Institute of Technology, Madras, for the award of the degree of **BACHELOR OF TECHNOLOGY**, is a bona fide record of the research work done by him under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

KEYWORDS: Spectrum sharing; loss networks; Erland fixed-point approximation; game theory

We consider a pairwise spectrum sharing model inspired by recently released regulations on spectrum sharing. We first establish a revenue-sharing model for the mobile service providers (MSPs) involved using principles of coalitional game theory. Acknowledging that the considered model can also be represented as a stochastic loss network, we proceed to analyze the system using the concept of shadow prices. The question of how much spectrum should optimally be shared by each MSP is considered and answered analytically. The results obtained indicate that it is optimal for atleast one of the MSPs to always completely share its licensed spectrum, and in certain cases, for both MSPs to fully share their respective spectrums. In order to set up a complete framework, we finally consider how MSPs should set prices to maximize their profits.

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NOTATION

p_i	Price per call for MSP i
N_i	Number of spectrum slots available to MSP i
k_i	Number of spectrum slots shared by MSP i
λ_i	Traffic incoming to MSP i
C_i	Capacity of a link i in the loss network
s_i	Shadow price of a link i in the loss network
b_i	Blocking probability for a link i in the loss network
B_i	SBlocking probability for a route i in the loss network
$E(\rho, C)$	Erlang-B loss function
$\eta(\rho, C)$	$E(\rho, C-1) - E(\rho, C)$

CHAPTER 1

Spectrum Sharing

Conventionally, frequency spectrum bands are licensed to mobile service providers (MSPs) by the government under strict rules that allow only that MSP to use the band. This ensures a certain level of Quality-of-Service for every MSP by ensuring that no interference from other MSPs occurs. However, this rigidity is often not optimal. Faced with a shortage of spectrum to auction off amongst MSPs ([1]), and an ever increasing consumer base, regulatory authorities in India have, in recent years, started to relax rules and allow MSPs to share their spectrum bands with others ([6], [7]).

This raises a number of interesting questions related to how MSPs should share spectrum. For exmaple, from a competitive viewpoint, there is an immediately obvious tradeoff - sharing more of one's own spectrum will lead to increased interference for one's own users, but not sharing enough spectrum could lead to other MSPs reciprocating and reducing the benefits intended from sharing. There is hence a large body of literature on the topic of spectrum sharing.

1.1 Previous Work

[3] provides a policy-level overview of spectrum sharing today. A majority of the work in this field has been in the subfield of *dynamic* spectrum sharing - where the excess spectrum of a single MSP is auctioned/allocated to other MSPs dynamically over time. Auction mechanisms are studied in [8] and Niyato et al. ([9], [10]) study the game theoretic aspects of such a model. [14] takes a mechanism design approach to spectrum sharing and investigates how incentives may be used to achieve a socially optimal sharing paradigm.

Another area of work - one that we are more interested in - is that of spectrum sharing with coalitions. [11] and [12] consider the questions of optimal spectrum allocation in a coalition to achieve a Nash Bargaining Solution and coalition formation for spectrum sharing. [13] considers the specific problem of how MSPs in a coalition will share their profits. [14] considers a mechanism design approach to spectrum sharing. [15] - [17] also consider the game theoretic aspects of wireless networks models, with [17] considering a model very close to our own.

In this report, we establish a framework of spectrum sharing between two MSPs who have already formed a coalition, the model for which is detailed in section 1.2. Our framework is established in three parts. In section 1.3, we address the issue of MSP inter-payments using cooperative game theory, establishing a profit-splitting paradigm that ensures the stability of the coalition. In section 1.4, we consider how much of its allotted spectrum each MSP should share. Here we present our main result: that atleast one MSP must fully share its spectrum (and in certain cases, both MSPs should) in order to maximize profits. In section 1.5, we present a variety of consumer pricing games that MSPs can play in order to decide price per call.

1.2 Model

We consider the circuit-switched telephony model presented in [1], one that is relevant while considering GSM networks. There are 2 mobile service providers, or MSPs, who enter into an agreement to share portions of their spectrum. We do not consider the question of how such sharing pairs are formed; rather, we assume that MSPs have already formed mutually beneficial partnerships, where both of them share some spectrum. It is assumed that the spectrum allocated to each MSP is split into multiple sub-bands, or *slots*, of equal size, each of which is used to service a call. Let the number of such slots for MSP i be N_i, $i \in \{1, 2\}$. Calls arrive for each MSP as a Poisson process with mean λ_i, and have unit mean exponentially distributed service times.

A call takes up the entirety of a single slot, i.e., we preclude the possibility of a slot being used for multiple calls. When all the slots available for the use of an MSP are occupied, any incoming calls do not wait for a slot to free up, and are dropped. Customers are charged only for a successful call, i.e., one that is not blocked. We assume that MSP i charges a constant price p_i for a successful call. We further assume that an MSP charges a price $\tilde{p}_i$ before entering the coalition.

MSP i can choose to *share* k_i slots with MSP j. This implies that a total of $N_1 + k_2$ slots are available for MSP 1's use. Note that there is no guarantee that all k_2 slots will be free for MSP 1's use - MSP 2 still treats the shared k_2 slots as it would any non-shared slot for its own calls. Either MSP will use its own slots before starting to use the shared slots. If any of MSP 1's own slots free up while one of its calls is being serviced in a shared sub-band of MSP 2's, the call will be instantaneously switched to MSP 1's own spectrum - a process known as call repacking.

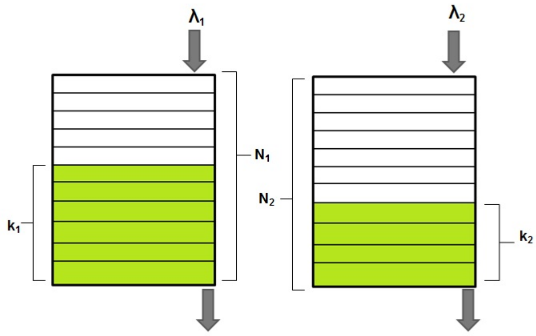

Figure 1.1: Sharing model

Note that:

$$0 < k_i \leq N_i, i \in \{1, 2\} \tag{1.1}$$

The strict inequality exists due to our initial assumption that MSPs would have formed pairs where sharing would be mutually beneficial and that no pair exists where only one MSP contributes to the shared pool.

None of the mobile service providers that participate in spectrum sharing are altruistic - each requires payment from any other MSP when they use any of its shared spectrum. Similar to the pricing model for customers, we assume that MSP i is paid a fixed price, r_i by MSP j whenever a slot that MSP i has shared is used by MSP j for one of its customers.

The quality of service that an MSP offers to customers while in a coalition is measured using the MSP's blocking probability, B_i, defined as the probability that a call incoming from any customer is blocked. While acting by itself, we denote the MSP's blocking probability as $\tilde{B}_i$.

Lemma 1.2.1. *The time-averaged numbers of calls being served by MSP* i*,* T_i*, is given* *by the equation:*

$$T_i = \lambda_i(1 - B_i) \tag{1.2}$$

The expression follows directly from Little's Law.

Using Lemma 1.2.1, the time-averaged revenue R_i that an MSP receives can be expressed as:

$$R_i = p_i\lambda_i(1 - B_i) \tag{1.3}$$

The next issue that must be addressed is that of the mathematical analysis of the above system. Let $\mathbf{u} = (u_1, u_2)$ represent the state of the system at any time instant, where u_i is the number of calls of MSP i's customers that are being serviced in the system. Then, the feasible set of such ordered pairs is given by:

$$\mathcal{F} = \{\mathbf{u} \mid u_1 \leq N_1 + k_2, u_2 \leq N_2 + k_1, u_1 + u_2 \leq N_1 + N_2\} \tag{1.4}$$

The system can now be modeled as a continuous time Markov process, with state space $\mathcal{F}$. Analysis now is possible, and results in the following limiting distribution.

Lemma 1.2.2. *The limiting distribution for a state $\mathbf{u} \in \mathcal{F}$ can be expressed as*

$$\pi(\mathbf{u}) = \frac{1}{\mathcal{D}} \frac{\lambda_1^{u_1}}{u_1!} \frac{\lambda_2^{u_2}}{u_2!} \tag{1.5}$$

where

$$\mathcal{D} = \sum_{\mathbf{u} \in \mathcal{F}} \frac{\lambda_1^{u_1}}{u_1!} \frac{\lambda_2^{u_2}}{u_2!} \tag{1.6}$$

The blocking probability for MSP i can now be expressed as:

$$B_i = \frac{1}{\mathcal{D}} \sum_{\mathbf{u} \in \mathcal{B}_i} \frac{\lambda_1^{u_1}}{u_1!} \frac{\lambda_2^{u_2}}{u_2!} \tag{1.7}$$

where

$$\mathcal{B}_i = \{\mathbf{u} \mid u_i = N_i + k_j\} \cup \{\mathbf{u} \mid u_1 + u_2 = N_1 + N_2\} \tag{1.8}$$

This distribution is identical to that of an appropriately chosen stochastic loss network, which will help us analyze the system further.

1.2.1 Stochastic Loss Networks

Stochastic loss networks have been used for almost a century to model systems that involve the simultaneous usage of a fixed amount of resources, without allowing for backlogging. They have found extensive applications in the fields of telecommunications and computer networks, as well as in a diverse range of other fields (see [18] - [20]).

In general, they are defined by a set of nodes, $\mathcal{N}$, and a set of edges or *links*, $\mathcal{J}$ between nodes. Routes, ($R_i \in \mathcal{R}$, $i \in \{1, 2, 3...\}$ are defined as subsets of links in the network that form a path. Each link has a capacity C_i allotted to it. Call/job arrivals & service times for each of the routes can be modeled as random processes. If the number of calls/jobs being served by a route is equal to the capacity of any of the links in the route, all incoming calls/jobs to that route will subsequently be blocked.
It is possible to construct a stochastic loss network to obtain the exact distribution seen in (1.5) and (1.6).

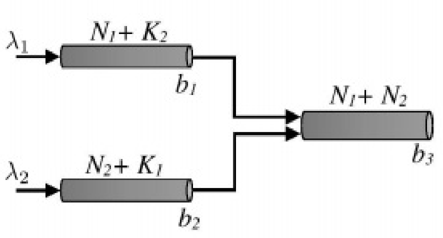

Figure 1.2: Equivalent loss network (Source: [1])

Such a network will have 3 links, say $\{1, 2, 3\}$ and two routes: $A = \{1, 3\}$, corresponding to MSP 1 and $B = \{2, 3\}$, corresponding to MSP 2. The capacities of the links are given by:

$$\begin{aligned} C_1 &= N_1 + k_2 \\ C_2 &= N_2 + k_1 \\ C_3 &= N_1 + N_2 \end{aligned} \tag{1.9}$$

1.3 Inter-MSP Payments (Profit Sharing)

The question of the quantum of inter-MSP payments is not one that has an immediately obvious solution. Intuitively, a high value of r_i would dissuade other MSPs from sharing spectrum with MSP i. Lowering the value of r_i, however, is detrimental to MSP i, who is aiming to maximize profits and offset operating costs. To address the problem of inter-MSP payments, we turn to cooperative game theory.

1.3.1 Introduction

Cooperative, or coalitional game theory analyzes the behaviour of rational players when they cooperate. The actors in cooperative game theory are not individual players - rather, they are coalitions, or groups of individual players. It is usually assumed that coalitions have already been formed beforehand ([2]), as has been done in our model. The choice of cooperative game theory is a natural one, since MSPs in our model have essentially entered into an agreement to share spectrum, thus forming a coalition. We do not address the issue of coalition formation further in this paper, apart from reiterating (1.1). Rather, the following analysis focuses on how the total revenue generated by a coalition is split among its members. We present a formal definition of the game being played below ([4], [5]).

1.3.2 Game Characterization

The game can be represented as $(v, \mathcal{P})$, where $\mathcal{P}$ is the set of players, and v is the characteristic function, defined later.

Let the set of players be defined by $\mathcal{P} = \{P_1, P_2\}$, and let the power set of $\mathcal{P}$ be $\mathbb{P}$. Any non-empty subset $\mathcal{S}$ of $\mathcal{P}$ is called a coalition. Each coalition is assigned a value or revenue, defined by the characteristic function $v : 2^{|\mathcal{P}|} \mapsto \mathbb{R}^+$, that denotes the worth of the coalition. We consider a coalitional game in *characteristic form*, where the value of a coalition depends only on its members, and not on the split of the coalition's nonmembers. Further, we assume that the game has a transferrable utility, i.e, revenue from the game can be shared freely among members of a coalition. These assumptions are natural for 2 MSPs with customer bases and revenue obtained from customers.

Lemma 1.3.1. *The characteristic function* v *is superadditive ([5]), i.e., for every* $S, T \in \mathbb{P}$ *such that* $S \cap T = \phi$,

$$v(S \cup T) \geq v(S) + v(T) \tag{1.10}$$

Proof. *Case 1:* When S or T is ϕ (and remarking that $v(\phi) = 0$), (1.10) is trivially satisfied.
Case 2: Without loss of generality, when $S = \{P_1\}$ and $T = \{P_2\}$, (1.10) is:

$$\begin{aligned} v(\{P_1\} \cup \{P_2\}) &\geq v(\{P_1\}) + v(\{P_2\}) \\ \implies p_1\lambda_1(1 - B_1) &+ p_2\lambda_2(1 - B_2) \\ &\geq \tilde{p_1}\lambda_1(1 - \tilde{B}_1) + \tilde{p_2}\lambda_2(1 - \tilde{B}_2) \end{aligned} \tag{1.11}$$

This condition must necessarily be satisfied by all coalitions that form in our model. It is easy to see why: if (1.11) does not hold, at least one member of the coalition formed must receive less revenue while in the coalition than if it acted on its own. Hence, the concerned MSP would have no incentive to participate in spectrum sharing, and would not form a coalition to begin with.

■

Since the game is superadditive, and from our initial assumptions, the only coalition that will form is the *grand coalition*: $\{P_1\} \cup \{P_2\}$.
While superadditivity is a necessary condition for a stable coalition (one whose members have no incentive to split off and act individually), it is not sufficient. To ensure stability, we must find a suitable payoff vector, $[f_1 \; f_2]^T$ (note that f_i is the total payoff for player i), so that:

$$\begin{aligned} f_1 + f_2 = v(\{P_1\} \cup \{P_2\}) \\ and \\ f_i \geq v(\{P_i\}), i \in \{1, 2\} \end{aligned} \tag{1.12}$$

Such a payoff vector is said to belong to the core of the game being played.

Definition 1.3.1. Core. The core of a coalitional game with two players is defined as the set of all payoff vectors satisfying 1.12, i.e, the set of stable payoff vectors - those that will ensure that the players of the game have no incentive to leave the grand coalition.

The Shapely Value

The Shapely value is a well-known solution concept to coalitional games, named after its inventor, Lloyd Shapely. It is a 'fair' distribution, in that it satisfies a set of axiomatic properties that make it a desirable payoff distribution (see [5]).

Definition 1.3.2. Shapely Value. For a game $(v, \mathcal{P})$, the Shapely value payoff is defined as:

$$f_i(v) = B_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S)) \tag{1.13}$$

where $n = |\mathcal{P}|$.

For our model, the Shapely value payoffs simplify to:

$$\begin{aligned} f_1 &= \frac{1}{2}(R_1 + R_2 + \tilde{R}_1 - \tilde{R}_2) \\ f_2 &= \frac{1}{2}(R_1 + R_2 + \tilde{R}_2 - \tilde{R}_1) \end{aligned} \tag{1.14}$$

where R_i and $\tilde{R}_i$ are the revenues that MSP i receives after and before joining the coalition respectively.

Definition 1.3.3. Convex Game. A game is said to be convex if:

$$v(S) + v(T) - v(S \cap T) \leq v(S \cup T) \tag{1.15}$$

, where $S \in \mathbb{P}, T \in \mathbb{P}, S \neq T$.

We state another lemma from [4] that will be useful:

Lemma 1.3.2. *If a game is convex, the Shapely value belongs to the core of the game.*

We now state the main result of this section:

Theorem 1.3.3. *The Shapely value belongs to the core of the game relevant to our model.*

Proof. Clearly, from 1.3.2, it is sufficient to prove that our game is convex in order to prove this theorem.

Note that $\mathbb{P} = \{\phi, \{P_1\}, \{P_2\}, \{P_1\} \cup \{P_2\}\}$.

Case 1. Without loss of generality, assume that $S = \phi$. We then trivially get $v(T) \geq v(T)$ (since $v(\phi) = 0$).

Case 2. Without loss of generality, assume $S = \{P_1\} \cup \{P_2\}$ and $T \neq \phi$. We then trivially get $v(S) \geq v(S)$.

Case 3. Without loss of generality, assume $S = P_1$ and $T = P_2$. Then, from 1.10, we get $v(S \cup T) \geq v(S) + v(T)$.

Hence, proved.

■

It is clear that the payoffs defined by 1.14 belong to the core and ensure a stable coalition. Hence, the quantum of inter-operator payments (r_i) can be obtained from the Shapely value split.

1.4 Quantum of sharing

In this section, we consider the question of how much spectrum each MSP should optimally share with the other, i.e, we will derive optimal values of k_1 and k_2. To begin, we will state the following assumptions.

- We assume that the incoming traffic to each MSP as well as the prices charged per successful call are known and fixed, i.e, λ_i and p_i are fixed.
- We further assume that since the MSPs have agreed to share spectrum, they will share in such a way as to maximize the total revenue they receive:

$$P_t = R_1 + R_2 = p_1\lambda_1(1 - B_1) + p_2\lambda_2(1 - B_2) \tag{1.16}$$

This can also be intuited by looking at 1.14, the revenue split we established before. In order to maximize individual revenues, MSPs must maximize total revenue.
In order to obtain an optimal sharing policy, we consider the equivalent loss network that we described in Section 1.2.1 (refer Fig 1.2). The question of an optimal sharing policy now reduces to dimensioning links 1 & 2 in the loss network previously described. The optimal dimensioning of loss networks is a problem that has been considered previously in the context of circuit-switched phone networks (see [21]). To analyze our model, we will utilize the concept of *shadow prices*, which were introduced by Kelly ([22]).

1.4.1 The Erlang fixed-point distribution

As stated before, the blocking probability distributions for the routes on the loss networks are equivalent to the blocking probabilities for the MSPs, and are given in 1.2.2. The expressions, however, are somewhat intractable to analyze, and hence we turn to an approximation called the Erlang fixed-point approximation.

The Erlang approximation considers the question: what if call blocking on each link in a loss network was independent of any other link (though this is obviously not the case)? In this case, the blocking probability of a single link would be given by the well-known Erlang-B function:

For traffic λ and link capacity C,

$$E(\lambda, C) = \frac{\frac{\lambda^C}{C!}}{\sum_{i=0}^{C} \lambda^i / i} \tag{1.17}$$

The dependence of the blocking probability of a link on the states of the other links in the network is taken into account by the approximation by way of the concept of 'reduced traffic'. The traffic offered to each link on a route is assumed to be decreased by the effects of blocking in the other links on that route. The Erlang fixed-point approximation for our loss network can hence be expressed as:

$$\begin{aligned} b_1 &= E(\lambda_1(1-b_3), N_1 + k_2) \\ b_2 &= E(\lambda_2(1-b_3), N_2 + k_1) \\ b_3 &= E(\lambda_1(1-b_1) + \lambda_2(1-b_2), N_1 + N_2) \\ & \quad and \\ B_1 &= 1 - (1-b_1)(1-b_3) \\ B_2 &= 1 - (1-b_2)(1-b_3) \end{aligned} \tag{1.18}$$

where b_i is the blocking probability for link i and B_i is the blocking probability for route i. (recall that routes correspond to MSPs, and that links 1 & 3 make up route 1, and links 2 & 3 make up route 2). The 'reduced traffic' assumption of the approximation can be clearly seen in the above expression. The Erlang fixed-point approximation has been largely used to compute approximate blocking probabilities for large loss networks numerically till date. Kelly ([22]) proves that the fixed point (i.e, the solution to 1.18) both exists and is unique via Brouwer's fixed point theorem. Hence, it is always possible to iteratively compute approximate blocking probabilities from 1.18. However, we will now use the fixed-point approximation to derive analytical results on dimensioning our loss network.

To do so, we will now introduce the concept of shadow prices.

1.4.2 Shadow prices

The concept of shadow prices was introduced by Kelly in [22], as a tool to aid in the dimensioning of loss networks.

Definition 1.4.1. Shadow price. The shadow price of a link is defined as the derivative of the total revenue generated by the loss network with respect to the capacity of that link. Using previously established notation, the shadow price s_i of a link i is:

$$s_i = \frac{P_t}{C_i} \tag{1.19}$$

As with the fixed-point approximation, shadow prices are defined by a set of equations that yield a unique fixed point ([22]). The general form can be found in [22], and the set of equations that yield shadow prices for our loss networks are:

$$\begin{aligned} s_1 &= \eta_1 \lambda_1 (1 - b_3)(p_1 - s_3) \\ s_2 &= \eta_2 \lambda_2 (1 - b_3)(p_2 - s_3) \\ s_3 &= \eta_3 (\lambda_1 (1 - b_1)(p_1 - s_1) + \lambda_2 (1 - b_2)(p_2 - s_2)) \end{aligned} \tag{1.20}$$

where

$$\eta_i = \eta(\rho_1, C_i) = E(\rho_i, C_i - 1) - E(\rho_i, C_i) \tag{1.21}$$

and

$$\begin{aligned} \rho_1 &= \lambda_1 (1 - b_3) \\ \rho_2 &= \lambda_2 (1 - b_3) \\ \rho_3 &= \lambda_1 (1 - b_1) + \lambda_2 (1 - b_2) \end{aligned} \tag{1.22}$$

1.4.3 Sharing paradigm (Equal Prices)

We now assume that the price per call of both MSPs is equal, i.e, $p_1 = p_2 = p$.

Theorem 1.4.1. *When the above assumption on equality of the price per call holds, the revenue-maximizing sharing policy is for the MSPs to share their spectrum fully, i.e,* $k_1 = N_1$ *and* $k_2 = N_2$.

Proof. We start our proof by re-emphasizing our assumption of superadditivity (1.10). We assume that the loss network starts from an under-dimensioned state, i.e, initially, $k_1 = 0$ and $k_2 = 0$. The superadditivity condition guarantees that in this under-dimensioned state, the shadow prices $s_1 > 0$ and $s_2 > 0$, since MSPs have an incentive to share, and so the gradient will be positive. To prove the theorem, we will prove that the shadow prices s_1 and s_2 never hit zero, which in turn implies that the MSPs always have an incentive to share more. Hence, they will share as much of their spectrum as they can - resulting in full sharing. Note that this proof method is valid only when considering continuous C_i, not discrete. Kelly ([22]) however assumes continuous link capacities in his shadow price analysis, and our proof doesn't assume discrete capacities anywhere.

The shadow price equations now become:

$$\begin{aligned} s_1 &= \eta_1 \lambda_1 (1 - b_3)(p - s_3) \\ s_2 &= \eta_2 \lambda_2 (1 - b_3)(p - s_3) \\ s_3 &= \eta_3 (\lambda_1 (1 - b_1)(p - s_1) + \lambda_2 (1 - b_2)(p - s_2)) \end{aligned} \tag{1.23}$$

In order to obtain optimal dimensioning results, we assume:

$$\begin{aligned} s_1 &= 0 \\ s_2 &= 0 \end{aligned} \tag{1.24}$$

Note that by the equations that define s_1 and s_2, if we assume one to be zero, the other also must be zero. Also note from the definition of b_3 (1.18) that $b_3 < 1$ unless:

$$\begin{aligned} N_1 + N_2 = 0 \\ or \\ \lambda_1(1-b_1) + \lambda_2(1-b_2) \to \infty \end{aligned} \tag{1.25}$$

Since we have $N_1 > 0$ and $N_2 > 0$, and assuming finite λ_1 and λ_2 for now, we get:

$$s_3 = p \tag{1.26}$$

Substituting into the last equation in 1.23, we get:

$$\begin{aligned} p = p\eta_3(\lambda_1(1-b_1) + \lambda_2(1-b_2)) \\ \implies 1 = \eta_3(\lambda_1(1-b_1) + \lambda_2(1-b_2)) \\ \implies \rho_3\eta_3 = 1 \end{aligned} \tag{1.27}$$

Note that an immediate implication of the above equation is that the sharing policy of the MSPs will be constant as long as prices per call are equal, regardless of the magnitude of that price. (η is defined in 1.21.)

Theorem 1.4.2. *$\rho\eta < 1$, where $\eta = E(\rho, C-1) - E(\rho, C), \forall \rho, C$.*

Proof. We know that the Erlang-B function $E(\rho, C)$ is convex in C (see Theorem 1 in [24]).
Hence, we can say that:

$$E(\rho, C-1) - E(\rho, C) > E(\rho, C) - E(\rho, C+1) \tag{1.28}$$

Hence, it follows that $\eta(\rho, C)$ is a decreasing function of C.
$\implies$ $\rho\eta$ is also a decreasing function of C.
Now, for a finite ρ, we have:

$$\begin{aligned} E(\rho, 0) = 1 \\ E(\rho, 1) = \frac{\rho}{\rho+1} \\ E(\rho, 2) = \frac{\rho^2/2}{1+\rho+\rho^2/2} \end{aligned} \tag{1.29}$$

It is easy to show that for finite ρ,

$$\begin{aligned}\rho(E(\rho,0)-E(\rho,1)) &= \frac{\rho}{\rho+1} < 1\\ \rho(E(\rho,1)-E(\rho,2)) &= \frac{\rho^2/2}{1+\rho+\rho^2/2} < 1\end{aligned} \tag{1.30}$$

Hence, $\forall$ finite ρ and C, $\rho\eta < 1$. ■

We'll now, for completeness, consider the system when $\lambda_1 \to \infty$ and $\lambda_2 \to \infty$. The equation 1.27 can be rewritten as:

$$\rho_3(e(\rho_3, C_3 - 1) - E(\rho_3, M_3)) = 1 \tag{1.31}$$

Since $C_3 = N_1 + N_2$ is known, the only variable is ρ_3. Assume now that a solution to this equation exists, say $\rho_3{}^*$. Then the value of $b_3{}^* = E(\rho_3{}^*, C_3)$ is also fixed. We now have:

$$b_1 = E(\lambda_1(1 - b_3{}^*), N_1 + k_2)b_2 = E(\lambda_2(1 - b_3{}^*), N_2 + k_1) \tag{1.32}$$

We'll now make use of the main result in [25] that is restated below:

Theorem 1.4.3. *For any $\epsilon \in (0, 1)$, if*

$$\rho \geq n + \frac{1}{\epsilon} \tag{1.33}$$

then

$$1 - \frac{n}{\rho} < E(\rho, n) < 1 - \frac{n}{\rho} + \epsilon \tag{1.34}$$

This theorem implies that as traffic goes to infinity, the Erlang-B function tends to the LHS of the above inequality. Hence, as $\lambda_1 \to \infty$ and $\lambda_2 \to \infty$, we have:

$$\begin{aligned}b_1 &\to 1 - \frac{N_1 + k_2}{\lambda_1(1 - b_3{}^*)}\\ b_2 &\to 1 - \frac{N_2 + k_1}{\lambda_2(1 - b_3{}^*)}\end{aligned} \tag{1.35}$$

Hence,

$$\lambda_1(1 - b_1)(1 - b_3{}^*) + \lambda_2(1 - b_2)(1 - b_3{}^*) \to N_1 + N_2 + k_1 + k_2 \tag{1.36}$$

The LHS of this limit, however, is the average number of calls being served at a given time. Hence,

$$\lambda_1(1-b_1)(1-b_3{}^*) + \lambda_2(1-b_2)(1-b_3{}^*) \leq N_1 + N_2 \implies N_1 + N_2 + k_1 + k_2 \leq N_1 + N_2 \tag{1.37}$$

This is a contradiction, since our model assumes that $k_1 > 0$, $k_2 > 0$ (since some sharing must occur). hence, our assumption (1.27) is wrong.

We have arrived at a contradiction, and hence our initial assumption (1.24) is wrong. Hence, the shadow prices s_1 and s_2 can never be zero, and it is always in the best interest of the MSPs to dimension their loss network to the fullest, i.e

$$k_1 = N_1$$
$$k_2 = N_2$$

■

1.4.4 Sharing paradigm (Unequal Prices)

For this section, we will assume without loss of generality that $p_1 < p_2$.

Theorem 1.4.4. *When the above assumption on prices per call holds, the revenue-maximizing sharing policy is for the MSP charging the lower price (here, MSP 1) to share its spectrum fully (i.e, $k_1 = N_1$), and for the other MSP (here, MSP 2) to share some portion of its spectrum.*

Proof. Our proof method is: assume that initially the prices are equal, ie. $p_1 = p_2 = p$.The network is also hence fully dimensioned, i.e, $k_1 = N_1$ and $k_2 = N_2$. We will show that as p_2 is increased (and link dimensioning is kept static), s_2 stays positive, but s_1 does not - instead becoming increasingly negative, implying a larger decrease in k_2. Writing the shadow price equations again:

$$\begin{aligned} s_1 &= \eta_1\lambda_1(1-b_3)(p_1-s_3) \\ s_2 &= \eta_2\lambda_2(1-b_3)(p_2-s_3) \\ s_3 &= \eta_3(\lambda_1(1-b_1)(p_1-s_1)+\lambda_2(1-b_2)(p_2-s_2)) \end{aligned} \tag{1.38}$$

Upon inspection of these equations, it is clear that dividing all 3 equations by a constant value will result in a set of three equations with appropriately scaled shadow prices. We hence divide all 3 equations by p_1, and in a slight abuse of notation, continue to refer to the scaled shadow prices as s_i. We also define

$$R = \frac{p_2}{p_1} > 1 \tag{1.39}$$

The shadow price equations are now:

$$\begin{aligned} s_1 &= \eta_1\lambda_1(1-b_3)(1-s_3) \\ s_2 &= \eta_2\lambda_2(1-b_3)(R-s_3) \\ s_3 &= \eta_3(\lambda_1(1-b_1)(1-s_1)+\lambda_2(1-b_2)(R-s_2)) \end{aligned} \tag{1.40}$$

We can treat the above set of equations as a set of 3 linear equations in $\{s_1, s_2, s_3\}$.

Solving for s_1 and s_2, we get:

$$s_1 = \frac{D(1-AB)R + D(-A+AB-CR)}{-(AB+CD-1)}$$
$$s_2 = \frac{B(1-CD) + B(-A-RC+RCD)}{-(AB+CD-1)} \tag{1.41}$$

where

$$\begin{aligned} A &= \eta_3\lambda_1(1-b_1) \\ B &= \eta_1\lambda_1(1-b_3) = \eta_1\rho_1 \\ C &= \eta_3\lambda_2(1-b_2) \\ D &= \eta_2\lambda_2(1-b_3) = \eta_2\rho_2 \end{aligned} \tag{1.42}$$

From 1.4.2, we observe the following:

$$\begin{aligned} A+C = \eta_3\rho_3 &< 1 \\ B &< 1 \\ D &< 1 \end{aligned} \tag{1.43}$$

$$\implies AB + CD < 1$$

$\implies$ the denominators of the expressions in 1.41 are positive. Consider now the numerators of:

- $\mathbf{s_1}$:

  $B(1-CD) + B(-A-RC+RCD) = B(1-A-CD+R(C(D-1))$

  When $R = 1$, i.e, equal prices, this simplifies to $B(1-A-C) > 0$.

  Note that $D < 1 \implies C(D-1) < 0$. Hence, as R increases, s_1 decreases until it crosses 0 at some value of R.

- $\mathbf{s_2}$:

  $A(B-1) + R(1-C-AB)$

  When $R = 1$, this simplifies to $1-A-C > 0$.

  As R increases, s_2 becomes increasingly positive (since $1-C-AB > 0$).

Hence, from the above behaviour of the shadow prices s_1 and s_2, we can say that the optimal sharing policy (for $p_1 < p_2$) is:

$$\begin{aligned} k_1 &= N_1 \\ k_2 &= {k_2}^* \end{aligned} \tag{1.44}$$

where ${k_2}^*$ is such that:

$$\begin{aligned} 1 - A - CD + R(C(D-1)) &= 0 \\ \implies 1 - A - CD = RCD - RC &= 0 \\ \implies 1 - A + (R-1)CD - RC &= 0 \\ \implies 1 + (R-1)CD &= RC + A \end{aligned} \tag{1.45}$$

$$\implies 1 + (R-1)\eta_3\eta_2{\lambda_2}^2(1-b_2)(1-b_3) = R\eta_3\lambda_2(1-b_2) + \eta_3\lambda_1(1-b_1)$$

■

1.5 Call Pricing

With knowledge of how MPSs will optimally share their spectrum with each other, we'll now consider the following question: MSPs now know their optimal sharing strategies given prices. How will they strategically set prices to maximize their profit?

1.5.1 Non-strategic customers

In this section, we will assume that customers are non-strategic, i.e, they cannot make any decisions regarding whether or not to use a certain MSP. Instead, MSPs can choose how much traffic they accept to service, i.e, MSP i can choose λ_i. This model is studied in [26], where it is posed as an optimization problem - the objective function is the revenue earned by the entire loss network:

$$\begin{aligned} \text{min } -\sum p_k \lambda_k (1 - B_k) \\ \text{over } \lambda_k \forall k \\ \text{subject to} \\ \lambda_k \geq 0 \ \forall k \\ \text{and the Erlang fixed-point distribution} \end{aligned} \tag{1.46}$$

Karush-Kuhn-Tucker conditions are used to establish necessary conditions for an optimal traffic allocation. The main result of [26] is stated below.

Theorem 1.5.1. *When the traffic allocation in a loss network is optimal, then:*

- *For every route k to which traffic offered is nonzero,*

$$c_k + \sum_{l \in r} \eta_l v_l = 0 \tag{1.47}$$

- *For every route k to which traffic offered is zero,*

$$c_k + \sum_{l \in r} \eta_l v_l \leq 0 \tag{1.48}$$

Here, we assume a route in a loss network is a set of links.

Also,

$$v_i = \frac{-s_i}{\eta_i} \tag{1.49}$$

Assume also, without loss of generality, that $p_1 > p_2$. We'll now state the main theorem of this section:

Theorem 1.5.2. *In order to maximize revenue, MSPs should direct all traffic towards the MSP charging the higher price per call, and the other MSP should only cooperate by sharing spectrum slots.*

Proof. Recall that as per the sharing paradigm previously established, $k_2 = N_2$, and:

$$\begin{aligned} s_1 &> 0 \\ s_2 &= 0 \end{aligned} \tag{1.50}$$

- Case 1: $\boldsymbol{\lambda_1 > 0, \lambda_2 > 0}$
  The simplified necessary conditions for optimality are:

  $$\begin{aligned} s_1 + s_3 &= p_1 \\ s_2 + s_3 &= p_2 \end{aligned} \tag{1.51}$$

  $\implies p_2 = s_3$ (1.50)

  But from the definition of shadow prices,
  $s_1 = \eta_1 \lambda_1 (1 - b_3)(p_1 - s_3)$.
  Together with 1.52,
  $\implies p_1 - s_3 = \eta_1 \lambda_1 (1 - b_3)(p_1 - s_3)$
  $\implies p_1 - p_2 = \eta_1 \lambda_1 (1 - b_3)(p_1 - p_2)$
  $\implies \eta_1 \lambda_1 (1 - b_3) = 1$ (since $p_1 > p_2$)
  $\implies \rho_1 \eta_1 = 1$.
  This isn't possible (1.4.2). Hence, such a case cannot be a optimal traffic allocation.

- Case 2: $\boldsymbol{\lambda_1 = 0, \lambda_2 > 0}$
  The simplified necessary conditions for optimality are:

  $$\begin{aligned} s_1 + s_3 &\geq p_1 \\ s_2 + s_3 &= p_2 \end{aligned} \tag{1.52}$$

  $\implies p_2 = s_3$ (1.50)

  But from the definition of shadow prices,
  $s_1 = \eta_1 \lambda_1 (1 - b_3)(p_1 - s_3)$.
  Together with 1.52,
  $\implies p_1 - s_3 \leq \eta_1 \lambda_1 (1 - b_3)(p_1 - s_3)$

$\implies p_1 - p_2 \leq \eta_1\lambda_1(1-b_3)(p_1-p_2)$
$\implies \eta_1\lambda_1(1-b_3) \geq 1$ (since $p_1 > p_2$)
$\implies \rho_1\eta_1 \geq 1.$
This isn't possible (1.4.2). Hence, such a case cannot be a optimal traffic allocation either.

- Case 3: $\boldsymbol{\lambda_1 > 0, \lambda_2 = 0}$
  We have:
  $s_1 + s_3 = p_1$
  $s_2 + s_3 \geq p_2$
  . 1.50 yields $s + 3 \geq p_2$
  $\implies p_1 - s_1 \geq p_2$
  $\implies p_1 - p_2 \geq s_1$
  $\implies \eta_1\rho_1 \leq 1.$

  Hence, an optimal traffic allocation could exist here, and it would be defined by the equation: $s_1 + s_3 = p_1$

■

1.5.2 Strategic Customers

The question of how MSPs should set prices when customers are strategic is as yet unanswered (see [27] - [31] for references being used in current work). This question straddles the realms of mechanism design and economics - for example, one can assume that customers behave according to a demand curve based on the price offered to them by MSPs - alternatively, they could consider *perceived price* instead. Customers could be selfish, aiming to minimize their own perceived prices - this could be modelled using the concept of a Wardrop equilibrium (a good reference is [30]). Customers could also be forced to cooperate to achieve a social optimum instead.

1.6 Conclusion

We see that the model we are analyzing heavily skews towards the MSP that is charging a higher price per call - optimally necessitating the complete spectrum of the other MSP to be shared with it, as well as the entirety of traffic to be redirected towards it. We note that this entire analysis has been carried out uner the Erlang fixed-point approximation, and hence, may only be an approximate optimum. These are, however, one of the first analytical results on the dimensioning of loss networks that have been established. Further work remains to be done on optimal pricing policies when customers are strategic.

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