QUANTUM FIELD THEORY IN DE SITTER SPACETIME

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under the guidance of

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CERTIFICATE

This is to certify that the project entitled **Quantum field theory in de Sitter spacetime** submitted by S. Sunil Kumar is a bona fide record of work done by him towards the partial fulfilment of the requirements for the award of the Degree in Bachelor in Electrical Engineering at Indian Institute of Technology, Madras, Chennai, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

De Sitter spacetime is a cosmological solution to field equations of general relativity and has been studied extensively as it is a maximally symmetric solution. It models the universe by neglecting ordinary matter considering the contribution only due to positive cosmological constant in describing the dynamics of the universe. This report is aimed at studying certain aspects of quantum field theory in de Sitter spacetime. After getting familiar with the essential classical aspects of the de Sitter spacetime, we investigate the behaviour of a massive quantum scalar field to understand some of the important phenomena associated with the de Sitter spacetime.

Contents

Chapter 1

Introduction

Study of maximally symmetric solutions of Einstein's equation have assumed great importance in the recent past. A few important ones among them are the Minkowski (flat) spacetime, de Sitter spacetime (driven by positive cosmological constant) and Anti de Sitter spacetime (sourced by negative cosmological constant). From the view point of physics, de Sitter spacetime is different from Minkowski spacetime due to the fact that it is the solution for Einstein's equations with positive cosmological constant and no matter sources in contrast to Minkowski spacetime which is the solution with no cosmological constant and also no matter sources. However, the maximally symmetric nature of both of these spacetimes implies that they both have the same number of independent components of Riemann tensor.

De Sitter spacetime is the maximally symmetric, vacuum solution of Einstein's equations with a positive cosmological constant Λ (corresponding to a positive vacuum energy density and negative pressure). De Sitter spacetime has been studied vastly as it highly symmetric curved space which makes it easier to quantize fields and obtain simple exact solutions. It is also used to describe the phase of accelerated expansion referred to as inflation that occurs in the early universe. De Sitter model is widely used for pedagogical purpose as it assumes the matter contribution to be zero which is a close approximate although not completely true in the real universe we live in.

This report is aimed at studying certain aspects of quantum field theory in de Sitter spacetime. The report has been divided into four main chapters. In the second chapter, we review the classical properties of de Sitter spacetime. This includes study of various useful coordinate systems that exploit the symmetry properties of the spacetime. We also study the transformations among these various coordinate systems as not all are equally convenient at all times. In the process, we establish the expansion rate of the universe, H in terms of the cosmological constant and discuss the implications of it. In the classical properties, we also study the causal structure of the de Sitter spacetime in various coordinate systems using the Penrose diagrams.

In the next chapter, we study the quantum field theory in flat spacetime as a preface to more rigorous study of quantum field theory in curved spacetime. This includes the canonical quantization of the field in the Heisenberg picture. We present the quantization in two different basis, one being the plane wave basis and the other is the spherical basis. We use different coordinate systems in the process, viz the conventional Cartesian coordinate systems for the plane wave basis and the spherical polar coordinates for the spherical harmonics. Following this, we introduce Green functions in the last section of this chapter and present a detailed picture of them in flat spacetime.

After understanding the essential aspects of de Sitter spacetime and the quantum field theory, we proceed to discuss the quantum field theory in curved spacetime, more specifically in de Sitter spacetime. We try to understand the ambiguity in the choice of vacuum in the curved spacetime and in the process, present a brief description of the Bogoliubov transformations. Further, we describe the de Sitter invariant vacua for massive scalar fields. We show that a unique vacuum is not selected only by requiring that it be de Sitter invariant as all the invariant states form a one parameter family. We show how the entire family of states can be generated from a single vacuum state called Euclidean vacuum by trivial frequency independent Bogoliubov transformations. In the later parts of the chapter, we present a proof of how a massless scalar field has no de Sitter invariant vacuum state.

In the penultimate chapter, we discuss an exotic phenomenon that is a characteristic of curved spacetimes, viz particle production. We derive the equations for Green functions in a de Sitter invariant form, both in closed as well as flat coordinates. We solve the wave equations to obtain the non-trivial Bogoliubov transformations for the mode expansions at past and future infinity. We establish quantitatively, the probability amplitudes for pair production and also compute the decay rate. This informally marks the end of this report.

The report is formally concluded by presenting an overall picture with all the results summarised in the last chapter and a brief discussion about their implications.

Chapter 2

Classical Aspects of de Sitter spacetime

In this chapter, we study the classical geometry of de Sitter spacetime in arbitrary dimension. Two methods are employed for this. One is directly by solving the Einstein equation for the metric ansatz and the second is by using various useful coordinate systems with different transformations among them. The metric signature that we are going to use in this report is $(-1, 1, 1, \ldots)$

2.1 Solution by Einstein equation

In d-spacetime dimensions, the Einstein- Hilbert action coupled to matter is given by

$$S[g_{\mu\nu}] = \frac{1}{16\pi G} \int \mathrm{d}^d x \sqrt{-g}(R - 2\Lambda) + S_m,$$

where S_m is the matter action of interest, which vanishes for the limit of pure gravity. The cosmological constant Λ is positive for de Sitter spacetime (dS_d) . The above actions yields the Einstein equations

$$G_{uv} + \Lambda g_{uv} = T_{uv}. \tag{2.1}$$

The energy-momentum tensor is T_{uv} is given by

$$T_{uv} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{uv}}.$$

For pure dS_d , the energy- momentum tensor vanishes so that the Einstein equations become

$$G_{uv} = -\Lambda g_{uv}.$$

For an empty spacetime with a positive constant vacuum energy ($\Lambda > 0$) we get

$$T_{uv}^{vacuum} = \frac{\Lambda}{8\pi G} g_{uv}. \tag{2.2}$$

The only non-trivial component of the Einstein equations is Ricci Scalar, R . From (2.1), we get

$$G_{uv} = -\Lambda g_{uv},$$
$$g^{uv}(R_{uv} - \frac{1}{2} g_{uv} R) = -\Lambda g^{uv} g_{uv}.$$

Since the spacetime we are working with is d dimensional, $g_{uv}g^{uv} = d$ which gives

$$R = \frac{2\Lambda d}{d-2} \ . \tag{2.3}$$

Ricci scalar being positive implies that de Sitter spacetime is maximally symmetric, of which the local structure is characterized by a positive constant curvature scalar alone such as

$$R_{\mu v\rho\sigma} = \frac{1}{d(d-1)} (g_{\mu\rho} g_{v\sigma} - g_{\mu\sigma} g_{v\rho}) R \ . \tag{2.4}$$

Computing the Kretschmann scalar

$$\begin{aligned} R_{\mu v\rho\sigma} R^{\mu v\rho\sigma} &= \left(\frac{R}{d(d-1)} \right)^2 (g_{\mu\rho} g_{v\sigma} - g_{\mu\sigma} g_{v\rho})(g^{\mu\rho} g^{v\sigma} - g^{\mu\sigma} g^{v\rho}) \\ &= \left(\frac{2R^2}{d(d-1)} \right) \ . \end{aligned}$$

Scalar curvature being constant everywhere implies the fact that dS_d is free from physical singularities which is confirmed by calculating the Kretschmann scalar which also turns out to be constant.

2.2 Coordinate systems

In this section, we shall discuss various coordinate systems that can be constructed to understand the properties of de Sitter spacetime. Four different coordinate systems are employed and various transformations among them are studied.

2.2.1 Global coordinates (τ, θ_i)

De Sitter spacetime can be viewed as an embedding of the dS_d into flat $(d+1)$ dimensional Minkowski spacetime. We know, that for a Minkowski spacetime, the Einstein equation is trivially satisfied. For a Minkowski spacetime of $(d+1)$ dimensions, we have

$$\begin{aligned} 0 &= {}^{d+1}R, \\ &= g^{AB}R_{AB}, \\ &= R + {}^{d}R. \end{aligned}$$

The capital indices A, B run from 0 to d representing the $(d+1)$ Minkowski spacetime. Setting ${}^{d}R = -2\Lambda d/d-2$, we recover the Einstein equation of dS_d. This implies a positive constant curvature of the embedding space. Topology of such embedding can be visualised as an algebraic constraint of a hyperboloid given by

$$\eta_{AB}X^A X^B = l^2, \tag{2.5}$$

$$-X^0X^0 + X^1X^1 + \ldots + X^dX^d = l^2. \tag{2.6}$$

η_{AB} is the metric for $(d+1)$ dimensional Minkowski spacetime and so is diag. $(-1, 1, 1 \ldots 1)$. The metric for the $(d+1)$ Minkowski is

$$\mathrm{d}s^2 = \eta_{AB}\mathrm{d}X^A\mathrm{d}X^B\,. \tag{2.7}$$

This metric constrained by (2.5) represents the dS_d. Using (2.6) to eliminate the last spatial coordinate X^d from the metric (2.7) we get

$$\mathrm{d}X^d = \mp\frac{\eta_{\mu v}X^\mu \mathrm{d}X^v}{\sqrt{l^2 - \eta_{\alpha\beta}X^\alpha X^\beta}}.$$

The Greek indices μ, v, α, β run from 0 to $d-1$. From this we get the induced metric $g_{\mu v}$ of the curved de Sitter spacetime due to the embedding as

$$\begin{aligned} g_{\mu v} &= \eta_{\mu v} + \frac{X_\mu X_v}{l^2 - \eta_{\alpha\beta}X^\alpha X^\beta}, \\ g^{\mu v} &= \eta^{\mu v} - \frac{X^\mu X^v}{l^2}. \end{aligned}$$

From this metric, the induced connection, the Riemann tensor and the Ricci tensor can be obtained to be

$$\Gamma^{\mu}_{v\rho} = \frac{1}{l^2}\left(\eta_{v\rho}X^{\mu} + \frac{X^{\mu}X_{\rho}X_{v}}{l^2 - \eta_{\alpha\beta}X^{\alpha}X^{\beta}}\right), \tag{2.8}$$

$$R_{\mu v} = \frac{d-1}{l^2}\left(\eta_{\mu v} + \frac{X_{\mu}X_{v}}{l^2 - \eta_{\alpha\beta}X^{\alpha}X^{\beta}}\right) = \left(\frac{d-1}{l^2}\right) g_{\mu v}, \tag{2.9}$$

$$R = R_{\mu v}g^{\mu v} = \frac{d(d-1)}{l^2}. \tag{2.10}$$

Using (2.3) and (2.10), the cosmological constant Λ can be written in terms of length l as

$$\Lambda = \frac{(d-1)(d-2)}{l^2}. \tag{2.11}$$

From the constraint of the de Sitter spacetime as the hyperboloid embedding in the flat Minkowski spacetime, it can be seen that the relation between X^0 and the spatial sections $(X^1, X^2 \ldots X^d)$ is hyperbolic of the form $X^2 - Y^2 = C^2$. It can also be seen that spatial sections of constant X^0 form a sphere of the radius $\sqrt{l^2 + (X^0)^2}$. A convenient choice of coordinate system satisfying the constraint would be

$$\begin{aligned} X^0 &= l \sinh\left(\frac{\tau}{l}\right), \\ X^{\alpha} &= l\,\omega^{\alpha} \cosh\left(\frac{\tau}{l}\right),\ (\alpha = 1, 2, ..., d). \end{aligned}$$

where $-\infty < \tau < \infty$ and $\omega^{\alpha'} s$ satisfy the relation $\sum_1^d \omega^{\alpha} = 1$. Hence, the spatial coordinates can be expressed in terms of $(d-1)$ angle variables as

$$\begin{aligned} \omega^1 &= \cos\theta_1, \\ \omega^2 &= \sin\theta_1 \cos\theta_2, \\ &\vdots \\ \omega^d &= \sin\theta_1 \sin\theta_2 ..... \sin\theta_{\mathrm{d}-2} \sin\theta_{\mathrm{d}-1}, \end{aligned}$$

where $0 < \theta_{(1 \ldots d-2)} < \pi$ and $0 < \theta_{d-1} < 2\pi$. Using the above coordinates system we can rewrite the metric given by (2.7) as

$$\begin{aligned} \mathrm{d}s^2 = &-\cosh^2\left(\frac{\tau}{l}\right)\mathrm{d}\tau + \sinh^2\left(\frac{\tau}{l}\right)\left(\sum \omega^{2\alpha}\right)\mathrm{d}\tau + l^2\cosh^2\left(\frac{\tau}{l}\right)[(-\sin\theta_1 \mathrm{d}\theta_1)^2 \\ &+ (\cos\theta_1 \cos\theta_2 \mathrm{d}\theta_1 - \sin\theta_1 \sin\theta_2 \mathrm{d}\theta_2)^2 + \ldots] \\ = &-\mathrm{d}\tau^2 + l^2\cosh^2\left(\frac{\tau}{l}\right)\mathrm{d}\Omega^2_{d-1}, \end{aligned}$$

where $\mathrm{d}\Omega_{d-1}^2 = \sum_{j=1}^{d-1}\left(\Pi_{i=1}^{j-1}\sin^2\theta_{\mathrm{i}}\right)\mathrm{d}\theta_j^2$. The singularities in the above metric are not the physical singularities but just the singularities associated with this specific choice of coordinate system. This is confirmed by Ricci scalar as well as Kretschmann curvatures being positive. A Killing vector easily seen from this form of the metric is $\partial/\partial\theta_{d-1}$ as the metric is invariant under the rotation of the coordinate θ_{d-1} . The spatial hypersurfaces in this coordinate system are $(d-1)$ spheres of radius $l\cosh\left(\tau/l\right)$. Another way to obtain the above form of the metric is by assuming the metric with an unknown function$f\left(\tau/l\right)$ as

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + l^2 f^2\left(\frac{\tau}{l}\right)\mathrm{d}\Omega_{d-1}^2.$$

From this, we calculate the Ricci scalar and equate it to the form given by (2.3). Refer to Appendix A.1 for more detailed calculation of the intermediate steps. The Ricci scalar is

$$R = (d-1)\frac{(d-2)(1+\dot{f}^2) + 2f\ddot{f}}{l^2 f^2}, \tag{2.12}$$

where a single over-dot represents a single derivative and a double dot represents a double derivative with respect to τ. Equating this form of the Ricci scalar to the form obtained by computing it from the hyperboloid constraint, we obtain that

$$2(f\ddot{f} - \dot{f}^2 - 1) = d(-\dot{f}^2 + f^2 - 1).$$

A solution for the above second order differential equation will be in terms of d. However for the solution to be independent of d, the following couple of equations have to be solved, i.e

$$f\ddot{f} - \dot{f}^2 - 1 = 0,$$
$$-\dot{f}^2 + f^2 - 1 = 0.$$

A non trivial solution to the above set of simultaneous equation is

$$f\left(\frac{\tau}{l}\right) = \pm\cosh\left(\frac{\tau}{l}\right). \tag{2.13}$$

It has to be noted that this is equivalent to the metric obtained by a specific choice of coordinates mentioned previously, which cover the entire de Sitter spacetime.

2.2.2 Conformal coordinates (T, θ_i)

An interesting property of the dS_d can be observed by evaluating the Weyl (conformal) tensor, which is given by

$$C_{\mu v \rho \sigma} = R_{\mu v \rho \sigma} + \frac{1}{d-2}(g_{\mu\sigma}R_{v\rho} + g_{v\rho}R_{\mu\sigma} - g_{\mu\rho}R_{v\sigma} - g_{v\sigma}R_{\mu\rho})$$
$$+\frac{1}{(d-1)(d-2)}(g_{\mu\rho}g_{v\sigma} - g_{\mu\sigma}g_{v\rho})R.$$

Using the argument that the dS_d is a maximally symmetric spacetime, its Ricci tensor $R_{\mu v \rho \sigma}$ can be written as

$$R_{\mu v \rho \sigma} = \frac{1}{d(d-1)}(g_{\mu\rho}g_{v\sigma} - g_{\mu\sigma}g_{v\rho})R. \tag{2.14}$$

A straight forward computation of Ricci tensor $R_{\mu v}$ from above yields

$$R_{\mu v} = \left(\frac{d-1}{l^2}\right) g_{\mu v},$$
$$R = \frac{d(d-1)}{l^2}.$$

A look at (2.10) shows that this has been already obtained by solving for the hyperboloid constraint in the previous section. Using the above results, we get

$$C_{\mu v \rho \sigma} = \frac{1}{d(d-1)}(g_{\mu\rho}g_{v\sigma} - g_{\mu\sigma}g_{v\rho})R + \frac{(d-1)}{l^2(d-2)}(g_{\mu\sigma}g_{v\rho} + g_{v\rho}g_{\mu\sigma} - g_{\mu\rho}g_{v\sigma} - g_{v\sigma}g_{\mu\rho})$$
$$+ \frac{1}{(d-1)(d-2)}(g_{\mu\rho}g_{v\sigma} - g_{\mu\sigma}g_{v\rho})R,$$
$$= \left\{\frac{1}{d(d-1)} - \frac{2}{d(d-2)} + \frac{1}{(d-1)(d-2)}\right\}(g_{\mu\rho}g_{v\sigma} - g_{\mu\sigma}g_{v\rho})R = 0.$$

Hence, maximally symmetric nature of dS_d has led to the fact that the conformal tensor vanishes for dS_d .

Using this result, dS_d can also be studied in terms of conformal coordiante system. Let the conformal time be T . The metric can be expressed as

$$\mathrm{d}s^2 = F^2\left(\frac{T}{l}\right)(-\mathrm{d}T^2 + l^2\mathrm{d}\Omega^2_{d-1}).$$

Again, a single over dot represents a single derivative and double dot represents a double derivative with respect to T. Upon a little computation, we get the Ricci scalar as

$$R = (d-1)\frac{(d-4)\dot{F}^2 + (d-2)F^2 + 2F\ddot{F}}{l^2F^4}.$$

Equating this form of the Ricci scalar to the form obtained by computing it from the hyperboloid constraint we get

$$2(F\ddot{F} - F^2 - 2\dot{F}^2) = d(F^4 - \dot{F}^2 - F^2).$$

The solution to the above equation, irrespective of d , is obtained by solving the simultaneous equations

$$\begin{aligned} F\ddot{F} - F^2 - 2\dot{F}^2 &= 0, \\ F^4 - \dot{F}^2 - F^2 &= 0. \end{aligned}$$

With the condition that $F(0) = 1$, the solution to the above is $F(T/l) = \sec(T/l)$. Another way to obtain the solution for $F(T/l)$ is comparing the conformal line element to the one that is dealt with in the global coordinates case. The coordinate transformation between the two coordinate systems can be captured in

$$\begin{aligned} F^2(T/l) &= \cosh^2(\tau/l), \\ \mathrm{d}T &= \pm \mathrm{d}\tau/\cosh(\tau/l), \\ \frac{\mathrm{d}}{\mathrm{d}T}(\ln F) &= \pm\sqrt{F^2 - 1}. \end{aligned}$$

Upon solving the above, we get $F(T/l) = \sec(T/l)$. As can be seen from the above transformation, there exists a one-to-one correspondence between the two coordinate systems. Since the global coordinates cover the entire dS_d , one-to-one correspondence between these two coordinates suggests that the conformal coordinate systems is a good coordinate systems which covers the entire dS_d. The metric is isometric under the rotation of θ_{d-1} and hence $\partial/\partial\theta_{d-1}$ is a Killing vector. Thus, there is axial symmetry.

Penrose diagrams are good tools to study the causal behaviour of the spacetimes. The distances are highly distorted and infinity points are mapped on to finite points and the whole information about the causal structure is studied. Penrose diagrams will be discussed in detail at the end of this chapter. From the conformal metric, it should be noted that the topology of the dS_d is cylindrical. So, the process to make the Penrose diagram is to change the hyperboloid into a d dimensional cylinder of finite height.

2.2.3 Planar coordinates

We use this coordinate system exploiting the property of maximally symmetric nature of dS_d. The line element in planar coordinates is of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t/l)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

where $a(t/l)$ is the cosmic scale factor. Since dS_d is maximally symmetric, the $(d-1)$ dimensional spatial hypersurface should also be maximally symmetric and hence the Ricci tensor for the this spatial hypersurface will be of the form

$${}^{d-1}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}),$$

where k is a constant. The metric for the spatial hypersurface is $a^2\gamma_{ij}$ which give the value of k as

$$k = {}^{d-1}Ra^4/(d-1)(d-2).$$

We try to solve for $a(t/l)$ by calculating the Ricci scalar. The Ricci scalar is

$$R = (d-1)\frac{2a\ddot{a} + (d-2)(\dot{a}^2 + k)}{a^2}.$$

In the above equation, a single over-dot and a double over-dot represent single and double derivatives with respect to t respectively. The pure de Sitter spacetime we are studying can be interpreted as solutions to the Friedmann equations driven by a perfect fluid. A perfect fluid has the stress-energy tensor as

$$T_{\mu v} = (p + \rho)u_a u_b + p\eta_{\mu v}, \tag{2.15}$$

where u_a is the velocity of the fluid as measured by a comoving observer (in other terms, as measured in a local rest frame of the fluid, so has the form $(1, 0, 0...0)$) , ρ is the energy density and p is the pressure of the fluid. The equation of state for the cosmological perfect fluid is characterised by a dimensionless number w given by $w = p/\rho$. The equation of state can be used in FLRW equations to describe the evolution of an isotropic universe fill with a perfect fluid. The equation of state for cosmological constant is $w = p/\rho = -1$. With this relation, we get $T^\mu_v = \mathrm{diag.}\ (-\rho, p, p \ldots p)$. Equating the expression for stress-energy tensor in (2.2), we get

$$\rho = -p = \frac{\Lambda}{8\pi G}. \tag{2.16}$$

The spatial part of the metric γ_{ij} can be written in terms of Friedmann-Lemaitre-Robertson-Walker (FLRW) metric with $(d-1)$ dimensional spherical coordinates $[(r,\theta_i),\ i=1,2,...d-2]$ due to its isotropy and homogeneity. The line element becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-k(r/l)^2} + r^2\mathrm{d}\Omega^2_{d-2}\right]. \tag{2.17}$$

For this form of the metric with spatial part replaced by the FLRW metric, k can take values -1 (open), 1 (closed), 0 (flat) . This form of the metric can be solved for $a(t/l)$ using the Einstein equations. The Friedmann equations obtained by using (2.1) are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{d-2}\left(\frac{d}{d-1}\rho - (d-4)p\right) = \frac{d-2}{2(d-1)}\Lambda, \tag{2.18}$$

$$\frac{\ddot{a}}{a} = -4\pi G\left(\frac{\rho}{d-1} + p\right) = \frac{d-2}{2(d-1)}\Lambda. \tag{2.19}$$

From the above equations, it can be seen that for acceleration parameter determined by $\ddot{a}$ is always positive. The quantity $\dot{a}$ being positive implies that the universe is expanding (eternally) and is true for $k=0$ and $k=1$. However, for $k=-1$ the universe decelerates, reaches a stage of critical a_{cr} such that $\dot{a}=0$ which gives $a_{cr}=\Lambda\sqrt{2(d-1)/(d-2}$ and starts eternally expanding. The solution for $a(t)$ depends on the value of k and is given by

$$a = \begin{cases} l\sinh(t/l), & for\ k=-1, \\ \alpha\exp(\pm t/l) & for\ k=0. \\ l\cosh(t/l) & for\ k=+1, \end{cases}$$

where α is an arbitrary proportionality constant. This is a very remarkable results which shows the expansion of universe for a pure cosmological constant with contributions from other matter considered to be zero.

The constraint of the hyperboloid dealt with in section 2.2.1 corresponding to the de Sitter embedding in the flat Minkowski coordinates can be decomposed into two constraints. Using these two constraints, we will construct a coordinate system in which the line elements resemble the one in (2.17). The constraints can be decomposed as

$$-\left(\frac{X^0}{l}\right)^2 + \left(\frac{X^d}{l}\right)^2 = 1 - \left(\frac{x^i}{l}\right)^2 e^{(2t/l)}. \tag{2.20}$$

This is a hyperbola of radius

$$\sqrt{1-\left(\frac{x^i}{l}\right)^2 e^{(2t/l)}}.$$

The second constraint turns out to be sphere of radius $(x^i/l)\, e^{t/l}$. It follows from (2.20) and (2.6) that

$$\left(\frac{X^1}{l}\right)^2 + \left(\frac{X^2}{l}\right)^2 + \ldots + \left(\frac{X^{d-1}}{l}\right)^2 = \left(\frac{x^i}{l}\right)^2 e^{2t/l}.$$

A good coordinate system that can be constructed from the above constraints is

$$\frac{X^0}{l} = \sinh\left(\frac{t}{l}\right) + \frac{1}{2}(x^i/l)^2 e^{t/l}, \tag{2.21}$$

$$\frac{X^d}{l} = -\cosh\left(\frac{t}{l}\right) + \frac{1}{2}(x^i/l)^2 e^{t/l}, \tag{2.22}$$

$$\frac{X^i}{l} = \frac{x^i}{l} e^{t/l}, \quad [i = 1, 2, \ldots d-1], \tag{2.23}$$

where range of x^i is $-\infty < x^i < \infty$ and that of t is $-\infty < t < \infty$. This follows in a straight forward manner from the range of X^i. Constructing the line element for the above choice of coordinate system, we obtain that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2t/l}(\mathrm{d}x^i)^2.$$

However, $-X^0 + X^d = -l\, e^{t/l} < 0$. This implies that the above choice of coordinates cover only one half of the de Sitter spacetime. A slightly modified coordinate system is used to cover the other half of the de Sitter spacetime. We can rewrite the constraint of the hyperboloid in (2.6) as

$$-\left(\frac{X^0}{l}\right)^2 + \left(\frac{X^d}{l}\right)^2 = 1 - \left(\frac{x^i}{l}\right)^2 e^{-2t/l},$$

$$\left(\frac{X^1}{l}\right)^2 + \left(\frac{X^2}{l}\right)^2 + \ldots + \left(\frac{X^{d-1}}{l}\right)^2 = \left(\frac{x^i}{l}\right)^2 e^{-2t/l}.$$

A good choice of coordinate system to implement the above constraints is

$$\frac{X^0}{l} = \sinh\left(\frac{t}{l}\right) - \frac{1}{2}(x^i/l)^2 e^{-t/l}, \tag{2.24}$$

$$\frac{X^i}{l} = \frac{x^i}{l} e^{-t/l}, \quad [i = 1, 2, \ldots d-1], \tag{2.25}$$

$$\frac{X^d}{l} = \cosh\left(\frac{t}{l}\right) - \frac{1}{2}(x^i/l)^2 e^{-t/l}. \tag{2.26}$$

We proceed further and calculate the line element as done above for upper half of the de Sitter spacetime and we have

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{-2t/l}(\mathrm{d}x^i)^2. \tag{2.27}$$

This choice of coordinates cover the lower half of the de Sitter spacetime governed by the equation $-X^0 + X^d > 0$. It can be observed that both the forms of the line elements are identical to the flat solutions obtained by solving (2.18) and (2.19).The metric is invariant under spatial translations since it is independent of any of the spatial coordinates x^i. Hence $\partial/\partial x^i s$ are the Killing vectors and there exists translational as well as rotational symmetries.

2.2.4 Static coordinates (t, r, θ_i)

Instead of the choice of pair of constraints used in planar coordinates, the hyperboloid constraint of (2.6) can be written as below by introducing an additional parameter r. By doing so, we have

$$-\left(\frac{X^0}{l}\right)^2 + \left(\frac{X^d}{l}\right)^2 = 1 - \left(\frac{r}{l}\right)^2,$$
$$\left(\frac{X^1}{l}\right)^2 + \ldots + \left(\frac{X^{d-1}}{l}\right)^2 = \left(\frac{r}{l}\right)^2.$$

One of the these constraints is a sphere and the other is a hyperbola as was in the case of planar coordinates. Now we develop a coordinate system that satisfies the above constraints and obtain the line element in the corresponding coordinate system. The coordinates are

$$\frac{X^0}{l} = -\sqrt{1 - \left(\frac{r}{l}\right)^2}\sinh\left(\frac{t}{l}\right),$$
$$\frac{X^i}{l} = \frac{r}{l}\omega^i \quad [i = 1, 2, \ldots d-1],$$
$$\frac{X^0}{l} = -\sqrt{1 - \left(\frac{r}{l}\right)^2}\cosh\left(\frac{t}{l}\right),$$

where $\omega^i s$ are defined as

$$\omega^1 = \cos\theta_1,$$
$$\omega^2 = \sin\theta_1 \cos\theta_2,$$
$$\vdots$$
$$\omega^{d-1} = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-3} \sin\theta_{d-2}.$$

Hence $\sum_{i=1}^{d-1} \omega^i = 1$ and it follows that $\omega^i d\omega^i = 0$. Correspondingly,

$$\mathrm{d}s^2 = -\left(1 - \frac{r^2}{l^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{r^2}{l^2}\right)} + r^2\mathrm{d}\Omega^2_{d-2}, \tag{2.28}$$

where

$$\mathrm{d}\Omega^2 = \sum_{b=1}^{d-1}\prod_{a=1}^{b-1}\sin^2\theta_a \mathrm{d}\theta_b.$$

This form of the metric can also be obtained by solving the Einstein equations as it is done in the case of the other three coordinate systems in the previous sections. A static observer may introduce a static coordinate system where the metric involves two independent functions of the radial coordinate r which are $\Omega(r)$ and $A(r)$. Such a metric will be of the form

$$\mathrm{d}s^2 = -e^{2\Omega(r)}A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{A(r)} + r^2\mathrm{d}\Omega^2_{d-2}.$$

We proceed in the usual way of evaluating the components of Ricci tensor. As before, we would refer to appendix A.4 for the exact calculations.The Ricci scalar is

$$R = (d-2)\left[\frac{(d-2)(1-A)}{r^2} - \frac{2}{r}\left(\frac{\partial A}{\partial r} + A\frac{\partial \Omega}{\partial r}\right)\right] - \left[\frac{\partial^2 A}{\partial r^2} + 2A\frac{\partial^2 \Omega}{\partial r^2} + 2A\left(\frac{\partial \Omega}{\partial r}\right)^2 + 3\frac{\partial A}{\partial r}\frac{\partial \Omega}{\partial r}\right].$$

The Einstein equations of (2.6) can be summarised as

$$\frac{d-2}{r}\frac{\partial \Omega}{\partial r} = 0, \tag{2.29}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}[r^{d-3}(1-A)] = r^{d-2}\left(\frac{d-1}{l^2}\right), \tag{2.30}$$

for which the solutions are $\Omega = constant$ and $A = 1 - r^2/l^2 - 2GM/r^{d-3}$. The constant of Ω can be absorbed by a scale transformation and setting $M = 0$ gives back the metric given by (2.28).

2.3 Penrose diagrams

Penrose diagrams are the two dimensional figures that capture the causal relations between different points in spacetime. These two dimensional figures are finite in size in contrast to the actual spacetimes which can extend to infinity in space and time. The metric on the Penrose diagrams is conformally equivalent to the actual metric of the spacetime. If we consider a spacetime with a physical metric $g_{\mu v}$, we can introduce another metric $\tilde{g}_{\mu v}$ so that this is related to the actual physical metric by the relation

$$\tilde{g}_{\mu v} = \Omega^2 g_{\mu v}, \tag{2.31}$$

where Ω is called the conformal factor. This relation points out the fact that the distances are highly distorted since the whole spacetime is shrunk to a finite region. Through such conformal compactification, all the information on the causal structure of the spacetime is easily visualised in these finite diagrams. It can be proven that null geodesics (obtained by setting line element to zero) are conformally invariant since the conformal factor does not play any role in null geodesics. Infinities of actual physical metric or spacetime are represented by a finite hypersurface I which is obtained by setting $\Omega = 0$. This implies that the metric at I is stretched by an infinite factor. Since I represents the infinities of the actual metric, it forms the boundary for the Penrose diagrams. Accounting for the time direction, this hypersurface I can be split into I^+ and I^- corresponding to future and pass null infinities respectively. All the null geodesics originate on I^- and terminate on I^+. Penrose diagrams are analogous to the Minkowski diagram, a graphic depiction of Minkowski spacetime, in which the vertical dimension represents time and horizontal direction represents space and the slanted lines represent the null geodesics in general. Penrose diagrams are drawn as two-dimensional squares. For a positive cosmological constant, the hypersurface I is spacelike.

A very useful coordinate system that can be used to draw Penrose diagrams is the Kruskal coordinate system obtained by transformations from static coordinates and Penrose diagrams for any other system can be easily visualised by obtaining the transformations among them with this Kruskal system.

We will understand the Penrose diagrams in different coordinate systems starting with conformal coordinate systems as it is convenient for study. We further proceed to understand the diagrams in other coordinates as well.

2.3.1 Conformal coordinates

The conformal line element describing de Sitter spacetime reads as

$$\mathrm{d}s^2 = F^2\left(\frac{T}{l}\right)(-\mathrm{d}T^2 + l^2\mathrm{d}\Omega_{d-1}^2).$$

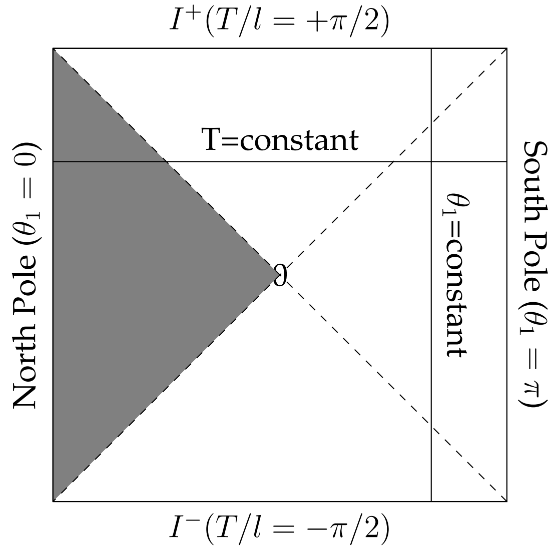

Figure 2.1: Penrose diagram in conformal coordinates

From the previous section, it can be noted that $\Omega = \cos(T/l)$ and equating it to zero gives the hypersurfaces I^+ and I^- as the surfaces $T/l = +\pi/2$ and $T/l = -\pi/2$ respectively. The hypersurfaces $\theta_1 = 0$ and $\theta_1 = \pi$ are called the north and south poles respectively and form the boundaries of the Penrose diagrams to the left and right respectively whereas the hypersurfaces $T/l = -\pi/2$ and $T/l = \pi/2$ form the boundaries on bottom and top respectively. Since the Penrose diagram is a two dimensional figure, each point on the Penrose diagram corresponds to a $(d-2)$ dimensional sphere. Since, the line element in the conformal system is (excluding the conformal factor) is given by

$$ds^2 = -dT^2 + l^2 d\Omega_{d-1}^2, \tag{2.32}$$

the cylindrical topology is manifest in this line element. Cutting this cylinder along constant T surfaces described above and unwrapping it to form a 2-d diagram gives the Penrose diagram with top and bottom boundaries as $T = \pm\pi/2$ surfaces and left and right boundaries as $\theta_1 = 0$ and $\theta = \pi$. The null geodesics are obtained by setting $ds^2 = 0$ which gives lines at 45^o. The timelike surfaces are more vertical than the null geodesics and the spacelike surfaces are more horizontal. Every horizontal slice corresponds to $T = constant$ surface and every vertical slice corresponds to a $\theta_1 = constant$.

Although the conformal coordinates cover the entire de Sitter spacetime, not any single observer can observe the whole spacetime. The de Sitter spacetime has both particle horizon

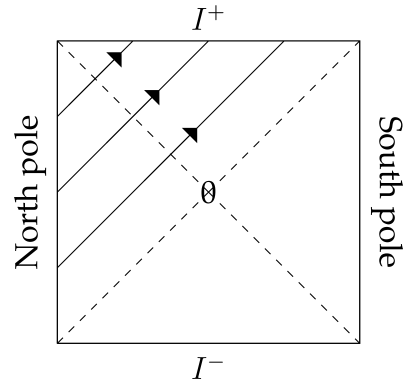

Figure 2.2: Causal future of an observer at North pole

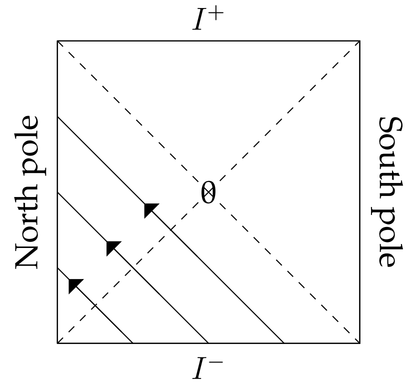

Figure 2.3: Causal past of an observer at North pole

and event horizon because both I^- and I^+ are spacelike. An event horizon is a boundary in spacetime beyond which events cannot affect an observer. Particle horizon is the maximum distance from which the particles could have travelled to the observer in the age of universe. This restricts the accessible region for any observer. An observer at north pole cannot receive anything from the south pole, or in other words, anything beyond his past null cone due to the presence of his particle horizon. In the same way, he cannot send anything to any region beyond his future null cone or to an observer at south pole due to his future event horizon. Hence, the information that is totally accessible to an observer is the intersection of these two regions which is only one fourth of the entire spacetime. All this is depicted diagrammatically in Figure (2.1). The dashed lines are the null geodesics which form the horizons and the shaded part is the causal region accessible to the observer at the north pole. Let us now try to understand the Penrose diagrams in another coordinate system, viz. static coordinates.

2.3.2 Static coordinates

In this section we introduce a couple of important coordinate systems which are useful in the study of Penrose diagrams. The first among them is the Eddington-Finkelstein coordinates parametrized by (x^+, x^-, θ_a). In terms of the static coordinates, these are given by

$$x^{\pm} = t \pm \frac{l}{2} \ln \left(\frac{1 + r/l}{1 - r/l} \right) . \tag{2.33}$$

Here the range of $x^{\pm} = (-\infty, +\infty)$. From the static coordinates, it can be noted that for the coordinates to be real, the range of r is $(0, l)$. However, rewriting the static line element in terms of the Eddington-Finkelstein coordinates, we get

$$\mathrm{d}s^2 = -\mathrm{sech}^2\left(\frac{x^+ - x^-}{2l}\right)\mathrm{d}x^+\mathrm{d}x^- + l^2\tanh^2\left(\frac{x^+ - x^-}{2l}\right)\mathrm{d}\Omega^2_{d-2}. \tag{2.34}$$

This line element is real for the whole range of r and covers the entire de Sitter spacetime as r ranges from 0 to ∞. We shall introduce another coordinate system called the Kruskal system parametrized by U and V which can be conveniently written in terms x^+ and x^- as $U = -e^{x^-/l}$, $V = e^{-x^+/l}$. The metric takes the form

$$\mathrm{d}s^2 = \frac{l^2}{(1-UV)^2}[-4\mathrm{d}U\mathrm{d}V + (1+UV)^2\mathrm{d}\Omega^2_{d-2}]. \tag{2.35}$$

From this form of the metric, it can be easily seen that the conformal factor Ω for the Penrose diagrams is $(1 - UV/l)^2$. Setting this to zero defines the gives $UV = 1$ which defines the hypersurfaces I^+ and I^- respectively. Rewriting the static coordinates in terms of the Kruskal coordinates, we get

$$\frac{r}{l} = \frac{1+UV}{1-UV}. \tag{2.36}$$

Setting $UV = 1$ gives $r = \pm\infty$ which form the boundaries at top and bottom. The left and right boundaries correspond to the $r/l = 0$. The left and right boundaries correspond to $r/l = 0$ which gives $UV = -1$. The static time t can also be written in terms of these as $-U/V = e^{2t/l}$. So, $t = \infty$ is equivalent to $V = 0$ and $t = -\infty$ to $U = 0$. These lines of $t = \pm\infty$ form the null geodesics. This can be seen from the mathematical expression $UV = 0$ which gives $r/l = 1$. This is the horizon in the static coordinates and results in the form of a compact equation in $UV = 0$ in these coordinates. The Penrose diagram in the Kruskal coordinates (equivalently in static coordinates) is shown in Figure(2.4). The arguments of the horizon and the information causality holds equally good in these coordinates as was in conformal coordinates. Any observer has only one fourth of the entire space for information exchange or is causally connected. Coordinate transformation between Kruskal coordinates and the conformal coordinates can be found easily by comparing the two metrics which

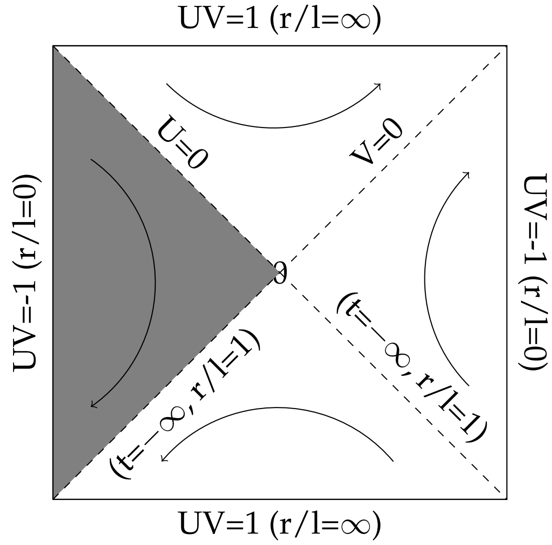

Figure 2.4: Penrose diagram in the Kruskal and the static coordinates

gives

$$\frac{(1+UV)^2}{(1-UV)^2} = \frac{\sin^2\theta_1}{\cos^2(T/l)},$$
$$\frac{4l^2}{(1-UV)^2}\mathrm{d}U\mathrm{d}V = \frac{1}{\cos(T/l)}(\mathrm{d}T^2 - l^2\mathrm{d}\theta_1^2).$$

Solving these equation gives

$$U = \tan\left[\frac{1}{2}\left(\frac{T}{l} + \theta_1 - \frac{\pi}{2}\right)\right], \tag{2.37}$$
$$V = \tan\left[\frac{1}{2}\left(\frac{T}{l} - \theta_1 + \frac{\pi}{2}\right)\right]. \tag{2.38}$$

The one-to-one correspondence between the Kruskal and the conformal coordinates implies that the Kruskal coordinates cover the entire de Sitter space.

2.3.3 Planar coordinates

By comparing the metrics in the Kruskal and the planar coordinates we get the coordinate transformations as

$$U = \frac{1}{2}(r/l - e^{-t/l}), \tag{2.39}$$
$$V = \frac{2}{e^{-t/l} + r/l} \tag{2.40}$$

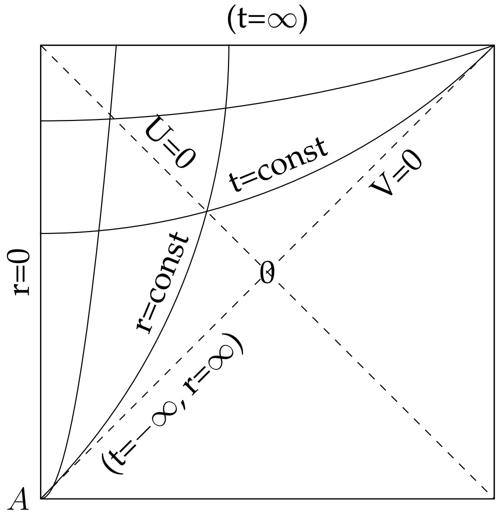

Figure 2.5: Penrose diagram in the spatially flat planar coordinates

Reversing the transformations, we get $r/l = U + 1/V$ and $t/l = -\ln(1/V - U)$. From these relations, it can be seen that $V > 0$. The expressions $r/l = 0$ and $r/l = \infty$ correspond to $UV = -1$ and hence the left and right boundaries which correspond to $UV = -1$ are the same as in the case of static coordinates. However, $t = -\infty$ corresponds to $V = 0$ and hence is the diagonal line as opposed to the bottom boundary in static coordinate case. But $t = \infty$ remains the boundary at the top in the Penrose diagram with $UV = 1$. Since $V > 0$ always, the planar coordinates cover only one half of the de Sitter spacetime as was also highlighted in the discussion of section 2.2.3. To cover the other half, the coordinate system has to be tinkered a little which gives new relations with the Kruskal coordinates to cover the lower triangular part of the Penrose diagram. The vertical lines does not correspond to $r/l = constant$ surfaces. Also, the horizontal slices are not the $t = constant$ hypersurfaces which is in contrast with the Penrose diagrams in the other coordinate systems. The Penrose diagrams in planar coordinates is shown in figure above. This discussion of planar coordinates is relevant only for spatially flat sections and not for open or closed systems.

Chapter 3

Quantum field theory in flat spacetime

Quantum field theory is the framework for the modern theoretical physics. This is a framework in which quantum mechanics and special relativity are successfully reconciled. In an informal way, it is an extension of quantum mechanics (dealing with particles) to fields, with infinite degrees of freedom. Quantum field theory has become an interesting and important mathematical and conceptual framework for contemporary elementary particle physics. In this chapter, we learn the basic ingredients of quantum field theory to use it in the case of fields with background de Sitter spacetime.

3.1 Brief introduction

Before starting to learn quantum field theory, we should understand the need for the quantization of the fields rather than just quantization of the particles. In order to understand the process that occur at small scales and at high energies it is simply not enough to quantize the relativistic particles just the way it was done for non-relativistic particles. The latter method leads to a number of inconsistencies. A fairly simple example to assert this point would be to consider the amplitude for a free particle to propagate from $\boldsymbol{x}_0$ to $\boldsymbol{x}$ given by $U(t) = \langle \boldsymbol{x}|e^{-iHt}|\boldsymbol{x}_o\rangle$. In non-relativistic quantum mechanics, for a free particle $E = \boldsymbol{p}^2/2m$, so that

$$U(t) = \langle \boldsymbol{x}|e^{-ip^2t/2m}|\boldsymbol{x}_o\rangle.$$

Using one particle identity relation$\int \frac{\mathrm{d}^3\boldsymbol{p}}{2m}|\boldsymbol{p}\rangle\langle\boldsymbol{p}| = I$ we get,

$$U(t) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\langle \boldsymbol{x}|e^{-ip^2t/2m}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\boldsymbol{x}_o\rangle,$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{p}\, e^{-i\frac{p^2 t}{2m}}\langle \boldsymbol{x}|\boldsymbol{p}\rangle\langle \boldsymbol{p}|\boldsymbol{x}_0\rangle, \\
&= \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\boldsymbol{x}-\boldsymbol{x}_0)^2/2t}.
\end{aligned}
$$

Since the above expression shows that the amplitude to propagate from one point to the other is non-zero for any $\boldsymbol{x}$ and t, it implies that the particle can propagate between any two points in arbitrarily short time violating the principle of causality. Using the relativistic expression for the energy $E = \sqrt{\boldsymbol{p}^2 + m^2}$ we get the amplitude as

$$
U(t) \sim e^{-m\sqrt{x^2 - t^2}},
$$

which is still non-zero outside the light cone implying that particle can travel faster than the speed of light.

Let us begin our formal study of quantum field theory with the simplest type of field: the real Klein-Gordon field. We start by considering a classical field theory and proceed to quantize this classical field. Let us consider the simple case of a real field with Lagrangian density given by

$$
\mathcal{L} = \frac{1}{2}(\partial_\mu \partial^\mu \phi) - \frac{1}{2}m^2\phi^2. \tag{3.1}
$$

The Euler-Lagrangian equations for the field are

$$
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}, \tag{3.2}
$$

which leads to the equation

$$
\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{3.3}
$$

This is the well known Klein-Gordon equation for a simple real field $\phi(x)$. The operator $\partial^\mu \partial_\mu$ is called the D'Alembertian operator and is often denoted as $\Box$. Lagrangian formulation of field theory is well suited to relativistic dynamics because all the expressions are manifestly Lorentz invariant. Conjugate momentum density is defined as $\pi(x) = \partial\mathcal{L}/\partial\dot{\phi}$. For the Lagrangian density considered above, it gives, $\pi(x) = \dot{\phi}(x)$. The dot here represents the first derivative with respect to x_0 component of x_μ vector. The Hamiltonian density is given by

$$
\begin{aligned}
\mathcal{H} &= \sum_i \pi_i \dot{\phi}^i - \mathcal{L}, &(3.4)\\
&= \frac{1}{2}\left(\pi^2(x) + (\nabla\phi)^2 + m^2\phi^2\right). &(3.5)
\end{aligned}
$$

The above formulae for the Hamiltonian density and conjugate momentum density can be derived as the components of the Noether charge which involves a rigorous derivation by exploiting the relationship between symmetries and conservation laws. Let us try to solve the equations of motion for the real Klein-Gordon field given by

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0.$$

Since the Klein-Gordon field is real, $\phi^*(\boldsymbol{p}) = \phi(-\boldsymbol{p})$ where $\phi(\boldsymbol{p})$ is the Fourier transformation of $\phi(\boldsymbol{x})$. The Fourier decomposition of $\phi(t, \boldsymbol{x})$ is

$$\phi(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(t, \boldsymbol{p}). \tag{3.6}$$

Under Fourier transformation, the equation of motion given by (3.3) becomes

$$\left(\frac{\partial^2}{\partial t^2} - (i\boldsymbol{p})^2 + m^2 \right) \phi(t, \boldsymbol{p}) = 0. \tag{3.7}$$

This is a familiar equation corresponding to simple harmonic oscillator with frequency given by $\omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$. In case of the simple harmonic oscillator, the Fock space is constructed by raising and lowering operators given by $\hat{a}$ and $\hat{a}^\dagger$ with commutation relations given by $\left[\hat{a}, \hat{a}^\dagger\right] = 1$. In the same way, we can determine the spectrum of the Klein-Gordon Hamiltonian using the raising and lowering operators. It is to be noted that in case of simple harmonic oscillator there was only one mode. However, here we have infinite number of modes with each corresponding to the frequency given as above. Hence, we have raising and lowering operators $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}^\dagger_{\boldsymbol{p}}$ corresponding to each of the modes. The Klein-Gordon field can be thought of as being composed of infinite number of oscillators which are independent of each other. Hence, we can write the expansion for the field ϕ as

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 \vec{\boldsymbol{p}}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right), \tag{3.8}$$

$$\pi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 \vec{\boldsymbol{p}}}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left(a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right). \tag{3.9}$$

Upto now, we have been treating the field in view of classical field theory. We now impose the commutation relations for the fields as

$$[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{x}')] = i\delta^3(\boldsymbol{x} - \boldsymbol{x}'). \tag{3.10}$$

From this commutation relation, we can obtain the commutation relations of annihilation and creation operators and proceed to find the expansions for Hamiltonian and momentum operators. This is called the Schrodinger picture in which the operators are constant in time but the basis states are not. However, it is very advantageous to work in the Heisenberg picture in which the operators are varying in time with the basis fixed. Nevertheless, a few important remarks can be made. The operator $\hat{a}_{\boldsymbol{p}}^{\dagger}$ can be interpreted as the one creating states with energy $\omega_{\boldsymbol{p}}$ and momentum p. Any general state $\hat{a}_{\boldsymbol{p}}^{\dagger}\hat{a}_{\boldsymbol{q}}^{\dagger}\dots|0\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue(energy) given by $\omega_{\boldsymbol{p}}+\omega_{\boldsymbol{q}}+\dots$ and is also an eigenstate of $\hat{\boldsymbol{P}}$ with eigenvalue $\boldsymbol{p}+\boldsymbol{q}+\dots$. We also have the relation $\omega_{\boldsymbol{p}}=\sqrt{|\boldsymbol{p}|^2+m^2}$. Hence, we can consider these states as states containing particles since these are discrete entities with proper relativistic energy-momentum relation. We can now look at the statistics of these particles. Since any general state is formed as $\hat{a}_{\boldsymbol{p}}^{\dagger}\hat{a}_{\boldsymbol{q}}^{\dagger}\dots|0\rangle$ and that all $\hat{a}^{\dagger}$'s commute with each other, their order can be interchanged which implies that the particles can be interchanged. Also, we can also have a state as $(\hat{a}_{\boldsymbol{p}}^{\dagger})^n|0\rangle$ which has the interpretation of a single mode $\boldsymbol{p}$ with n particles. Thus, Klein-Gordon particles obey the Bose-Einstein statistics and are bosons. But, the quantization of Dirac fields force us to impose anti-commutation relations rather than commutation relations and hence their $\hat{a}^{\dagger}$'s cannot be interchanged. Such particles follow Fermi-Dirac statistics and are called Fermions. However, we would not be discussing the quantization of the Dirac fields in this report.

3.2 Heisenberg representation

The above discussion was done in Schrodinger representation in which the state are evolving in time and the operators remain independent of time. But, it is more convenient to work in the Heisenberg picture in which the operators are varying in time with the basis fixed, i.e the states do not vary with time. The time dependent Schrodinger equation reads

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle=\hat{H}|\psi(t)\rangle.$$

In the Schrodinger the representation the operators being independent of time implies that $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$. For any operator $\hat{B}$, we have

$$\begin{aligned}\langle \hat{B}\rangle_t &= \langle\psi(t)|\hat{B}|\psi(t)\rangle \\ &= \langle\psi(0)e^{i\hat{H}t/\hbar}|\hat{B}|e^{-i\hat{H}t/\hbar}\psi(0)\rangle \\ &= \langle\psi(0)|\hat{B}(t)|\psi(0)\rangle,\end{aligned}$$

where $\hat{B}(t) = e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}$. Time evolution of $\hat{B}(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{B}(t) = \frac{i}{\hbar}e^{i\hat{H}t/\hbar}\hat{H}\hat{B}e^{-i\hat{H}t/\hbar} - \frac{i}{\hbar}e^{i\hat{H}t/\hbar}\hat{B}\hat{H}e^{-i\hat{H}t/\hbar} = \frac{i}{\hbar}[\hat{H},\hat{B}(t)]. \tag{3.11}$$

If $\hat{B}$ itself was dependent on time, we would have

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{B}(t) = \frac{i}{\hbar}[\hat{H},\hat{B}(t)] + e^{i\hat{H}t/\hbar}\left(\frac{\partial\hat{B}}{\partial t}\right)e^{-i\hat{H}t/\hbar}. \tag{3.12}$$

The commutation relations of $\hat{a}_p(t)$ and $\hat{a}_p^\dagger(t)$ would become

$$\begin{aligned}[\hat{a}_p(t),\hat{a}_p^\dagger(t)] &= e^{i\hat{H}t/\hbar}[\hat{a}_p,\hat{a}_p^\dagger]e^{-i\hat{H}t/\hbar}, \\ &= e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar} = 1.\end{aligned}$$

Hence the commutation relations remain unchanged for the raising and lowering operators. The Heisenberg picture is convenient as it will be easier to discuss time-dependent quantities and questions of causality. In this picture we have

$$\begin{aligned}\hat{\phi}(x) &= \hat{\phi}(t,\boldsymbol{x}) = e^{i\hat{H}t}\hat{\phi}(\boldsymbol{x})e^{-i\hat{H}t}, \\ \hat{\pi}(x) &= \hat{\pi}(t,\boldsymbol{x}) = e^{i\hat{H}t}\hat{\pi}(\boldsymbol{x})e^{-i\hat{H}t}.\end{aligned}$$

As derived above, the equation describing the time evolution of $\hat{B}$ is called the Heisenberg equation of motion. Using it , we can compute

$$\begin{aligned}i\frac{\partial}{\partial t}\hat{\phi}(t,\boldsymbol{x}) &= [\hat{\phi}(x),\hat{H}] \\ &= \left[\hat{\phi}(t,\boldsymbol{x}),\frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}'\left(\hat{\pi}^2(t,\boldsymbol{x}') + (\nabla\hat{\phi}(t,\boldsymbol{x}'))^2 + m^2\hat{\phi}^2(t,\boldsymbol{x}')\right)\right] \\ &= \int \mathrm{d}^3\boldsymbol{x}(-i\delta^3(\boldsymbol{x}-\boldsymbol{x}')\hat{\pi}(t,\boldsymbol{x}') \\ &= i\hat{\pi}(t,\boldsymbol{x}),\end{aligned}$$

and also

$$\begin{aligned}
i\frac{\partial}{\partial t}\hat{\pi}(t,\boldsymbol{x}) &= [\hat{\pi}(x), H], \\
&= \left[\hat{\pi}(t,\boldsymbol{x}), \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}' \left(\hat{\pi}^2(t,\boldsymbol{x}') + (\nabla\hat{\phi}(t,\boldsymbol{x}'))^2 + m^2\hat{\phi}^2(t,\boldsymbol{x}')\right)\right] \\
&= \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}' \left(-i\nabla\delta^3(\boldsymbol{x}-\boldsymbol{x}')\nabla\hat{\phi}(t,\boldsymbol{x}') - m^2 i\delta^3(\boldsymbol{x}-\boldsymbol{x}')\hat{\phi}(t,\boldsymbol{x}')\right) \\
&= \int \mathrm{d}^3\boldsymbol{x}' \left(i\delta^3(\boldsymbol{x}-\boldsymbol{x}')\nabla^2\hat{\phi}(t,\boldsymbol{x}') - m^2 i\delta^3(\boldsymbol{x}-\boldsymbol{x}')\hat{\phi}(t,\boldsymbol{x}')\right) \\
&= i(\nabla^2 - m^2)\hat{\phi}.
\end{aligned}$$

In the above derivation we have used the distributional derivative property of Dirac delta function which is written mathematically as

$$\int_{-\infty}^{\infty} \delta'(x)f(x)dx = -\int_{-\infty}^{\infty} \delta(x)f'(x)dx.$$

The above two Heisenberg equations of motion for $\hat{\phi}$ and $\hat{\pi}$ can be combined into one by differentiating any one of them and using the relation for the other which leads to

$$\frac{\partial^2}{\partial t^2}\phi = (\nabla^2 - m^2)\phi, \tag{3.13}$$

which is the familiar Klein-Gordon equation. Let us construct the formal solution for the above Klein-Gordon equation with $\hat{\phi}$ being dependent on time. Let the space part of the solution be $u_{\boldsymbol{p}}(\boldsymbol{x}) = N_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ and let $\hat{\phi}(t,\boldsymbol{x}) = \int \mathrm{d}^3\boldsymbol{p}\, N_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\hat{a}_{\boldsymbol{p}}(t)$. Substituting this into the above Klein-Gordon equation, we obtain that

$$\ddot{\hat{a}}_{\boldsymbol{p}}(t) = -(\boldsymbol{p}^2 + m^2)\hat{a}_{\boldsymbol{p}}(t),$$

In the above equation, the double dot is the second derivative with respect to t which is same as the one in (3.13)

$$\hat{a}_{\boldsymbol{p}}(t) = \hat{a}^{(1)}_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t} + \hat{a}^{(2)}_{\boldsymbol{p}}e^{i\omega_{\boldsymbol{p}}t}.$$

The condition of real field implies $\phi^* = \phi$ which translates to $\hat{\phi}^\dagger = \hat{\phi}$ leading to

$$\begin{gathered}
\left(\hat{a}^{(1)}_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t} + \hat{a}^{(2)}_{\boldsymbol{p}}e^{i\omega_{\boldsymbol{p}}t}\right)e^{i\boldsymbol{p}.\boldsymbol{x}} = \left(\hat{a}^{\dagger(1)}_{\boldsymbol{p}}e^{i\omega_{\boldsymbol{p}}t} + \hat{a}^{\dagger(2)}_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t}\right)e^{-i\boldsymbol{p}.\boldsymbol{x}}, \\
a^{\dagger(1)}_{\boldsymbol{p}} = a^{(2)}_{-[\boldsymbol{p}}.
\end{gathered}$$

The field operator now becomes

$$\hat{\phi}(t,\boldsymbol{x}) = \int \mathrm{d}^3\boldsymbol{p} N_{\boldsymbol{p}}(\hat{a}_{\boldsymbol{p}} e^{i(\boldsymbol{p}.\boldsymbol{x}-\omega_{\boldsymbol{p}}t)} + \hat{a}^\dagger_{\boldsymbol{p}} e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p}}t)}) \tag{3.14}$$

We define the four-vector inner product as $p\cdot x = (\omega_{\boldsymbol{p}}t - \boldsymbol{p}\cdot\boldsymbol{x})$. Also redefine $u_p(x) = u_{\boldsymbol{p}}(t,\boldsymbol{x}) = N_{\boldsymbol{p}} e^{-ip\cdot x}$ with $N_{\boldsymbol{p}} = \sqrt{1/2\omega_{\boldsymbol{p}}(2\pi)^3}$. We will soon notice that this specific choice of $N_{\boldsymbol{p}}$ will give us back the original commutation relations for $\hat{\phi}$ and $\hat{\pi}$. Calculating the equal time commutation relation $[\hat{\phi}(t,\boldsymbol{x}), \hat{\pi}(t,\boldsymbol{x}')]$, we get

$$\begin{aligned}
[\hat{\phi}(t,\boldsymbol{x}), \hat{\pi}(t,\boldsymbol{y})] = \int\int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{q}\, N_{\boldsymbol{p}}N_{\boldsymbol{q}}(-i\omega_{\boldsymbol{q}})\{&-[\hat{a}_{\boldsymbol{p}},\hat{a}^\dagger_{\boldsymbol{q}}]e^{-ip\cdot x+iq.y} \\
&+ [\hat{a}_{\boldsymbol{p}},\hat{a}_{\boldsymbol{q}}]e^{-ip.x-iq.y} - [\hat{a}^\dagger_{\boldsymbol{p}},\hat{a}^\dagger_{\boldsymbol{q}}]e^{ip\cdot x+iq.y} + [\hat{a}^\dagger_{\boldsymbol{p}},\hat{a}_{\boldsymbol{q}}]e^{ip\cdot x-iq\cdot y}\} \\
= \int \mathrm{d}^3\boldsymbol{p}\, N^2_{\boldsymbol{p}}(i\omega_{\boldsymbol{q}})&\left\{e^{ip\cdot(x-y)} + e^{ip\cdot(y-x)}\right\} \\
= i\delta^3(\boldsymbol{x}-\boldsymbol{y})&.
\end{aligned}$$

Let us define the operation of scalar product of two functions as

$$\begin{aligned}
(\phi,\chi) &= i\int \mathrm{d}^3\boldsymbol{x}\,\phi^*(t,\boldsymbol{x})\overleftrightarrow{\partial_0}\chi(t,\boldsymbol{x}) \\
&= i\int \mathrm{d}^3\boldsymbol{x}\left(\phi^*\frac{\partial\chi}{\partial t} - \frac{\partial\phi^*}{\partial t}\chi\right)
\end{aligned}$$

With this definition, we have

$$\begin{aligned}
(u_{\boldsymbol{p}'}, u_{\boldsymbol{p}}) &= i\int \mathrm{d}^3\boldsymbol{x}\left(u^*_{\boldsymbol{p}'}\frac{\partial u_{\boldsymbol{p}}}{\partial t} - \frac{\partial u^*_{\boldsymbol{p}'}}{\partial t}u_{\boldsymbol{p}}\right) \\
&= \delta^3(\boldsymbol{p}-\boldsymbol{p}'), \\
(u^*_{\boldsymbol{p}'}, u^*_{\boldsymbol{p}}) &= -\delta^3(\boldsymbol{p}-\boldsymbol{p}'), \\
(u^*_{\boldsymbol{p}'}, u_{\boldsymbol{p}}) &= 0, \quad (u_{\boldsymbol{p}'}, u_{\boldsymbol{p}^*}) = 0.
\end{aligned}$$

The expansion of $\hat{\phi}$ can be written compactly as

$$\hat{\phi} = \int \mathrm{d}^3\boldsymbol{p}\,(\hat{a}_{\boldsymbol{p}}u_{\boldsymbol{p}}(t,\boldsymbol{x}) + \hat{a}^\dagger_{\boldsymbol{p}}u^*_{\boldsymbol{p}}(t,\boldsymbol{x})) \tag{3.15}$$

Let us look at the scalar product of $(u_{\boldsymbol{p}}, \hat{\phi})$

$$\begin{aligned}
(u_{\boldsymbol{p}}, \hat{\phi}) &= i\int \mathrm{d}^3\boldsymbol{x}\,u^*_{\boldsymbol{p}}\overleftrightarrow{\partial_0}\phi \\
&= i\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{p}\,\hat{a}_{\boldsymbol{p}}(u^*_{\boldsymbol{p}}\dot{u}_{\boldsymbol{p}} - \dot{u}^*_{\boldsymbol{p}}u_{\boldsymbol{p}}) \\
&= \int \mathrm{d}^3\boldsymbol{p}\,\hat{a}_{\boldsymbol{p}}\delta^3(\boldsymbol{p}-\boldsymbol{p}') = \hat{a}_{\boldsymbol{p}}.
\end{aligned}$$

Similarly $-(u_{\boldsymbol{p}}^*, \hat{\phi}) = \hat{a}_{\boldsymbol{p}}^\dagger$. From these, we can calculate the commutation relationships for $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}_{\boldsymbol{p}}^\dagger$. These are given by

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p}'}] = [(u_{\boldsymbol{p}}, \phi), -(u_{\boldsymbol{p}'}, \phi)] = -(u_{\boldsymbol{p}}, u_{\boldsymbol{p}'}^*) = 0, \tag{3.16}$$

$$[\hat{a}_{\boldsymbol{p}}^\dagger, \hat{a}_{\boldsymbol{p}'}^\dagger] = [(u_{\boldsymbol{p}}^*, \phi), -(u_{\boldsymbol{p}}^*, \phi)] = -(u_{\boldsymbol{p}}^*, u_{\boldsymbol{p}'}) = 0, \tag{3.17}$$

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p}'}^\dagger] = [(u_{\boldsymbol{p}}, \phi), -(u_{\boldsymbol{p}'}^*, \phi)] = (u_{\boldsymbol{p}}(t, \boldsymbol{x}), u_{\boldsymbol{p}'}(t, \boldsymbol{x})) = \delta^3(\boldsymbol{p} - \boldsymbol{p}'). \tag{3.18}$$

We are now ready to compute the Hamiltonian. We have

$$\hat{\phi}(t, \boldsymbol{x}) = \int \mathrm{d}^3\boldsymbol{p}\, N_{\boldsymbol{p}}(\hat{a}_{\boldsymbol{p}} u_{\boldsymbol{p}}(t, \boldsymbol{x}) + \hat{a}_{\boldsymbol{p}}^\dagger u_{\boldsymbol{p}}^*(t, \boldsymbol{x})), \tag{3.19}$$

$$\hat{\pi}(t, \boldsymbol{x}) = -i \int \mathrm{d}^3\boldsymbol{p}\, N_{\boldsymbol{p}} \omega_{\boldsymbol{p}}(\hat{a}_{\boldsymbol{p}} u_{\boldsymbol{p}}(t, \boldsymbol{x}) - \hat{a}_{\boldsymbol{p}}^\dagger u_{\boldsymbol{p}}^*(t, \boldsymbol{x})). \tag{3.20}$$

The Hamiltonian becomes

$$\begin{aligned}
\hat{H} &= \frac{1}{2} \int \mathrm{d}^3\boldsymbol{x}(\hat{\pi}^2(t, \boldsymbol{x}) + (\nabla\hat{\phi}(t, \boldsymbol{x}))^2 + m^2\hat{\phi}^2(t, \boldsymbol{x})) \\
&= \frac{(2\pi)^3}{2} \int \mathrm{d}^3\boldsymbol{p} N_{\boldsymbol{p}}^2 \{[\hat{a}_{-p}\hat{a}_{\boldsymbol{p}} e^{-2i\omega_{\boldsymbol{p}} t} + \hat{a}_{-p}^\dagger \hat{a}_{\boldsymbol{p}}^\dagger e^{2i\omega_{\boldsymbol{p}} t}](-\omega_{\boldsymbol{p}}^2 + |\boldsymbol{p}|^2 + m^2) \\
&\quad + (\hat{a}_{\boldsymbol{p}'}\hat{a}_{\boldsymbol{p}}^\dagger + \hat{a}_{\boldsymbol{p}'}^\dagger \hat{a}_{\boldsymbol{p}})(\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'} + \boldsymbol{p} \cdot \boldsymbol{p}' + m^2)\} \\
&= \int \mathrm{d}^3\boldsymbol{p}\, \omega_{\boldsymbol{p}}(\hat{a}_{\boldsymbol{p}}^\dagger \hat{a}_{\boldsymbol{p}} + \frac{1}{2}\delta^3(0)).
\end{aligned}$$

We again get the $\delta^3(0)$ which diverges upon integration. Hence we introduce a normal ordering operation defined as follows

$$: \hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}^\dagger + \hat{a}_{\boldsymbol{p}}^\dagger \hat{a}_{\boldsymbol{p}} := 2\hat{a}_{\boldsymbol{p}}^\dagger \hat{a}_{\boldsymbol{p}}. \tag{3.21}$$

The normal ordering operation moves all the annihilation operators ($\hat{a}_{\boldsymbol{p}}$) to the right of all creation operators ($\hat{a}_{\boldsymbol{p}}^\dagger$). This is, in a way, equivalent to subtracting the diverging Dirac-delta term. Similarly, we can calculate the momentum operator $\hat{\boldsymbol{P}}$ as follows (we do symmetrizing due to non-commutativity of $\hat{\pi}$ and $\hat{\phi}$)

$$\begin{aligned}
\hat{\boldsymbol{P}} &= -\frac{1}{2} \int \mathrm{d}^3\boldsymbol{x}\, (\hat{\pi}(t, \boldsymbol{x})\nabla\hat{\phi}(t, \boldsymbol{x}) + \nabla\hat{\phi}(t, \boldsymbol{x})\hat{\pi}(t, \boldsymbol{x})) \\
&= -\frac{1}{2} \int \mathrm{d}^3\boldsymbol{p}\, \mathrm{d}^3\boldsymbol{p}'\, N_{\boldsymbol{p}} N_{\boldsymbol{p}'} \{(\hat{a}_p \hat{a}_{\boldsymbol{p}'} e^{-2i\omega_p t} + \hat{a}_{\boldsymbol{p}}^\dagger \hat{a}_{\boldsymbol{p}'}^\dagger e^{2i\omega_{\boldsymbol{p}} t})(\omega_p\, \boldsymbol{p}' + \omega_{\boldsymbol{p}'}\, \boldsymbol{p})\delta^3(\boldsymbol{p} + \boldsymbol{p}') \\
&\quad - (\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}^\dagger + \hat{a}_{\boldsymbol{p}}^\dagger \hat{a}_{\boldsymbol{p}'})(\omega_{\boldsymbol{p}}\, \boldsymbol{p}' + \omega_{\boldsymbol{p}'}\, \boldsymbol{p})\delta^3(\boldsymbol{p} - \boldsymbol{p}')\} \\
&= \frac{1}{2} \int \mathrm{d}^3\boldsymbol{p}\, \boldsymbol{p}(\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}^\dagger + \hat{a}_{\boldsymbol{p}}^\dagger \hat{a}_{\boldsymbol{p}'}).
\end{aligned}$$

Normal ordering can again be imposed above and we can rewrite it as

$$:\hat{\boldsymbol{P}} := \int \mathrm{d}^3\boldsymbol{p}\,\boldsymbol{p}(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}). \tag{3.22}$$

Detailed calculations of the above and also the derivations of angular momentum operators in different basis can be found in [8].

3.3 Decomposition in terms of spherical harmonics

For all the derivations we have done upto now, we have used a particular choice of solutions (basis), namely the plane wave basis $e^{-ik\cdot x}$ and $e^{ik\cdot x}$ and constructed the expansions in terms of this basis. However, it is sometimes helpful to consider a suitable choice of basis according to the ease of solving and hence we shall discuss another choice of basis, spherical basis, $R_{pl}(r)Y_{lm}(\Omega)$. Here, $Y_{lm}(\Omega) = Y_{lm}(\theta,\phi)$ where θ and ϕ are the polar and azimuthal angle respectively which one encounters when working in the spherical polar coordinates. We shall redo all the calculations again in this basis. The field mode in spherical basis is written as $\phi_{plm} = N_p R_{pl} Y_{lm}(\Omega)$. The index m is not be confused with the mass term. In spherical coordinates, $\nabla^2 = \triangle = \triangle_r + \triangle_\Omega$. The radial functions $R_{pl}(r)$ satisfy the radial equation

$$\left(\triangle_r - \frac{l(l+1)}{r^2} + p^2\right) R_{pl}(r) = 0, \tag{3.23}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2} + p^2\right) R_{pl}(r) = 0. \tag{3.24}$$

The solution for the above equation for radial part is $R_{pl}(r) = (\sqrt{2/\pi})pj_l(pr)$ where j_l is the spherical Bessel function. These functions satisfy the property

$$\int_o^\infty \mathrm{d}r\, r^2 j_l(pr) j_l(p'r) = \frac{\pi}{2}\frac{1}{p^2}\delta(p-p'). \tag{3.25}$$

Similarly, the angular part of the solutions satisfy the equation

$$\left(\triangle_\Omega + \frac{l(l+1)}{r^2}\right) Y_{lm}(\Omega) = 0. \tag{3.26}$$

It can be recognised that $Y_{lm}(\Omega)$ are the eigenfunctions of the $\boldsymbol{L}^2$ operator where $\boldsymbol{L}$ is the angular momentum operator. The basis functions satisfy the following orthogonality and completeness relations

$$\begin{aligned}\int \mathrm{d}^3\boldsymbol{x}\phi^*_{plm}\phi_{p'l'm'} &= \int r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \, N_p N_{\boldsymbol{p'}} R_{pl}(r) R_{p'l'}(r) Y^*_{lm}(\Omega) Y_{l'm'}(\Omega) \\ &= N_p N_{\boldsymbol{p'}} \int r^2 \mathrm{d}r \, R_{pl}(r) R_{p'l'}(r) \int \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \, Y^*_{lm}(\Omega) Y_{l'm'}(\Omega) \\ &= N_p^2 \delta(p-p')\delta_{ll'}\delta_{mm'}.\end{aligned}$$

The orthogonality relation for the modes is

$$\int \mathrm{d}p \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} R_{pl}(r) Y^*_{lm}(\Omega_r) R_{pl}(r') Y_{lm}(\Omega_{r'}) = \delta^3(\boldsymbol{r}-\boldsymbol{r'}). \tag{3.27}$$

Hence the field $\hat{\phi}$ in this basis can be written as

$$\hat{\phi}(t,\boldsymbol{x}) = \int \mathrm{d}p \, N_p \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} R_{pl}(r) Y_{lm}(\Omega) \hat{a}_{plm}(t). \tag{3.28}$$

Since the field $\hat{\phi}(t,\boldsymbol{x})$ satisfies the equation $\ddot{\hat{\phi}} = (\triangle - m^2)\hat{\phi}$, we have

$$\begin{aligned}\ddot{\hat{\phi}} &= (\triangle_r + \triangle_\Omega - m^2) \int \mathrm{d}p \, N_p \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} R_{pl}(r) Y_{lm}(\Omega) \hat{a}_{plm}(t) \\ &= -\int \mathrm{d}p \, (p^2+m^2)\hat{\phi}, \\ \ddot{\hat{a}} &= -(p^2+m^2)\hat{a}.\end{aligned}$$

Defining $\omega_p = \sqrt{p^2+m^2}$, we get the solutions as

$$\hat{a}_{plm} = \hat{a}_{p+} e^{-i\omega_p t} + \hat{a}_{p-} e^{i\omega_p t}.$$

The condition of real scalar field implies $\hat{\phi} = \hat{\phi}^\dagger$ which translates to the following equation

$$Y_{lm}(\Omega)(\hat{a}_{p+} e^{-i\omega_p t} + \hat{a}_{p-} e^{i\omega_p t}) = Y^*_{lm}(\Omega)(\hat{a}^\dagger_{p+} e^{i\omega_p t} + \hat{a}^\dagger_{p-} e^{-i\omega_p t}).$$

Since $e^{-i\omega_p t}$ and $e^{i\omega_p t}$ are two independent solutions, upon rearraging the above terms we have $\hat{a}_{p-} = (Y^*_{lm}/Y_{lm})\hat{a}^\dagger_{p+}$. Using this relation, we have

$$\hat{\phi} = \int \mathrm{d}p \, N_p \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} R_{pl}(r)(Y_{plm}(\Omega)\hat{a}_{plm} e^{-i\omega_p t} + Y^*_{plm}(\Omega)\hat{a}^\dagger_{plm} e^{i\omega_p t}), \tag{3.29}$$

$$\hat{\pi} = -i \int \mathrm{d}p \, N_p \omega_p \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} R_{pl}(r)(Y_{plm}(\Omega)\hat{a}_{plm} e^{-i\omega_p t} - Y^*_{plm}(\Omega)\hat{a}^\dagger_{plm} e^{i\omega_p t}). \tag{3.30}$$

From the above relations, we can solve for $\hat{a}_{plm}$ and $\hat{a}^\dagger_{plm}$ by using the orthogonality conditions for spherical basis as was done for plane wave basis. This method is simply that of finding the inverse Fourier transforms. Instead, we can choose to impose the commutation relations for $[\hat{\phi}, \hat{\pi}]$. Imposing the commutation relations

$$[\hat{a}_{plm}, \hat{a}_{p'l'm'}] = 0 = [\hat{a}^\dagger_{plm}, \hat{a}^\dagger_{p'l'm'}], \tag{3.31}$$

$$[\hat{a}_{plm}, \hat{a}^\dagger_{p'l'm'}] = \delta(p-q)\delta_{ll'}\delta_{mm'}, \tag{3.32}$$

we get

$$[\hat{\phi}(t,\boldsymbol{x}), \hat{\pi}(t,\boldsymbol{y})] = 2iN_p^2\delta^3(\boldsymbol{r}-\boldsymbol{r'})\omega_p. \tag{3.33}$$

If we choose $N_p^2 = 1/2\omega_p$, we get equal time commutation relation as $[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{r}-\boldsymbol{r'})$, required. We can also find the relation between the creation operators in different basis by equating the field expansions. Since, the field does not depend on the particular choice of coordinates chosen for representation, we have

$$\phi_{spherical} = \phi_{plane},$$

$$\int \mathrm{d}p\, \frac{1}{\sqrt{2\omega_p}} \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} R_{pl}(r)Y_{plm}(\Omega)\hat{a}_{plm}e^{-i\omega_p t} = \int p^2\mathrm{d}p \int d\Omega \frac{1}{\sqrt{(2\pi)^3 2\omega_p}}\hat{a}_p e^{ip\cdot x - i\omega_p t}$$

$$\text{and also } e^{ip\cdot x} = 4\pi\sum_{lm} i^l j_l(pr)Y^*_{lm}(\Omega_p)Y_{lm}(\Omega_r).$$

So we have

$$\int \mathrm{d}p\, \frac{1}{\sqrt{2\omega_p}} \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} R_{pl}(r)Y_{plm}(\Omega)\hat{a}_{plm} = \int p^2\mathrm{d}p \int \mathrm{d}\Omega \frac{1}{\sqrt{(2\pi)^3 2\omega_p}}\hat{a}_p 4\pi$$

$$\times \left(\sum_{lm} i^l j_l(pr)Y^*_{lm}(\Omega_p)Y_{lm}(\Omega_r)\right),$$

$$\hat{a}_{plm} = \int \mathrm{d}\Omega_p p i^l Y^*_{lm}(\Omega_p)\hat{a}_p.$$

We can now proceed to calculate the hamiltonian in this basis and see if its form changes as the one compared to the plane wave basis.

$$\begin{aligned} H &= \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}(\hat{\pi}^2(t,\boldsymbol{x}) + (\nabla\hat{\phi}(t,\boldsymbol{x}))^2 + m^2\hat{\phi}^2(t,\boldsymbol{x})), \\ &= \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}(\hat{\pi}^2(t,\boldsymbol{x}) + m^2\hat{\phi}^2(t,\boldsymbol{x})) + \left(\frac{1}{2}\hat{\phi}(t,\boldsymbol{x})\nabla\hat{\phi}(t,\boldsymbol{x})\right)_{limits} - \int \frac{1}{2}\mathrm{d}^3\boldsymbol{x}\hat{\phi}(t,\boldsymbol{x})\nabla^2\hat{\phi}(t,\boldsymbol{x}), \\ &= \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}(\hat{\pi}^2(t,\boldsymbol{x}) + \hat{\phi}(t,\boldsymbol{x})(m^2-\nabla^2)\hat{\phi}(t,\boldsymbol{x})). \end{aligned}$$

But, the Klein-Gordon equation for $\hat{\phi}$ reads as $\ddot{\hat{\phi}} = (\nabla^2 - m^2)\,\hat{\phi}$. Hence the Hamiltonian becomes

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}(\hat{\pi}^2 - \hat{\phi}\ddot{\hat{\phi}}) \tag{3.34}$$

For an operator $\hat{A}$ which can be written in terms of plane wave basis, we can define the positive and negative modes as $\hat{A}^+$ and $\hat{A}^-$ as the ones with terms containing $e^{-i\omega t}$ and $e^{i\omega t}$ respectively. Using this notation, we can write

$$\begin{aligned}
\hat{\pi}(t,\boldsymbol{x}) = \hat{\pi}^+ + \hat{\pi}^- &= \sum_{plm} N_p R_{pl}(r)(-i\omega_p)(Y_{plm}(\Omega)\hat{a}_{plm}e^{-i\omega_p t} - Y^*_{plm}(\Omega)\hat{a}^\dagger_{plm}e^{i\omega_p t}),\\
\hat{\pi}^- &= \sum_{plm} N_p R_{pl}(r)(i\omega_p)Y^*_{plm}(\Omega)\hat{a}^\dagger_{plm}e^{i\omega_p t},\\
\hat{\pi}^+ &= \sum_{plm} N_p R_{pl}(r)(-i\omega_p)Y_{plm}(\Omega)\hat{a}_{plm}e^{-i\omega_p t}.
\end{aligned}$$

Similar notation can be used for $\hat{\phi}$ to write it as sum of $\hat{\phi}^+$ and $\hat{\phi}^-$ and $\ddot{\hat{\phi}}$ as $\ddot{\hat{\phi}}$ and $\ddot{\hat{\phi}}$. The expansions for $\hat{\phi}$ and $\ddot{\hat{\phi}}$ are as follows

$$\begin{aligned}
\phi(t,\boldsymbol{x}) = \hat{\phi}^+ + \hat{\phi}^- &= \sum_{plm} N_p R_{pl}(r)(Y_{plm}(\Omega)\hat{a}_{plm}e^{-i\omega_p t} + Y^*_{plm}(\Omega)\hat{a}^\dagger_{plm}e^{i\omega_p t}),\\
\ddot{\hat{\phi}}(t,\boldsymbol{x}) = \ddot{\hat{\phi}}^+ + \ddot{\hat{\phi}}^- &= -\sum_{plm} \omega_p^2 N_p R_{pl}(r)(Y_{plm}(\Omega)\hat{a}_{plm}e^{-i\omega_p t} + Y^*_{plm}(\Omega)\hat{a}^\dagger_{plm}e^{i\omega_p t}).
\end{aligned}$$

We use the normal ordering operation to ease the simplification of the steps involved. In general, for an operator A, A^- term corresponds to the one with term $e^{i\omega_p t}$ and will have the creation operator in its expansion and A^+ to the one with $e^{-i\omega_p t}$ and will have annihilation operator in its expansion. Hence

$$\begin{aligned}
A_1A_2 &= A_1^-A_2^- + A_1^+A_2^+ + A_1^-A_2^+ + A_1^+A_2^-,\\
:A_1A_2: &=: A_1^-A_2^- + A_1^+A_2^+ + A_1^-A_2^+ + A_1^+A_2^- :,\\
&= A_1^-A_2^- + A_1^+A_2^+ + A_1^-A_2^+ + A_2^-A_1^+.
\end{aligned}$$

Applying normal ordering to Hamiltonian, we get,

$$\begin{aligned}
:\hat{H}: &=: \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}(\hat{\pi}^2 - \hat{\phi}\ddot{\hat{\phi}}) :,\\
&= \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}(\hat{\pi}^-\hat{\pi}^- + \hat{\pi}^+\hat{\pi}^+ + 2\hat{\pi}^-\hat{\pi}^+ - \hat{\phi}^-\ddot{\hat{\phi}}^- - \hat{\phi}^+\ddot{\hat{\phi}}^+ - \hat{\phi}^-\ddot{\hat{\phi}}^+ - \ddot{\hat{\phi}}^-\hat{\phi}^+).
\end{aligned}$$

Radial functions have the orthogonal property that

$$\int \mathrm{d}^3\boldsymbol{x} R_{pl}(r) R_{p'l'}(r') = \delta(p - p').$$

Hence, in the Hamiltonian, upon integrating with $\mathrm{d}^3\boldsymbol{x}$ we get $\delta^3(\boldsymbol{p} - \boldsymbol{p'})$ terms for all the terms and it can be easily seen that this delta functions can be removed by integrating with one of $\mathrm{d}^3\boldsymbol{p}$ or $\mathrm{d}^3\boldsymbol{p}'$. Hence, the terms $\hat{\pi}^-\hat{\pi}^-$ and $\hat{\phi}^-\ddot{\hat{\phi}}^-$ have same terms with opposite signs and cancel away. Same is the case with $\hat{\phi}^+\ddot{\hat{\phi}}^+$ and $\hat{\pi}^+\hat{\pi}^-$. The terms remaining the Hamiltonian are

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}(2\hat{\pi}^-\hat{\pi}^+ - \hat{\phi}^-\ddot{\hat{\phi}}^+ - \ddot{\hat{\phi}}^-\hat{\phi}^+).$$

We also have the orthogonality relation for ϕ_{plm} as

$$\int \mathrm{d}^3\boldsymbol{x}\phi^*_{plm}\phi_{p'l'm'} = N_p^2\delta(p - p')\delta_{ll'}\delta_{mm'}.$$

Unlike the terms $\hat{\pi}^-\hat{\pi}^-$, $\hat{\phi}^-\ddot{\hat{\phi}}^-$ and $\hat{\phi}^+\ddot{\hat{\phi}}^+$, $\pi^+\pi^-$, the remaining terms in the Hamiltonian add up giving the result

$$\hat{H} = \int \mathrm{d}p \sum_{lm} \omega_p \hat{a}^\dagger_{plm}\hat{a}_{plm}. \tag{3.35}$$

The result is of great importance as it shows the form invariance of the Hamiltonian in any basis we chose to work with.

3.4 Green functions in flat spacetime

At the beginning of the study of the quantum field theory, we examined the amplitude of a relativistic particle to go from x to y and found an inconsistency that mere quantization of particles lead to problems arising with causality as the amplitude to propogate is non-zero outside light cone. Now, let us try to look at the problem in the formalism of quantum field theory we have understood. The amplitude for a particle to propogate from y to x is given by $\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle$. Before calculating the amplitude we have to work on the renormalization of the particle states. The vaccuum is defined as state which is annihilated by all annihilation operators i.e $\hat{a}_{\boldsymbol{p}}|0\rangle = 0$. We choose this vaccuum such that $\langle 0|0\rangle = 1$. The one particle state is obtained by using the creation operator i.e $|\boldsymbol{p}\rangle = \hat{a}^\dagger_{\boldsymbol{p}}|0\rangle$. The simplest normalization that we can think of is $\langle \boldsymbol{p}|\boldsymbol{q}\rangle = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q})$.

But this normalization is not Lorentz invariant, as we can demonstrate by considering a Lorentz boosted frame. For a four-momentum p_μ, considering a Lorentz boost in p_3 we have $p_3^{'} = \gamma(p_3 + \beta E)$, $E^{'} = \gamma(E + \beta p_3)$. For the delta function, we have the identity

$$\delta(f(x) - f(x_0)) = \frac{1}{f'(x_0)}\delta(x - x_0).$$

Hence, using the relation $E^2 = \boldsymbol{p} \cdot \boldsymbol{p} + m^2$

$$\begin{aligned}
\delta^3(\boldsymbol{p} - \boldsymbol{q}) &= \delta^3(\boldsymbol{p}' - \boldsymbol{q}')\frac{\mathrm{d}p'_3}{\mathrm{d}p_3} \\
&= \delta^3(\boldsymbol{p}' - \boldsymbol{q}')\gamma(1 + \beta\frac{\mathrm{d}E}{\mathrm{d}p_3}) \\
&= \delta^3(\boldsymbol{p}' - \boldsymbol{q}')\frac{\gamma}{E}(E + \beta p_3) \\
&= \delta^3(\boldsymbol{p}' - \boldsymbol{q}')\frac{E'}{E}.
\end{aligned}$$

From this it can be seen that, it is not δ^3 quantities which are Lorentz invariant but $E\delta^3$ which are invariant. Hence we use the renormalization $\langle \boldsymbol{p}|\boldsymbol{q}\rangle = (2\pi)^3 2E_{\boldsymbol{p}}\delta^3(\boldsymbol{p} - \boldsymbol{q})$ and hence the one-particle states become $|\boldsymbol{p}\rangle = \sqrt{2E_{\boldsymbol{p}}}\hat{a}^\dagger_{\boldsymbol{p}}|0\rangle$. This can be also be understood from the fact that it is d^4x which is Lorentz invariant and not $\mathrm{d}^3\boldsymbol{x}$.

Let us study the quantity $G(x, y)$ which is the solution of the Klein-Gordon equation

$$(\Box_x + m^2)G(x, y) = -\delta^4(x - y). \tag{3.36}$$

The solution for this equation $G(x - y)$ is called the Green function. For any other source $\rho(x)$ such that

$$(\Box_x + m^2)\phi(x) = -\rho(x),$$

we can obtain the solution for ϕ in terms of $G(x, y)$ as follows

$$\phi(x) = \int \mathrm{d}^4y\, G(x, y)\rho(y).$$

However, $G(x, y)$ is not unique and any function satisfying the property $(\Box_x + m^2)f(x) = 0$ can be added to $G(x, y)$ to get a new G such that $G^{'} = G + f$. Uniqueness of the Green functions follows only if impose suitable boundary conditions. If the spacetime is translationally invariant, then $G(x, y) = G(x - y)$. We can solve the equation (3.36) by converting it into

Fourier space. Converting into Fourier space, we have

$$\delta^4(x-y) = \int \frac{\mathrm{d}^4x}{(2\pi)^4} e^{-ik\cdot(x-y)},$$
$$G(x-y) = \int \frac{\mathrm{d}^4x}{(2\pi)^4} e^{-ik\cdot(x-y)} \tilde{G}(k).$$

We get

$$(-k^2+m^2)\tilde{G}(k) = -1,$$
$$\tilde{G}(k) = \frac{1}{k^2-m^2},$$

and hence

$$G(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik\ddot{(}x-y)} \frac{1}{k^2-m^2}.$$

The k^2 term in the denominator is the square of the four momentum k_μ and not the three vector $\boldsymbol{k}$ i.e. $k^2 = k_\mu k^\mu = \omega^2 - |\boldsymbol{k}|^2$. Hence $k^2 - m^2 = \omega^2 - \omega_{\boldsymbol{k}}^2$ where $\omega_{\boldsymbol{k}}^2 = |\boldsymbol{k}|^2 + m^2$. Rewriting this way, we can perform the integration over the dk_o by identifying the singularities. So,

$$G(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{1}{\omega^2-\omega_k^2}. \tag{3.37}$$

This integral has two singularities at $\omega = \omega_{\boldsymbol{k}}$ and $\omega = -\omega_{\boldsymbol{k}}$. Since $\omega = k_0$ integral runs from $-\infty$ to ∞, these integration can be performed by slightly deforming the contour at the singularties, or equivalently shifting the singularities. This is mathematically expressed as , for $\epsilon > 0$ and arbitrarily small,

$$G(x-y) = \int \mathrm{d}\omega \int \frac{\mathrm{d}^3k}{(2\pi)^4} \left\{ \frac{e^{-ik\cdot(x-y)}}{\omega - \omega_k \pm i\epsilon} - \frac{e^{-ik\cdot(x-y)}}{\omega + \omega_k \pm i\epsilon} \right\}.$$

We now have four possibilities for performing the $d\omega$ integration. These four are

1. Shifting both the singularities in Upper half complex (UHP) plane (adding $-i\epsilon$ to both),

2. Shifting both the singularities in Lower half complex (LHP) plane (adding $+i\epsilon$ to both),

3. Shifting the singularity at $-\omega_k$ in UHP (adding $-i\epsilon$) and ω_k to LHP (adding $i\epsilon$),

4. Shifting the singularity at $-\omega_k$ in LHP (adding $i\epsilon$) and ω_k to UHP (adding $-i\epsilon$).

Consider $\omega = |\omega| e^{i\theta}$ and it follows that

$$e^{-i\omega(t-t')} = e^{-i|\omega|(t-t')\cos\theta} e^{|\omega|(t-t')\sin\theta}.$$

For the integral not to diverge we should have the condition that $e^{|\omega|(t-t')\sin\theta}$ does not blow up upon integration. Hence, if $t > t'$ then $\sin\theta < 0$ implying that we need to close the contour in the lower half plane and vice-versa. Summarising,

1. If $t > t'$ close the contour in LHP
2. If $t' > t$ close the contour in UHP

Depending on how the singularities are moved, we have different types of Green functions. If both the singularities are moved into the UHP, the corresponding Green function is called advanced Green function G_{adv} and if the singularities are moved into the LHP, it is called retarded Green function G_{ret}. For G_{adv} we have

$$G_{adv} = 0 \;\; if \; t > t',$$

$$\neq 0 \;\; if \; t < t'.$$

Similarly

$$G_{ret} = 0 \;\; if \; t < t',$$

$$\neq 0 \;\; if \; t > t'.$$

Using the Cauchy residue theorem, we can compute the dk_0 integral to get the expression for G_{adv} and G_{ret}. The above conditions for G_{ret} and G_{adv} will be imposed by appropriately multiplying it with Heavyside function $\Theta(t-t')$ or $\Theta(t'-t)$ which gives

$$G_{adv}(x-y) = -\Theta(t'-t)\int \frac{\mathrm{d}^3k}{(2\pi)^3\omega_{\boldsymbol{k}}}\sin\omega_k(t-t')e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}, \tag{3.38}$$

$$G_{ret}(x-y) = \Theta(t-t')\int \frac{\mathrm{d}^3k}{(2\pi)^3\omega_{\boldsymbol{k}}}\sin\omega_k(t-t')e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}. \tag{3.39}$$

The retarded Green function takes causality into account. $G_{ret}(x-y)$ vanishing outside light cone implies that only $\rho(y)$ that lie in the past light cone will contribute to the determination of $\hat{\phi}(x)$. The sign of $t-t'$ uniquely fixes whether the point y lies in the past or future light cone of x. This is the condition that we imposed above. Similarly , G_{adv} has support in the future light cone. There is also another Green function called the Feynman propagator denoted by $G_F(x-y)$ which has one singularity in UHP and one in LHP. We will discuss this in the case of Klein-Gordon field. We will now address the problems of causality due to propagation amplitude that was discussed at the beginning of this chapter.

The amplitude is given by $\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle$. This is the amplitude for the particle to propagate from spacetime point y to x. Let us denote this amplitude as $D(x-y)$. The field operator $\hat{\phi}(x)$ has both annihilation operators and creation operators. The only contributing term will involve the product of $\hat{a}$ from $\hat{\phi}(x)$ and $\hat{a}^\dagger$ from $\hat{\phi}(y)$. Hence,

$$\begin{aligned}
\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}}e^{-ip\cdot x+iq\cdot y}\langle 0|\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{q}}|0\rangle \\
&= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}}e^{-ip\cdot x+iq\cdot y}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q}), \\
D(x-y) &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{-ip\cdot(x-y)}.
\end{aligned}$$

Consider the case when $x-y$ is timelike. If the interval is timelike, we can always find a frame in which $\boldsymbol{x}-\boldsymbol{y}=0$. Let $x_0-y_0=t$. Using the relation, $E=\sqrt{\boldsymbol{p}^2+m^2}$, we can convert the integral in terms of E which will be

$$\begin{aligned}
D(x-y) &= \frac{1}{4\pi^2}\int_m^\infty dE\sqrt{E^2-m^2}e^{-iEt}, \\
&\sim_{t\longrightarrow\infty} e^{-imt}.
\end{aligned}$$

Now, let us look at the case when the interval $x-y$ is spacelike. If the interval is spacelike, we can always find a frame in which $x_0-y_0=r$. Let $\boldsymbol{x}-\boldsymbol{y}=r$. The propagation amplitude then becomes

$$\begin{aligned}
D(x-y) &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\sqrt{\boldsymbol{p}^2+m^2}}e^{-ip\cdot r} \\
&= \frac{1}{(2\pi)^3}\int \mathrm{d}p\,|\boldsymbol{p}|^2\mathrm{sin}\theta\mathrm{d}\theta\,\mathrm{d}\phi\frac{e^{ip\cdot r}}{2E_{\boldsymbol{p}}} \\
&= \frac{2\pi}{(2\pi)^3}\int \mathrm{d}p\,|\boldsymbol{p}|^2\mathrm{sin}\theta\,\mathrm{d}\theta\frac{e^{ipr\mathrm{cos}\theta}}{2E_{\boldsymbol{p}}} \\
&= \frac{1}{8\pi^2}\int \mathrm{d}p\,\frac{e^{ipr}-e^{-ipr}}{E_{\boldsymbol{p}}r}p\cdot \quad \text{let } \rho=-ip \\
&= \frac{1}{4\pi^2 r}\int_m^\infty \mathrm{d}\rho\frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \sim_{r\longrightarrow\infty} e^{-mr}.
\end{aligned}$$

We find that in both the cases when the interval is timelike as well as spacelike, the propagation amplitude is non-zero but exponentially decaying outside light cone. To understand causality, it is more relevant to ask if the measurements at one point can affect the ones at another rather than the propagation amplitudes. For this we could compute the commutator

$[\hat{\phi}(x), \hat{\phi}(y)]$ and check if it vanishes outside light cone or not. The commutator becomes

$$\begin{aligned}[\hat{\phi}(x), \hat{\phi}(y)] &= \int\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}}\{[\hat{a}_p, \hat{a}_q]e^{-ip\cdot x-iq\cdot y} \\ &+ [\hat{a}_p^\dagger, \hat{a}_q^\dagger]e^{+ip\cdot x+iq\cdot y} + [\hat{a}_p^\dagger, \hat{a}_q]e^{ip\cdot x-iq\cdot y} + [\hat{a}_p, \hat{a}_q^\dagger]e^{-ip\cdot x+iq\cdot y}\}, \\ &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}(e^{-ip\cdot(x-y)} - e^{ip\cdot x(-y)}). \\ &= D(x-y) - D(y-x)\end{aligned}$$

Outside the light cone, the interval $(x-y)$ is spacelike and hence we can find a frame in which $x_0 - y_0 = 0$ and let $\boldsymbol{x} - \boldsymbol{y} = r$. The term $\int \mathrm{d}^3\boldsymbol{p}e^{-i\boldsymbol{p}\cdot\boldsymbol{r}} = \int \mathrm{d}^3\boldsymbol{p}e^{i\boldsymbol{p}\cdot\boldsymbol{r}}$. Hence the commutator vanishes if the interval $(x-y)$ is spacelike. Interpreting in another way, when the interval is spacelike, we can perform a Lorentz transformation from $(x-y)$ to $-(x-y)$ and the terms become equal with opposite signs and so cancel away. The commutator has the form of the Green functions and by imposing the conditions on time components of x and y, we can construct the retarded and advanced Green function

$$D_{ret}(x-y) = \Theta(x_0 - y_0)\langle 0|[\hat{\phi}(x), \hat{\phi}(y)]|0\rangle. \tag{3.40}$$

Let us do the computation $(\partial^2 + m^2)D_{ret}(x-y)$. We have,

$$\begin{aligned}(\partial^2 + m^2)D_{ret}(x-y) &= \partial^2\Theta(x_0 - y_0)(\langle 0|[\hat{\phi}(x), \hat{\phi}(y)]|0\rangle) + (\partial^2 + m^2)\langle 0|[\hat{\phi}(x), \hat{\phi}(y)]|0\rangle \\ &+ 2\partial_\mu\Theta(x_0 - y_0)\partial^\mu\langle 0|[\hat{\phi}(x), \hat{\phi}(y)]|0\rangle \\ &= -\delta(x_0 - y_0)\langle 0|[\hat{\pi}(x), \hat{\phi}(y)]|0\rangle + 2\delta(x_0 - y_0)\langle 0|[\hat{\pi}(x), \hat{\phi}(y)]|0\rangle + 0 \\ &= -i\delta^4(x-y),\end{aligned}$$

which corroborates our statement that the above expression is indeed Green function.

Previously, in the discussion of Green functions, we have understood G_{ret} and G_{adv} by shifting the poles either into the UHP or into LHP. However, there is another Green function called the Feynman propagator which is obtained by moving one pole into UHP and another into LHP. Whether the contour is closed in UHP or LHP, only one pole is inside the contour and this determines the expression for the Green function. The mathematical form of Feynman propagator is

$$D_F(x-y) = \int \frac{\mathrm{d}^4x}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x-y)}, \tag{3.41}$$

$$D_F(x-y) = \begin{cases} D(x-y)\text{if } x_0 > y_0, \\ D(y-x)\text{if } y_0 < x_0 \end{cases}$$

$$D_F(x-y) = \Theta(x_0 - y_0)D(x-y) + \Theta(y_0 - x_0)D(y-x) = \langle 0|T(\hat{\phi}(x)\hat{\phi}(y)|0\rangle. \tag{3.42}$$

The symbol T is called the time ordering operator and is frequently encountered in calculating the scattering matrix elements. The time ordering arranges the operators in order with the latest to the left. Having understood quantum field theory in Minkowski spacetime and Green functions, we are now set to combine our understanding of de Sitter spacetime and quantization to understand quantum field theory in a curved spacetime.

Chapter 4

Quantum field theory in curved spacetime

We have comprehensively studied the Klein-Gordon field in flat spacetime. It would be interesting to look at the same in curved spacetime. So, now we proceed to understand the quantization in curved spacetime and the interesting phenomena associated with it. The flat spacetime Klein-Gordon equation can be written as

$$\phi^{,\mu}_{,\mu} + m^2\phi = 0.$$

In the presence of gravity, the normal derivative becomes the covariant derivative and hence the equation becomes,

$$\phi^{;\mu}_{;\mu} + m^2\phi = 0 \tag{4.1}$$

Using the expansion for the covariant derivative and the property that

$$g^{\mu v}\Gamma^{\sigma}_{\mu v} = -\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\sigma d}),$$

where Γ is the christoffel connection symbol, we get

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu v}\partial_v\phi) + m^2\phi = 0. \tag{4.2}$$

This form of the Klein-Gordon equation is very useful compared to the one with the covariant derivatives when solving for field ϕ.

4.1 Bogoliubov transformations

As we have seen in the quantization of the scalar field in the flat spacetime, the field operator was expressed in terms of modes $u(p)$ and $u^{\dagger}(p)$. Mathematically, it is

$$\phi(x) = \sum_i a_i u_i(x) + a_i^{\dagger} u_i^{\dagger}(x).$$

However, in curved spacetime, no natural mode of decomposition based on the separation of wave equation is possible. General relativity is based on the principle of general covariance.Although, coordinate systems are useful a lot of times in understand the various important properties of the spacetime, it might not be unique. There can exist various coordinate systems which can be used to describe the same coordinate system. Hence, there can exist a second complete set of orthonormal modes $\overline{u}_j(x)$ for the expansion of field operator. The expansion of $\hat{\phi}$ in these modes is expressed as

$$\hat{\phi}(x) = \sum_i \hat{\overline{a}}_i \overline{u}_i(x) + \hat{\overline{a}}_i^{\dagger} \overline{u}_i^{\dagger}(x).$$

The new vaccuum state is defined by $\hat{\overline{a}}_j|0\rangle = 0,\ \forall j$. The new modes $\overline{u}_j$ can be expressed in terms of the old ones u_j as follows

$$\overline{u}_j = \sum_i \alpha_{ji} u_i + \beta_{ji} u_i^*, \tag{4.3}$$

$$\overline{u}_j^* = \sum_i \alpha_{ji}^* u_i + \beta_{ji}^* u_i. \tag{4.4}$$

These transformations are called the Bogoliubov transformations. It is straightforward to find out the inverse relations to express u_i in terms of $\overline{u}_j$ as

$$u_i = \sum_i \alpha_{ji}^* \overline{u}_j - \beta_{ji}^* \overline{u}_j^*.$$

We have used the relation $\sum_k \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^* = \delta_{ij}$ in deriving the above. It is useful to take note of a few important results from the above which are

$$(\overline{u}_i, u_j) = \alpha_{ij},\ (u_i, u_j) = \delta_{ij},\ (u_i, u_j^*) = 0,\ -(\overline{u}_i, u_j^*) = \beta_{ij}.$$

Using all of these results derived above, it is not difficult to determine the relations between the annihilation and creation operators corresponding to the two modes. Since, the field operator can be expanded in more than one set of basis, but describes the same dynamics, it must be same expressed in any basis. Hence equating the expansion of the field operator in the two modes, we have

$$\sum_i \left(\hat{\overline{a}}_i \overline{u}_i(x) + \hat{\overline{a}}_i^\dagger \overline{u}_i^\dagger(x)\right) = \sum_i \left(\hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i^\dagger(x)\right).$$

Using the expressions for new modes in terms of old modes, we get

$$\hat{\overline{a}}_j = \sum_i \alpha_{ji}^* \hat{a}_i - \beta_{ji} \hat{a}_i^*. \tag{4.5}$$

It would be now interesting to see the action of old annihilation operator on the new vaccuum state defined by $\overline{a}_i$. With $|\overline{0}\rangle$ as the new vacuum, we get

$$\begin{aligned} \hat{a}_i|\overline{0}\rangle &= \sum_j (\alpha_{ij}\hat{\overline{a}}_j + \beta_{ji}^* \hat{\overline{a}}_j^\dagger)|\overline{0}\rangle, \\ &= \sum_j \beta_{ji}|\overline{1}_j\rangle \neq 0. \end{aligned}$$

This shows that the two vacua defined with respect two different modes are not equivalent unless the coefficient $\beta_{ij} = 0 \ \forall j$. This leads to the ambiguity in defining a vacuum state in curved spacetime. On account of this, it would be very captivating to study the various vacuum states in the de Sitter space. For instance, this choice of vacuum state is important in the construction of a realistic inflation model. It is important to understand the meaning of a vacuum state and to select a meaningful vacuum state in a general curved spacetime.

4.2 De Sitter invariant vacua for a massive scalar field

In de Sitter spacetime, we have two kinds of vacuum states. The ones which are de Sitter invariant and the ones which are not. The de Sitter invariant states are the ones which look the same to any freely falling observer, anywhere in de Sitter space. The symmetry group for de Sitter space $O(1,4)$ is connected of four disconnected components. One of the four components is the group G containing the identity element. This is analogous to the proper orthochronous Poincare group in flat space which consists of continuous Lorentz transformations. Similarly the other three components are also similar to the three components of the full Lorentz group in the flat spacetime. Hence the other three components are $(1+4)$ dimensions.

Time reversal (T) =diag. (-1,1,1,1,1)

Space reflection or Parity (S)=diag. (1,-1,1,1,1)

Time+ space reflection(TS)=diag. (-1,-1,1,1,1) The de Sitter invariant state is one which is invariant under the action of all four components of $O(1,4)$. There can also be states which are invariant under the action of connected part of the de Sitter group G, but which are invariant under the action of the other components. The antipodal transformation is defined as A=diag. (-1,-1,-1,-1,-1). This operation sends the point, x, to its antipodal point, $\overline{x}$ such that if the five vector corresponding to point x is $X(x)$ then its antipodal point has the five vector $X(\overline{x}) = -X(x)$.

As we have already seen in the section corresponding to the classical properties of the de Sitter space that it can be visualised as an embedding in the flat spacetime. If the flat spacetime metric is given by $\eta_{ab} = \text{diag.}\ (-1,1,1,1,1)$ for $(1+4)$ dimensions, we have the embedding given by the mathematical expression

$$X^a X^b \eta_{ab} = H^{-2}.$$

The geodesic distance between the two points x and y is given by

$$d(x,y) = H^{-1}\cos^{-1} Z, \tag{4.6}$$

where the function Z is defined as

$$Z(x,y) = H^{-2}\eta_{ab}X^a(x)X^b(y). \tag{4.7}$$

From this definition of Z it can be seen that

$$Z(x,y) = -Z(\overline{x},y) = -Z(x,\overline{y}). \tag{4.8}$$

It is important to use a convenient coordinate system that covers the de Sitter space. There are several well-known systems. As we have seen in the discussion of the classical properties of de Sitter space, we can use two spatially flat coordinate patches. The metric in such a patch is written as

$$\mathrm{d}s^2 = H^{-2}t^{-2}(-\mathrm{d}t^2 + \mathrm{d}x^2).$$

A notable feature of this coordinate system is that if a point has coordinates (t,x) then its antipodal point has the coordinates $(-t,x)$. If the coordinates of two points x and y are (t,x)

and (t', x'), then

$$Z(x,y) = \frac{t^2 + t'^2 - (x - x')^2}{2tt'}. \tag{4.9}$$

Now let us examine the de Sitter invariant states for a real scalar field. Consider the symmetric two-point function

$$G_\lambda^{(1)}(x,y) = \langle \lambda | \Phi(x)\Phi(y) + \Phi(y)\Phi(x) | \lambda \rangle, \tag{4.10}$$

in a de Sitter invariant state $|\lambda\rangle$. There will be more than one such invariant states. Since, $|\lambda\rangle$ is invariant under the full disconnected group $O(1,4)$, it implies that the symmetric two point function above can only depend on the separation between the spacetime points x and y via the geodesic distance $d(x,y)$. Since the geodesic distance $d(x,y)$ depends only on $Z(x,y)$, the two-point function $G^{(1)}(x,y) = F(Z)$. This two point function obeys the Klein-Gordon equation for a massive scalar field

$$(\Box_x + m^2)G(x,y) = 0.$$

This can be expressed as an equation in Z, (the derivation for which will be discussed later)

$$\left[(Z^2 - 1)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 4Z\frac{\mathrm{d}}{\mathrm{d}Z} + m^2 H^{-2} \right] F(Z) = 0 \tag{4.11}$$

This second order equation has two solutions. Let the first solution be $f(Z)$. The above equation being invariant under the change of $Z \to -Z$, the second solution would be $f(-Z)$. The fundamental real solution for the above differential equation is given by the hypergeometric function

$$f(Z) = {}_2F_1(c, 3 - c, 2, \frac{1}{2}(1 + Z)), \tag{4.12}$$

where c is given by the solutions for the equation

$$c(c - 3) + m^2 H^{-2} = 0.$$

For a massive field, the solutions $f(Z)$ and $f(-Z)$ are linearly independent and hence the general solutions can be written as $F(Z) = af(Z) + bf(-Z)$. The form of the solutions in terms of hypergeometric functions suggest that the general solution has two poles at $Z = -1$

and $Z = 1$. These singular points physically correspond to x being either on the light cone of $\overline{y}$ or y. The Euclidean vacuum is defined as the one with the coefficient $b = 0$ implying only one singular point when x is on the light cone of y. The constant a is determined by the canonical commutation relations between Φ and $\dot{\Phi}$ and is given by

$$a = (8\pi)^{-1}H^2(m^2H^{-2} - 2)\sec\left[\pi\left(\frac{9}{4} - m^2H^{-2}\right)^{\frac{1}{2}}\right].$$

Now, let us look at the other de Sitter invariant states. There must be a particular set of modes $\phi_n(x)$ which are orthonormal and which serve to define the above derived Euclidean vacuum. Using Bogoliubov transofrmation, we obtain new modes which, via canonical quantization, serve to define new vacuum and we understand the de Sitter invariance of such vacua. Let the new modes defined by the Bogoliubov transformations be defined as

$$\overline{\phi}_n(x) = A\phi(x) + B\phi^*(x).$$

Bogoliubov transformations are orthonormality preserving. This means that, though they mix the positive and negative frequency modes, they still satisfy the property of orthonormality in the new modes. Mathematically,

$$\begin{aligned}(\overline{\phi}_m, \overline{\phi}_n) &= (|A|^2 - |B|^2)(\phi_m, \phi_n),\\ &= (|A|^2 - |B|^2)\delta_{mn}.\end{aligned}$$

Since the constants A and B are frequency or mode independent, with the condition that $|A|^2 - |B|^2 = 1$, the general solution would be written as $A = e^{i\gamma}\cosh\alpha, B = e^{i(\gamma+\beta)}\sinh\alpha$. However, since the overall phase $e^{i\gamma}$ is irrelevant as it vanishes in the calculation of expectation values. Inserting the solution for A and B we get

$$\overline{\phi}_n = \cosh\alpha\,\phi(x) + e^{i\beta}\sinh\alpha\,\phi^*(x).$$

This is a two parameter family in α and β. The ranges of α and β are $[0, \infty]$ and $[-\pi, \pi]$ respectively. The Euclidean vacuum corresponding to $A = 1$ and $B = 0$ corresponds to $\alpha = 0$. We now study the de Sitter invariance of states with $\alpha \neq 0$. We will use a small trick to make our computations easier. Let $\phi_n(\overline{x}) = \phi_n^*(x)$ where $\overline{x}$ corresponds to the antipodal point of x. The set of modes used to describe Euclidean can be written in the spherical modes as $\psi_{klm}(x) = y_k(t)Y_{klm}(\Omega)$. The antipodal transformation changes $(t,\Omega) \to (-t,\overline{\Omega})$ manifests as $y_k(-t) = y_k^*(t)$ and $Y_{klm}(\overline{\Omega}) = (-1)^k Y^*_{kl-m}(\Omega)$ and so the transformation law for the modes become $\psi_{klm}(\overline{x}) = \psi^*_{kl-m}(x)$. Defining the new modes by Bogoliubov transformations

$$\phi_{klm}(x) = \frac{e^{ik\pi/2}}{\sqrt{2}}[e^{i\pi/4}\psi_{klm}(x) + e^{-i\pi/4}\psi_{kl-m}(x)].$$

As we have seen above, the general form of bogoliubov transformation is

$$\phi_n = \sum_m \alpha_{nm}\psi_m + \beta_{nm}\psi_m^*.$$

This transformation mixes the positive and negative frequency modes. The transformation is called non-trivial if and only if atleast one of β_{nm} is non-zero. In the above definition of ϕ_{klm} the transformation is trivial as the expansion just corresponds to two non-zero α's which are α_{klm} and α_{kl-m}. All the Bogoliubov transformation which are trivial define a physically equivalent vacuum state. The symmetric and antisymmetric two-point functions in (α,β) state is given by

$$G^{(1)}_{\alpha,\beta}(x,y) = \langle\alpha,\beta|\Phi(x)\Phi(y) + \Phi(y)\Phi(x)|\alpha,\beta\rangle.$$

$$iD_{\alpha,\beta}(x,y) = \langle\alpha,\beta|\Phi(x)\Phi(y) - \Phi(y)\Phi(x)|\alpha,\beta\rangle.$$

Expanding the above in the transformed basis, $\overline{\phi}_n$ we get

$$G^{(1)}_{\alpha,\beta}(x,y) = \sum_n \overline{\phi}_n(x)\overline{\phi}^*_n(y) + \overline{\phi}^*(x)\overline{\phi}_n(y),$$

$$iD_{\alpha,\beta}(x,y) = \sum_n \overline{\phi}_n(x)\overline{\phi}^*_n(y) - \overline{\phi}^*_n(x)\overline{\phi}_n(y).$$

Writing the Bogoliubov transformation between Euclidean vacuum modes and these modes, we get

$$G^{(1)}_{\alpha,\beta} = \cosh 2\alpha\left(\sum_n \phi_n(x)\phi_n^*(y) + \phi_n^*(x)\phi_n(y)\right) + \sinh 2\alpha\cos\beta\left(\sum_n \phi_n(x)\phi_n(y) + \phi_n^*(x)\phi_n^*(y)\right)$$
$$+ i\sinh 2\alpha\sin\beta\left(\sum_n \phi^*(x)\phi^*(y) - \phi(x)\phi(y)\right).$$

It can be recognised that the first term $\sum_n \phi_n(x)\phi_n^*(y) + \phi_n^*(x)\phi_n(y)$ is $G_0^{(1)}(x,y)$, the two point function corresponding to the Euclidean vacuum. The second and third terms can be appropriately written as $G_0^{(1)}$ and D_0 by choosing the Euclidean modes to obey $\phi_n(\overline{x}) = \phi_n^*(x)$. Using these properties, one obtains

$$G^{(1)}_{\alpha,\beta} = \cosh 2\alpha\, G_0^{(1)}(x,y) + \sinh 2\,\alpha \cos\beta\, G_0^{(1)}(\overline{x},y) - \sinh 2\alpha \sin\beta\, D_0(\overline{x},y), \tag{4.13}$$

$$iD_{\alpha,\beta}(x,y) = iD_0(x,y). \tag{4.14}$$

Since $G(x,y) = G(Z)$ and $G(\overline{x},y) = G(-Z)$, the above equation is de Sitter invariant if and only if $\beta = 0$ which makes the RHS of (4.13) only a function of Z. $D_0(x,y)$ cannot be a function of only Z as $D(x,y) = -D(y,x)$ but $Z(x,y) = Z(y,x)$. Hence $D(x,y)$ is not $O(1,4)$ invariant and non vanishing $\sin\beta$ term leaves the two-point symmetric function non-invariant under the disconnected de Sitter group $O(1,4)$.

It is also insightful to look at the time reversal property of two point symmetric functions

$$G_0^{(1)}(Tx,Ty) = G_0^{(1)}(\overline{x},\overline{y}) = G_0^{(1)}(x,y), \tag{4.15}$$

since $Z(Tx,Ty) = Z(x,y)$. Similarly, from the definition of the anti-commutator two point function $D(x,y)$ it can be seen that

$$D_0(Tx,Ty) = D_0(\overline{x},\overline{y}) = -D_0(x,y). \tag{4.16}$$

Using these one can see that $G^{(1)}_{\alpha,\beta}(Tx,Ty) = G^{(1)}_{\alpha,-\beta}(x,y)$. In fact, the time-reversal state of $(\alpha,\beta) = (\alpha,-\beta)$. Only the states with $\beta = 0$ are time-reversal invariant. Starting from the Euclidean, we have constructed two parameter (α,β) states that are invariant under the de Sitter group G. There is a one-real parameter $(\alpha,0)$ family of time symmetric states that are invariant under the full disconnected group $O(1,4)$.

4.3 Massless scalar field case

Till now, we have only considered the case of massive fields. We will also understand the treatment for $m^2 = 0$. For the case of $m = 0$ we will get the $c = 0$ as one solution. Hence

$$f(Z) = {}_2F_1\left[0,3;2;\frac{1}{2}(1+z)\right] = 1.$$

The other solution $f(-Z)$ is also the same and hence is not an independent solution. The constant is a trivial solution to $\Box G = 0$. Hence the hypergeometric functions are not the general solutions for the equation $\Box G = 0$. Though one of the solutions is a constant term, the second fundamental real solution is given by

$$P(Z) = (1+Z)^{-1} - (1-Z)^{-1} + \ln\left|\frac{Z-1}{Z+1}\right|$$

and hence the general solution becomes $\alpha P(Z) + \beta$. The solution $P(Z)$ has the property that $P(-Z) = -P(Z)$ which leads to the property

$$G^{(1)}(Z) + G^{(1)}(-Z) = 2\beta. \tag{4.17}$$

We now proceed to show that in the massless case there is no de Sitter invariant Fock vacuum state. In the process of establishing this result, we will prove some important results.

Let us start with a few definitions

1. The inner product of two scalar functions $\phi_1(x)$ and $\phi_2(x)$ in de Sitter space is defined as

$$(\phi_1, \phi_2) = i\int_\sigma (\phi_1^* \nabla_\mu \phi_2 - \phi_2^* \nabla_\mu \phi_1) \mathrm{d}\sigma^\mu$$

where σ is any Cauchy surface

2. Let $\phi_n(x)$ be a set of complex scalar functions satisfying $(\Box_x - m^2)\phi_n(x) = 0$ for real m. These functions are orthonormal implying $(\phi_n, \phi_m^*) = 0$ and $(\phi_n, \phi_m) = \delta_{mn}$

3. $G^{(1)}(x, y) = \sum_n \phi_n(x)\phi_n^*(y) + \phi_n^*(x)\phi_n(y)$

We now establish a few important results which eventually lead to the proof of non-existence of de Sitter invariant Fock vacuum.

1. $G^{(1)}(x, y) + G^{(1)}(x, \overline{y}) \neq 0$ everywhere

Let us prove this by contradiction. Assume $G^{(1)}(x, y) + G^{(1)}(x, \overline{y}) = 0$ everywhere. Expanding it in terms of the mode functions, we get

$$\sum_n \phi_n(x)(\phi_n^*(y) + \phi_n^*(\overline{y})) + \phi_n^*(x)(\phi_n(y) + \phi_n(\overline{y})) = 0.$$

Taking inner product with ϕ_m we get

$$\phi_m^*(y) + \phi_m^*(\overline{y}) = 0.$$

for each m. Defining $\dot{\phi_m} = (\partial/\partial t)\phi_m(x)$. Combining the conditions $\phi_m^*(y) + \phi_m^*(\overline{y}) = 0$ and $\phi_m^*(x) = \phi_m(\overline{x})$ we get $\phi_m(x) = -\phi_m(\overline{x})$. Since $\overline{x}$ is the antipodal point of x, we now have

$$\dot{\phi}_m(\overline{x}) = \phi_m^*(x),$$

and hence

$$\phi_m^*(x)\dot{\phi}_m(x) = -\phi_m^*(\overline{x})\dot{\phi}_m(\overline{x}).$$

With the inner product defined above we have

$$(\phi_m(x), \phi_m(x)) = i\int_{s^3}(\phi_m^*(x)\dot{\phi}_m(x) - \phi_m(x)\dot{\phi}_m^*(x))\mathrm{d}V$$

This integral has equal and opposite values on a pair of point antipodal to each other and since the summation is over the entire sphere, this integral vanishes contradicting our definition of inner product $(\phi_m(x), \phi_m(x)) = \delta_{mn}$. Hence we have proved by contradiction that our original assumption is wrong which implies

$$G^{(1)}(x, y) + G^{(1)}(x, \overline{y}) \neq 0 \ \text{ everywhere.}$$

2. $G^{(1)}(x, y) - G^{(1)}(x, \overline{y}) \neq 0$ everywhere.

The proof for this is exactly the same as above except that the intermediate terms differ in a sign. Following the same procedure as above, we would have

$$\phi_m(x) = \phi_m(\overline{x}),$$

$$\dot{\phi}_m(x) = -\dot{\phi}_m(\overline{x}).$$

With the above results, the property $\phi_m^*(x)\dot{\phi}_m(x) = -\phi_m^*(\overline{x})\dot{\phi}_m(\overline{x})$ still holds good and hence the integral vanishes again and leads to a contradiction.

3. If $m^2 > 0$ and C is a real constant, then $G^{(1)}(x, y) + G^{(1)}(x, \overline{y}) \neq C$ everywhere

We again follow the method of proof by contradiction. Let us assume that

$$G^{(1)}(x, y) + G^{(1)}(x, \overline{y}) = C.$$

Since C is a constant $\Box C = 0$. C being a solution to the wave equation can be expanded in the orthonormal basis, ϕ_n, as

$$C = \sum c_n\phi_n(x) + c_n^*\phi_n^*,$$

where $c_n = (\phi_n, C)$. By our assumption $G^{(1)}(x, y) + G^{(1)}(x, \overline{y}) - C = 0$ we have

$$\sum \phi_n(x)(\phi_n^*(y) + \phi_n^*(\overline{y}) - c_n) + \phi_n^*(x)(\phi_n(y) + \phi_n(\overline{y}) - c_n^*) = 0,$$

which leads to $\phi_n(y) + \phi_n(\overline{y}) = c_n^*$ for each mode. None of c_n should vanish since if $c_n = 0$ then $\phi_n(y) + \phi_n(\overline{y}) = 0$ and we can use the results of first part to prove that $(\phi_n, \phi_n) = 0$. Defining $\tilde{\phi}_n(x) = \phi_n(x) - c_n^*/2$. Then $\tilde{\phi}_n(\overline{x}) = -\tilde{\phi}_n(x)$ and it follows $\dot{\tilde{\phi}}_n(\overline{x}) = \dot{\tilde{\phi}}_n(x)$. Hence

$$\tilde{\phi}_n^*(x)\dot{\tilde{\phi}}_m(x) = -\tilde{\phi}_n^*(\overline{x})\dot{\tilde{\phi}}_m(\overline{x}),$$

$$(\tilde{\phi}_n(x), \tilde{\phi}_m(x)) = 0.$$

Taking $n \neq m$, we have

$$(\phi_n(x), \phi_m(x)) - \frac{1}{2}\left\{(c_n^*, \phi_m) + (\phi_n, c_m^*)\right\} + \frac{1}{4}(c_n^*, c_m^*) = 0.$$

The first and the last term in the above equation are equal to zero. The equation can be conveniently rewritten as

$$\frac{c_n}{2C}(C, \phi_m) + \frac{c_m^*}{2C}(\phi_n, C) = 0,$$
$$\frac{c_n c_m^*}{C} = 0.$$

This need atleast one of c_n to be equal to zero which contradicts our previous result. Hence our assumption $G^{(1)}(x,y) + G^{(1)}(x,\overline{y}) = C$ is wrong. We know

$$G^{(1)}(x,y) = \langle 0|\Phi(x)\Phi(y) + \Phi(y)\Phi(x)|0\rangle,$$

where $|0\rangle$ is defined as the vacuum annihilated by all annihilation operators i.e. $a_n|0\rangle = 0$. Expanding Φ we get

$$G^{(1)}(x,y) = \sum_n \left\{\phi_n(x)\phi_n^*(y) + \phi_n^*(x)\phi_n(y)\right\} = G^{(1)}(Z).$$

From results of (1) and (2) we can infer there exists no Fock vacuum state for which $G^{(1)}(Z) \pm G^{(1)}(-Z) = 0$ everywhere. Result from theorem (3) contradicts (4.17) and hence it can be understood that there exists no Fock vacuum state which is de Sitter invariant in the massless case. The massless case of $m = 0$ having no Fock vacuum has nothing to do with the fact that wave equation has a constant solution, often called "zero mode". It has been argued and showed that a small perturbation to the Euclidean vacuum for $m^2 > 0$ decays exponentially and the properties of the state approach those of Euclidean vacuum. However, it has been shown that this is not the case for $m = 0$ and a small spatially homogeneous perturbation grows linearly at first and then approaches a constant non-zero value causing spontaneous breaking of de Sitter invariance. More discussion on this can be found in [3]. Having understood a good deal of physics about the de Sitter invariant states, we now analyze the canonical quantization in de Sitter space which eventually lead to some of the interesting phenomena like particle production.

Chapter 5

Particle production in de Sitter spacetime

5.1 Canonical quantization in de Sitter Space

For a non-interacting scalar field, the field could be expanded in a Fock-space representation as follows:

$$\hat{\phi} = \sum_{\lambda} \hat{a}_\lambda \phi_{\lambda(+)} + \hat{a}^\dagger_\lambda \phi_{\lambda(-)}, \tag{5.1}$$

where $\phi_{\lambda(+)}$ and $\phi_{\lambda(-)}$ are the positive and negative frequency modes respectively. The vacuum state is then uniquely specified by $\hat{a}_\lambda|0\rangle = 0$. If there were interaction, the positive and negative frequency modes are not uniquely defined, but depends on the choice of time slicing. In case of an adiabatic switching of a background Klein-Gordon field, a preferred time slicing would be defined and the asymptotic non-interacting in and out states are found.

Schwinger proposed an alternative method of defining positive and negative frequency modes separately at $t = -\infty$ and $t = \infty$. This discussion can be found in [2]. Once these in and out states are defined, we could calculate the Bogoliubov mixing coefficients. The positive and negative modes are defined according to whether the inner product

$$(f, g)_\sigma = \int_\sigma \mathrm{d}\sigma \, i f^* (\overrightarrow{\partial}_a - \overleftarrow{\partial}^a) g$$

is positive or negative with σ being a spacelike surface. The corresponding operators $\hat{a}_\lambda, \hat{a}^\dagger_\lambda$ satisfy the commutation relations

$$[\hat{a}_\lambda, a^\dagger_{\lambda'}] = \delta(\lambda - \lambda'). \tag{5.2}$$

$$[\hat{a}_\lambda, \hat{a}_{\lambda'}] = 0 = [\hat{a}^\dagger_\lambda, \hat{a}^\dagger_{\lambda'}]. \tag{5.3}$$

and the $|in\rangle$ vacuum is defined as $\hat{a}_\lambda|in\rangle = 0$. The outgoing modes and the corresponding operators can also be defined in the same way and the modes are $\phi_\lambda^{(\pm)}$ are the ones which can be analytically continued into the lower half m^2 plane and are regular at future infinity. The corresponding modes for the outgoing operators are $\hat{b}_\lambda$,$\hat{b}_\lambda^\dagger$ satisfy the same commutation relations as the operators for incoming modes. The vacuum $|out\rangle$ is defined as one which satisfies $\hat{b}_\lambda|out\rangle = 0$.

Since the wave equation for ϕ is only second order and the incoming and the outgoing modes are just the asymptotic solutions at past and future infinity, there exists a linear transformation between them which is the same as the Bogoliubov transformations discussed earlier. Hence the modes can be related as

$$\phi_{\lambda(+)} = \alpha_\lambda \phi_\lambda^{(+)} + \beta_\lambda \phi_\lambda^{(-)}, \tag{5.4}$$

$$\phi_{\lambda(-)} = \beta_\lambda^* \phi_\lambda^{(+)} + \alpha_\lambda^* \phi_\lambda^{(-)}. \tag{5.5}$$

For a real scalar field $\phi_\lambda^{(+)} = \left(\phi_\lambda^{(-)}\right)^*$ and $\phi_{\lambda(+)} = \left(\phi_{\lambda(-)}\right)^*$ and hence equating the field operator expansion in both the basis, we get the relationship between the annihilation and creation operators as

$$\hat{a}_\lambda = \alpha_\lambda^* \hat{b}_\lambda - \beta_\lambda^* \hat{b}_\lambda^\dagger, \tag{5.6}$$

$$\hat{a}_\lambda^\dagger = \alpha_\lambda \hat{b}_\lambda^\dagger - \beta_\lambda \hat{b}_\lambda. \tag{5.7}$$

The commutation relations between annihilation and creation operators gives the relationship $|\alpha|^2 - |\beta|^2 = 1$.

Let us now try to compute these transformation quantities and corresponding modes in the de Sitter spacetime. We will calculate them in both spatially flat as well as spatially closed coordinates. We shall restrict ourselves to $(1{+}3)$ spacetime dimensions in this chapter. The metric in spatially closed coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + H^{-2}\cosh^2(Ht)(\mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega^2)$$

where $\mathrm{d}\Omega^2$ is the metric corresponding to two sphere. The wave equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu v}\partial_v\phi) + m^2\phi = 0,$$

translates as

$$\left[\frac{1}{\cosh^3(Ht)}\frac{\partial}{\partial t}\left(\cosh^3 Ht\frac{\partial}{\partial t}\right) - \frac{H^2\triangle_3}{\cosh^3(Ht)} + m^2\right]\phi = 0, \tag{5.8}$$

where

$$\triangle_3 = \frac{1}{\sin^2\chi}\left[\frac{\partial}{\partial\chi}\left(\sin^2\chi\frac{\partial}{\partial\chi}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\omega^2}\right].$$

5.2 Green function invariance in de Sitter spacetime

The above wave equation can be cast in terms of the more evident de Sitter invariant form with the equation being dependent only on the de Sitter invariant quantity

$$z = H^2\eta_{ab}X^a(x)Y^b(y).$$

Here X^a and Y^b are the vectors corresponding to the spacetime points x and y.

5.2.1 Spatially closed coordinates

In the case of the closed coordinates in $(1+3)$ dimensions, using the coordinate systems introduced in the second chapter, this becomes

$$\begin{aligned}
X^a(x) &= (H^{-1}\sinh(Ht),\ H^{-1}\cosh(Ht)\cos\chi,\ H^{-1}\cosh(Ht)_1\sin\chi\cos\theta,\\
&\quad H^{-1}\cosh(Ht)\sin\chi\sin\theta\cos\omega,\ H^{-1}\cosh(Ht)\sin\chi\sin\theta\sin\omega),\\
Y^b(y) &= (H^{-1}\sinh(Ht'),\ H^{-1}\cosh(Ht)'\cos\chi',\ H^{-1}\cosh(Ht)'\sin\chi'\cos\theta',\\
&\quad H^{-1}\cosh(Ht)'\sin\chi'\sin\theta'\cos\omega',\ H^{-1}\cosh(Ht)'\sin\chi'\sin\theta'\sin\omega'),\\
z &= -\sinh(Ht)\sinh(Ht') + \cosh(Ht)\cosh(Ht)'\cos\Omega,
\end{aligned}$$

where $\cos\Omega = \cos\chi\cos\chi' + \sin\chi\sin\chi'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\omega-\omega'))$. All the partial derivates involving t, χ, θ, ω can be written in terms of the full derivatives of z. Hence,

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{df}{dz}\frac{\partial z}{\partial t} = f'\left(-\cosh(Ht)\sinh(Ht') + \sinh(Ht)\cosh(Ht)'\cos\Omega\right),\\
\frac{\partial f}{\partial\chi} &= \frac{df}{dz}\frac{\partial z}{\partial\chi} = f'(\cosh(Ht)\cosh(Ht)'(-\sin\chi\cos\chi' + \cos\chi\sin\chi'\cos\beta)),\\
\frac{\partial f}{\partial\theta} &= \frac{df}{dz}\frac{\partial z}{\partial\theta} = f'(\cosh(Ht)\cosh(Ht)'\sin\chi\sin\chi'\left(-\sin\theta\cos\theta' + \cos\theta\sin\theta'\cos\left(\omega-\omega'\right)\right)),\\
\frac{\partial f}{\partial\omega} &= \frac{df}{dz}\frac{\partial z}{\partial\omega} = f'(\cosh(Ht)\cosh(Ht)'\sin\chi\sin\chi'\sin\theta\sin\theta'\left(-\sin(\omega-\omega')\right)),
\end{aligned}$$

where $\cos\beta = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\omega - \omega')$. Also, we have

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\frac{\partial f}{\partial t} = f'\ddot{z} + f''\dot{z}^2,$$

where the over prime denotes the derivative with respect to z and overdot represents the derivative with respect to t. The same double derivative can be extended to the other variables with overdot representing the derivatives with respect to the corresponding coordinates. The equation for the Green function is the wave equation already seen above

$$\left[\frac{1}{\cosh^3(Ht)}\frac{\partial}{\partial t}\left(\cosh^3(Ht)\frac{\partial}{\partial t}\right) - \frac{H^2\triangle_3}{\cosh^3(Ht)} + m^2\right] G(x,y) = 0.$$

De Sitter invariance of the Green function implies that $G(x,y) = G(z(x,y)) = G(z)$ and using the above relations for partial derivatives, the above equation can be written in the form

$$\left((z^2-1)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 4z\frac{\mathrm{d}}{\mathrm{d}z} + m^2\right) G(z) = 0. \tag{5.9}$$

In general for an arbitrary de Sitter space of $1 + d$ dimension the above equation can be generalised to be

$$\left((z^2-1)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (d+1)z\frac{\mathrm{d}}{\mathrm{d}z} + m^2\right) G(z) = 0. \tag{5.10}$$

5.2.2 Spatially flat coordinates

It is to be noted that the above calculation was performed in spatially closed coordinates of de Sitter spacetime. An interesting thing would be to perform the same calculation in other coordinates and interpret the result. As it turns out, this is equally true in case of spatially flat de Sitter space. For de Sitter spacetime of $(1+1)$ dimensions

$$X^a(x) = \left(H^{-1}\left(\sinh(Ht) + H^2\frac{x^2e^{Ht}}{2}\right),\, xe^{Ht},\, H^{-1}\left(-\cosh(Ht) + H^2\frac{x^2e^{Ht}}{2}\right)\right)$$

$$Y^b(y) = \left(H^{-1}\left(\sinh(Ht) + H^2\frac{x^2e^{Ht}}{2}\right),\, xe^{Ht},\, H^{-1}\left(-\cosh(Ht) + H^2\frac{x^2e^{Ht}}{2}\right)\right)$$

Hence $z = H^{-2}\cosh H(t-t') - \frac{1}{2}e^{H(t+t')}(x-x')^2$. The wave equation in the flat coordinates is

$$\left[\frac{\partial^2}{\partial t^2} + 2H\frac{\partial}{\partial t} - e^{-2tH}\frac{\partial^2}{\partial x^2}\right]\phi = 0.$$

In general for an arbitrary dimensional de Sitter spacetime of $(1+d)$ dimensions, we would have

$$\left[\frac{\partial^2}{\partial t^2} + 2H\frac{\partial}{\partial t} - e^{-2tH}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}\right]\phi = 0. \tag{5.11}$$

5.3 Particle production

Now, let us try to solve the wave equation and find the transformation coefficients between the modes at early and late times. Solving the wave equation in spherical harmonics basis $\phi(t,\chi,\theta,\omega) = y_k(t)Y_{klm}(\chi,\theta,\omega)$ we get

$$\triangle_3 Y_{klm} = -k(k+2)Y_{klm},$$
$$\left[\frac{1}{\cosh^3(Ht)}\frac{\partial}{\partial t}\left(\cosh^3(Ht)\frac{\partial}{\partial t}\right) + \frac{H^2k(k+2)}{\cosh^3(Ht)} + m^2\right]y_k(t) = 0$$

We define the quantity

$$\gamma = \left(\frac{m^2}{H^2} - \frac{9}{4}\right)^{\frac{1}{2}},$$

the solutions for $y_k(t)$ can be written as

$$y_k(t) = c_1(\tanh^2(Ht)-1)^{\frac{3}{4}}P^{i\gamma}_{(k+\frac{1}{2})}(\tanh(Ht)) + c_2(\tanh^2(Ht)-1)^{\frac{3}{4}}Q^{i\gamma}_{(k+\frac{1}{2})}(\tanh(Ht)),$$

where P and Q are the associated Legendre polynomials. Rewriting the associated Legendre polynomials in terms of hypergeometric functions using the relations

$$P^\mu_\lambda(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+z}{1-z}\right)^{\mu/2} {}_2F_1\left(-\lambda,\lambda+1;1-\mu;\frac{1-z}{2}\right), \tag{5.12}$$

$$Q^\mu_\lambda(z) = \frac{\sqrt{\pi}\Gamma(\lambda+\mu+1)}{2^{\lambda+1}\Gamma(\lambda+\frac{3}{2})}\frac{1}{z^{\lambda+\mu+1}}(1-z^2)^{\mu/2}{}_2F_1\left(\frac{\lambda+\mu+1}{2},\frac{\lambda+\mu+2}{2};\lambda+\frac{3}{2};\frac{1}{z^2}\right), \tag{5.13}$$

and using the relations

$$\begin{aligned}
{}_2F_1(\alpha,\beta;\gamma;z) &= (1-z)^{-\alpha}{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right),\\
{}_2F_1(\alpha,\beta;\gamma;z) &= \frac{(1-z)^{-\alpha}\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}{}_2F_1\left(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}\right)\\
&\quad+\frac{(1-z)^{-\beta}\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}{}_2F_1\left(\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}\right),
\end{aligned}$$

we can rewrite the solutions for $y_k(t)$ asymptotically as

$$y_k^{(\pm)}(t) \sim_{t\to-\infty} \cosh^k(Ht)\exp\left[(-k-\frac{3}{2}\mp i\gamma)Ht\right] {}_2F_1\left(k+\frac{3}{2}, k+\frac{3}{2}\pm i\gamma; 1\pm i\gamma; -e^{-2Ht}\right), \tag{5.14}$$

$$y_{k(\pm)}(t) \sim_{t\to+\infty} \cosh^k(Ht)\exp\left[(-k-\frac{3}{2}\mp i\gamma)Ht\right] {}_2F_1\left(k+\frac{3}{2}, k+\frac{3}{2}\mp i\gamma; 1\mp i\gamma; -e^{2Ht}\right). \tag{5.15}$$

Refer to [9] for the asymptotic forms of hypergeometric functions and various other transformations among them. From the above form of the solutions it can be noted that $y_k^{(-)}(t) = [y_k^{(+)}(t)]^*$, $y_{k(-)}(t) = [y_{k(+)}(t)]^*$ and $y_{k(\pm)}(t) = y_k^{(\pm)}(-t)$. Using the transformation laws for hypergeometric functions

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma;z) = {} & \frac{(-z)^{-\alpha}\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} {}_2F_1\left(\alpha, \alpha+1-\gamma; \alpha-\beta+1; \frac{1}{z}\right) \\ & + \frac{(-z)^{-\beta}\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} {}_2F_1\left(\beta, \beta+1-\gamma; \beta-\alpha+1; \frac{1}{z}\right),\end{aligned}$$

the transformation coefficients can be found to be

$$\alpha_k = \frac{\Gamma(1-i\gamma)\Gamma(-i\gamma)}{\Gamma(k+\frac{3}{2}-i\gamma)\Gamma(-k-\frac{1}{2}-i\gamma)}, \tag{5.16}$$

$$\beta_k = \frac{\Gamma(1-i\gamma)\Gamma(i\gamma)}{\Gamma(k+\frac{3}{2})\Gamma(-k-\frac{1}{2})} = \frac{i(-1)^k}{\sinh\pi\gamma} \tag{5.17}$$

These transformation coefficients satisfy the relation $|\alpha|^2 - |\beta|^2 = 1$ and hence can be written parametrically as $\alpha_k = e^{-2i\delta_k}\cosh 2\overline{\theta}$, $\beta_k = i(-1)^k \sinh 2\overline{\theta}$ and $\sinh 2\overline{\theta} = \mathrm{cosech}\pi\gamma$.

These results can also be derived for a spatially flat metric. The spatially flat metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\mathrm{d}x^2.$$

Let $\psi = \psi_k(t)e^{-ik\cdot x}$ i.e the temporal and the spatial parts of the solution are separated. The wave equation for $\psi_k(t)$ for this metric can be written by comparing the metric $g_{\mu v}$ from the above line element and using (4.2) is given by

$$\ddot{\psi}_k + H\dot{\psi}_k + (m^2 + k^2\exp(-2Ht))\psi_k = 0. \tag{5.18}$$

This form of the wave equation in terms of the cosmological time t is useful in finding the modes at future infinity. Recasting the equation in terms of $\phi_k(t) = \exp(-Ht/2)\psi_k(t)$ we obtain that

$$\ddot{\phi}_k + \left(m^2 - \frac{H^2}{4} + \frac{k^2}{a^2}\right)\phi_k = 0.$$

Considering the limit $m \gg H$, t_k is defined as the time when physical wavelength $a(t)k^{-1}$ is equal to the Compton wavelength m, i.e $ke^{-Ht_k} = m$. For $t \gg t_k$ the equation will have approximate solutions of the form

$$\psi_k = a^{-1/2}(c_k e^{-i\omega t} + d_k e^{i\omega t})$$

where $\omega = (m^2 - H^2/4)^{1/2}$. These are the solutions in de Sitter space of $(1+1)$ dimensions. This can be generalised to $(1+d)$ dimensions by replacing with

$$\omega = \left(M^2 - \frac{(d-1)^2}{4}H^2\right).$$

Hence, the outgoing modes are $\psi^{(\pm)}_{out,k} \propto a^{-1/2}e^{\mp i\omega t}$.

In terms of the conformal time $\eta = -H^{-1}e^{-Ht}$ and with $\psi = \psi_k(\eta)e^{ikx}$ the wave equation can be written as

$$\frac{\partial^2\psi_k}{\partial\eta^2} + \left(k^2 + \frac{m^2}{H^2\eta^2}\right)\psi_k = 0$$

The solutions for this equation are given by the general solution

$$\psi_k(\eta) = \left(\frac{\eta}{8}\right)^{1/2}[A_k H^{(2)}_\nu(k\eta) + B_k H^{(1)}_\nu(k\eta)] \tag{5.19}$$

where $\nu = ((d-1)^2/4 - m^2/H^2)$ for a d dimensional de Sitter space and $H^{(1)}_\nu$, $H^{(2)}_\nu$ are the Hankel functions of the first and second kind. We take the "in" vacuum as the Bunch-Davies vacuum which is given by $A_k = 1$ and $B_k = 0$ and hence

$$\psi_{in,k} = \left(\frac{\eta}{8}\right)^{1/2} H^{(2)}_\nu(k\eta).$$

This vacuum state is same as the one used in [5]. Using the asymptotic expression for Hankel function, we have

$$\psi^{(+)}_{in,k}(k\eta) = -\left(\frac{\eta}{8}\right)^{1/2}\frac{i}{\nu\pi}\left[\left(\frac{|k\eta|}{2}\right)^\nu \Gamma(1-\nu)e^{-\frac{i\pi\nu}{2}} - \left(\frac{|k\eta|}{2}\right)^\nu \Gamma(1+\nu)e^{\frac{i\pi\nu}{2}}\right]. \tag{5.20}$$

The "in" mode be expressed as the linear combination of "out" modes as given by

$$\psi^{(+)}_{in,k} = \alpha_k\psi^{(+)}_{out,k} + \beta_k\psi^{(-)}_{out,k}.$$

Using the relations for Gamma function $\Gamma(1+z)\Gamma(1-z) = \pi z/\sin(\pi z)$ we can find the transformation coefficients as

$$|\beta|^2 = \left(\exp(2\pi\omega H^{-1}) - 1\right)^{-1}, \tag{5.21}$$

$$|\alpha|^2 = \exp(2\pi\omega H^{-1})/\left(\exp(2\pi\omega H^{-1}) - 1\right). \tag{5.22}$$

If the decomposition of the solution space into positive and negative subspaces at $t = \infty$ and $t = -\infty$ is inequivalent, i.e if the linear transformation relating the two modes has off-diagonal elements, then particle creation occurs. We can now calculate the creation probabilities and decay rates as we have the transformation coefficients between the "in" and the "out" states. The relative amplitude for creation of a pair of particles in the final states (klm) and $(kl-m)$ if none were present in the initial state is

$$\begin{aligned} p &= \frac{\langle out|b_{klm}b_{kl-m}|in\rangle}{\langle out|in\rangle}, \\ &= \frac{1}{\alpha_k^*}\frac{\langle out|b_{klm}(a_{kl-m} + \beta_k^* b_{kl-m}^\dagger)|in\rangle}{\langle out|in\rangle}, \\ &= \frac{\beta_k^*}{\alpha_k^*}\frac{\langle out|\delta_{m,m} + b_{kl-m}^\dagger b_{klm}|in\rangle}{\langle out|in\rangle}, \\ &= \frac{\beta_k^*}{\alpha_k^*}. \end{aligned}$$

The square of this amplitude is $w_{klm} = |\beta_k/\alpha_k|^2$. This gives the relative amplitude of creating a pair in the given mode. Let the absolute probability of $\langle out|in\rangle$ in a given mode be denoted by N_{klm}. Then the absolute probabilities are obtained by imposing the condition that the sum of all the probabilities must equal one i.e the total sum of probabilities of creating $n, n\forall Z$ pairs be unity. Mathematically, we have

$$N_{klm}(1 + w_{klm} + w_{klm}^2 + \ldots) = 1,$$

$$\begin{aligned} N_{klm} &= 1 - w_{klm}, \\ &= 1 - |\beta_k/\alpha_k|^2. \end{aligned}$$

Let us work in the closed coordinates, which gives $N_{klm} = 1 - \mathrm{sech}^2\pi\gamma$. The term $1 - w_{klm}$ is also the probability of creating no particles in a given mode. Hence the probability of creating no particles in any mode is just the product of the probabilities of no particle production in all the available modes

$$\begin{aligned}|\langle out|in\rangle|^2 &= \Pi_{klm}N_{klm}\\ &= \exp\left(\sum_{klm}\ln(\tanh^2\pi\gamma)\right)\end{aligned}$$

However, since the summation in the exponential is independent of k, it is divergent and hence a cutoff has to be imposed. Let the sum be cutoff at $k = N$. We will consider a differential change in the sum in the exponential.

$$\begin{aligned}\triangle\sum_{k=0}^{N}\sum_{l=0}^{k}\sum_{m=-l}^{l}1 &\sim_{N\to\infty} N^2\triangle N,\\ &\sim e^{3\ln N\frac{\triangle N}{N}}.\end{aligned}$$

On the other hand, the decay rate is given by the expression

$$|\langle out|in\rangle|^2 \to \exp(-\Gamma V_4),$$

$$\Gamma = -\lim_{v_4\to\infty}\frac{1}{V_4}\ln|\langle out|in\rangle|^2 \tag{5.23}$$

To equate the exponentials from the two expressions, we need to consider the differential change in the four volume element $\triangle V_4$. The two changes can be equated by calculating the term $\triangle N$ and $\triangle V$ using the following considerations. The physical momentum of the state with quantum number N is given by

$$k_{phys} \to \frac{N}{\cosh(Ht)} \quad \text{as N} \to \infty.$$

For a fixed k_{phys}, as N and t becomes very large, we can write

$$\frac{\triangle N}{N} = \frac{\triangle(\cosh(Ht))}{\cosh(Ht)} \to H\triangle t.$$

This can be also seen as $\ln N \to Ht$. The three volume corresponding to the spatial part is

$$\begin{aligned}V_3 &= \int\frac{\cosh^3(Ht)}{H^3}\sin^2\chi\,\sin\theta\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\phi,\\ &= 2\pi^2\frac{\cosh^3(Ht)}{H^3}.\end{aligned}$$

Hence the differential four volume for some time slicing would be $\triangle V_4 = V_3 \triangle t$. Equating both, we get

$$-\Gamma \triangle V_4 = e^{3 \ln N} \ln(\tanh^2 \pi\gamma) \frac{\triangle N}{N},$$

$$\Gamma = \frac{8H^2}{\pi^2} \ln(\coth \pi\gamma). \tag{5.24}$$

We can carry out the same calculation in the spatially flat coordinates of de Sitter spacetime of $(1+3)$ dimensions. We have already calculated the transformation coefficients. Considering the limit of $m \gg H$ we get the result of decay rate similar to (5.24). The above few results are quite remarkable which gives a quantitative understanding of an important phenomenon characteristic of a curved spacetime.

Chapter 6

Conclusion

This brings us to the end of this report. The introduction of cosmological constant by Einstein led to some of the very important advancements in cosmology. The cosmological constant is a strong contender for the explanation of dark energy that accounts for most of the energy density of the universe today. This study of de Sitter spacetime with positive cosmological constant brings to the fore some of the important theoretical observations. Hence, de Sitter spacetime has been a good candidate for pedagogical study.

To summarise, we have understood the important aspects of de Sitter spacetime and employed various coordinate systems for the same. As can be seen from the results, global coordinates is a special case of planar coordinates which provides a good system to understand the expansion of the universe quantitatively. Each of the coordinate system has its own importance in presenting the important features of de Sitter spacetime. One of the most remarkable result was the expansion of universe whose dynamics are dominated by the cosmological term. A possible theoretic understanding of the structure of the universe which is observable today is based on de Sitter geometry. In the process, we have understood an important mathematical tool used for describing the causal structures of spacetimes. Although, only the case of spatially flat sections of de Sitter spacetime is presented in the section of Penrose diagrams, it would be instructive to study them in spatially closed as well as spatially open planar coordinates and understand the horizons.

Continuing further, we studied the theoretic framework of quantum field theory in flat spacetime. We explored canonical quantization in different basis and various Green functions. This formed the basis for the review of quantum field theory in a curved spacetime. We analyzed the symmetry properties of de Sitter spacetime in detail and arrived at an important result of non existence of de Sitter invariant vacuum for a massless scalar field. For more elaborate discussion on de Sitter invariant states, the reader can refer to [3].

In the final chapter, we explored couple of interesting aspects which are Green function invariance and particle production of a massive scalar field in de Sitter spacetime. The Green function invariance is proved in spatially closed as well as spatially flat coordinates systems. This suggests that the amplitude of propagation between any two spacetime points only depends on the de Sitter invariant distance between them. Another important phenomenon characteristic of a curved spacetime is particle production. We have derived the probability amplitude for producing particles in any state and also the decay rate. An interested reader can explore more about the recent developments and about various interesting phenomenon characteristic of a curved spacetime.

Appendix A

A

A.1 Global coordinates

The line element is

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + l^2 f\left(\frac{\tau}{l}\right)\mathrm{d}\Omega^2_{d-1}.$$

The covariant components of the metric above line element are

$$g_{\tau\tau} = -1,\;\; g_{ii} = l^2 f^2 \prod_{j=1}^{i-1} \sin^2\theta_j.$$

Its inverse $g^{\mu v}$ has the components

$$g^{\tau\tau} = -1,\;\;\; g^{ii} = \frac{1}{l^2 f^2 \prod_{j=1}^{i-1} \sin^2\theta_j}.$$

Calculating the Christoffel symbols from the above metric, we get

$$\Gamma^{\tau}_{ii} = \frac{1}{2} g^{\tau d}(-g_{ii,d}) = l^2 f\, \dot{f} \prod_{j=1}^{i-1} \sin^2\theta_j.$$

$$\Gamma^{i}_{\tau i} = \frac{1}{2} g^{id}(g_{di,\tau}) = \frac{\dot{f}}{f},\; \Gamma^{i}_{ij} = \frac{1}{2} g^{id}(g_{di,j}) = \frac{\cos\theta_j}{\sin\theta_j}.$$

$$\Gamma^{i}_{jj} = \frac{1}{2} g^{id}(-g_{jj,d}) = -\sin\theta_i \cos\theta_i \prod_{k=i+1}^{j-1} \sin^2\theta_k.$$

From these non-zero components of Christoffel symbols, non-zero components of Riemann curvature tensor can be evaluated

$$
\begin{aligned}
R^{\tau}_{i\tau i} &= \Gamma^{\tau}_{ii,\tau} - \Gamma^{\tau}_{id}\Gamma^{d}_{\tau i} = l^2 f \partial^2_{\tau} f \prod_{j=1}^{i-1} \sin^2\theta_j, \\
R^{i}_{\tau i\tau} &= -\Gamma^{i}_{\tau i,\tau} - \Gamma^{i}_{\tau d}\Gamma^{d}_{i\tau} = -\frac{\partial^2_{\tau} f}{f}, \\
R^{i}_{jij} &= \left(1 + l^2 \left(\frac{\partial f}{\partial \tau}\right)^2\right) \Pi_1^{i-1} \sin^2\theta_k, \\
R_{\tau\tau} &= -(d-1)\partial^2_{\tau} f, \\
R_{ii} &= \{l^2 f \partial^2_{\tau} f + (d-2)[1 + l^2(\partial_{\tau} f)^2]\} \prod_{k=1}^{i-1} \sin^2\theta_k, \\
R &= g^{\mu v} R_{\mu v}, \\
&= (d-1)\frac{(d-2)(1+\dot{f}^2) + 2f\ddot{f}}{l^2 f^2}.
\end{aligned}
$$

A.2 Conformal coordinates

The line element in the conformal coordinates is given by

$$
\mathrm{d}s^2 = F^2\left(\frac{T}{l}\right)(-\mathrm{d}T^2 + l^2 \mathrm{d}\Omega^2_{d-1}).
$$

The metric components for this line element are

$$
\begin{aligned}
g_{TT} &= -F^2 \text{ and } g_{ii} = l^2 F^2 \prod_{j=1}^{i-1} \sin^2\theta_j, \\
g^{TT} &= -F^{-2} \text{ and } g^{ii} = l^{-2} F^{-2} \prod_{j=1}^{i-1} \sin^{-2}\theta_j, \\
g &= -\frac{(lF)^{2d}}{l^2} \prod_{j=1}^{d-1} \prod_{k=1}^{i-1} \sin^2\theta_k.
\end{aligned}
$$

As was done in the case of global coordinate system assuming metric being dependent on the function $f(\tau/l)$, the non-vanishing components of the Christoffel symbols are

$$
\begin{aligned}
\Gamma^{\tau}_{\tau\tau} &= \frac{1}{2}g^{TT}g_{TT,T} = \frac{\partial_T F}{F},\\
\Gamma^{i}_{Ti} &= \frac{1}{2}g^{ii}g_{ii,T} = \frac{\partial_T F}{F},\\
\Gamma^{\tau}_{ii} &= \frac{1}{2}g^{\tau d}(-g_{ii,d}) = l^2 f\,\dot{f}\prod_{j=1}^{i-1}\sin^2\theta_j,\\
\Gamma^{i}_{jj} &= \frac{1}{2}g^{id}(-g_{jj,d}) = -\sin\theta_i\cos\theta_i\prod_{k=i+1}^{j-1}\sin^2\theta_k,
\end{aligned}
$$

$$
\Gamma^{i}_{ij} = \frac{1}{2}g^{id}(g_{di,j}) = \frac{\cos\theta_j}{\sin\theta_j}.
$$

Again, performing the same calculations as was done in the case of global coordinates case will yield

$$
\begin{aligned}
R^{T}_{iTi} &= \Gamma^{T}_{ii,T} + \Gamma^{T}_{Td}\Gamma^{d}_{ii} - \Gamma^{T}_{id}\Gamma^{d}_{Ti},\\
&= \frac{1}{F^2}\{F\partial_T^2 F - (\partial_T F)^2\}\prod_{j=1}^{i-1}\sin^2\theta_j,\\
R^{i}_{TiT} &= -\Gamma^{i}_{Ti,T} - \Gamma^{i}_{Td}\Gamma^{d}_{iT} + \Gamma^{i}_{id}\Gamma^{d}_{iT},\\
&= \frac{-1}{F^2}[F\partial_T^2 F - (\partial_T F)^2]\\
R^{i}_{jij} &= \frac{1}{F^2}\left[F^2 + \left(\frac{\partial F}{\partial T}\right)^2\right]\Pi_1^{i-1}\sin^2\theta_j.\\
R_{TT} &= -\frac{(d-1)}{F^2}[F\partial_T^2 F - (\partial_T F)^2],\\
R_{ii} &= \frac{1}{l^2F^2}[F\partial_T F + (d-2)F^2 + (d-3)(\partial_T F)^2]\prod_{k=1}^{i-1}\sin^2\theta_k,\\
R &= (d-1)\frac{(d-4)\dot{F}^2 + (d-2)F^2 + 2F\ddot{F}}{l^2F^4}.
\end{aligned}
$$

A.3 Planar coordinates

The non-vanishing components of the d dimensional Riemann curvature $R_{\mu\upsilon\rho\sigma}$ can be expressed by the $(d-1)$ dimensional metric γ_{ij} as below. We proceed by calculating the non-vanishing components of the Christoffel symbols followed by Riemann curvature tensor and finally arrive at the Ricci scalar. The metric for the line element written in planar coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t/l)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

The metric becomes

$$\begin{aligned} g_{tt} &= -1 \ \ and \ \ g_{ij} = a^2\gamma_{ij}, \\ g^{tt} &= -1 \ \ and \ \ g^{ij} = a^{-2}\gamma^{ij}, \\ g &= -a^{2(d-1)}\gamma. \end{aligned}$$

Now, we evaluate the Christoffel symbols as below

$$\begin{aligned} \Gamma^t_{ij} &= \frac{1}{2}g^{tt}(-g_{ij,t}) = a\dot{a}\gamma_{ij}, \\ \Gamma^i_{tj} &= \frac{1}{2}g^{id}g_{dj,t} = \frac{\dot{a}}{a}\delta^i_j, \\ \Gamma^i_{jk} &= \frac{1}{2}g^{id}(g_{dj,k} + g_{dk,j} - g_{jk,d}) = \frac{\gamma^{id}}{2}(\gamma_{dk,j} + \gamma_{dj,k} - \gamma_{jk,d}). \end{aligned}$$

The components of Ricci tensor are

$$\begin{aligned} R^t_{itj} &= \Gamma^t_{ij,t} - \Gamma^t_{ik}\Gamma^k_{it} = a\ddot{a}\gamma_{ij}, \\ R^i_{tjt} &= -\Gamma^i_{tj,t} - \Gamma^i_{tk}\Gamma^k_{it} = -\frac{\ddot{a}}{a}\delta^i_j, \\ R^i_{jkl} &= \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{kd}\Gamma^d_{jl} - \Gamma^i_{ld}\Gamma^d_{jk}. \end{aligned}$$

The d index in the calculation of R^i_{jkl} is the summation over all the coordinates i.e t, all $\theta's$. Just separating the t component in the summation allows us to write

$$\begin{aligned} R^i_{jkl} &= \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{k\theta}\Gamma^\theta_{jl} - \Gamma^i_{l\theta}\Gamma^\theta_{jk} + \Gamma^i_{kt}\Gamma^t_{jl} - \Gamma^i_{lt}\Gamma^t_{jk}, \\ &=^{d-1} R^i_{jkl} + \frac{\dot{a}}{a}\delta^i_k a\dot{a}\gamma_{jl} - \frac{\dot{a}}{a}\delta^i_l a\dot{a}\gamma_{jk}, \\ &= (\dot{a}^2 + k)(\delta^i_k\gamma_{jl} - \delta^i_l\gamma_{jk}). \end{aligned}$$

Upon contracting the two indices in the above non-vanishing components of $R^\mu_{v\rho\sigma}$ we get

$$\begin{aligned} R_{tt} &= -\frac{(d-1)}{a}\ddot{a}, \\ R_{ij} &= [a\ddot{a} + (d-2)\dot{a}^2 + (d-2)k]\gamma_{ij}, \\ R &= (d-1)\frac{2a\ddot{a} + (d-2)(\dot{a}^2 + k)}{a^2}. \end{aligned}$$

A.4 Static coordinates

The line element in static coordinates is

$$\mathrm{d}s^2 = -e^{2\Omega(r)}A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{A(r)} + r^2\mathrm{d}\Omega_{d-2}^2.$$

The metric components are

$$g_{tt} = -A(r)e^{2\Omega(r)}, \quad g_{rr} = \frac{1}{A(r)}, \quad g_{\theta_a\theta_a} = r^2\prod_{b=1}^{a-1}\sin^2\theta_b,$$
$$g^{tt} = -\frac{1}{A(r)}e^{-2\Omega(r)}, \; g^{rr} = A(r), \quad g^{\theta_a\theta_a} = 1/\left(r^2\prod_{b=1}^{a-1}\sin^2\theta_b\right),$$
$$g = -e^{2\Omega(r)}r^{2(d-2)}\prod_{b=1}^{d-2}\prod_{a=1}^{b-1}\sin^2\theta_a.$$

The non-vanishing christoffel symbols that follow from above are

$$\Gamma^t_{rt} = \frac{1}{2}g^{tt}g_{tt,r} = \frac{1}{2A}\left(\frac{dA}{dr} + 2A\frac{d\Omega}{dr}\right),$$
$$\Gamma^r_{tt} = -\frac{1}{2}g^{rr}(-g_{tt,r}) = \frac{1}{2}A\left(\frac{dA}{dr} + 2A\frac{d\Omega}{dr}\right),$$
$$\Gamma^r_{rr} = \frac{1}{2}g^{rr}g_{rr,r} = -\frac{1}{2A}\left(\frac{dA}{dr}\right),$$
$$\Gamma^{\theta_a}_{r\theta_a} = \frac{1}{2}g^{\theta_a\theta_a}(-g_{\theta_a\theta_a,r}) = \frac{1}{r},$$
$$\Gamma^r_{\theta_a\theta_a} = -\frac{1}{2}g^{rr}g_{\theta_a\theta_a,r} = -rA\prod_{b=1}^{a-1}\sin^2\theta_b,$$
$$\Gamma^{\theta_a}_{\theta_b\theta_a} = \frac{1}{2}g^{\theta_a\theta_a}g_{\theta_a\theta_a,\theta_b} = \frac{\cos\theta_b}{\sin\theta_b},$$
$$\Gamma^{\theta_a}_{\theta_b\theta_b} = -\frac{1}{2}g^{\theta_a\theta_a}g_{\theta_b\theta_b,\theta_a} = -\sin\theta_a\cos\theta_a\prod_{k=a+1}^{b-1}\sin^2\theta_k.$$

It is to be noted that the partial derivatives of A and Ω with respect to r are the same as the full derivatives as both are just functions of only one variable r. Non-vanishing components of Riemann tensor R^i_{jkl} are

$$\begin{aligned}
R^r_{trt} &= \Gamma^r_{tt,r} + \Gamma^r_{rk}\Gamma^k_{tt} - \Gamma^r_{tk}\Gamma^k_{tr}, \\
&= Ae^{2\Omega}\left(\partial_r^2 A + 2A\partial_r^2\Omega + 3\partial_r A\partial_r\Omega + 2A(\partial_r\Omega)^2\right), \\
R^t_{rtr} &= -\Gamma^t_{rt,r} + \Gamma^t_{tk}\Gamma^k_{rr} - \Gamma^t_{rk}\Gamma^k_{tr}, \\
&= \frac{1}{2A}\left[3\partial_r A\partial_r\Omega + 2A(\partial_r\Omega)^2 + \partial_r^2 A + 2A\partial_r^2\Omega\right], \\
R^t_{\theta_a t\theta_a} &= \Gamma^t_{tk}\Gamma^k_{\theta_a\theta_a} = -\frac{r}{2}\left[\partial_r A + 2A\partial_r\Omega\right]\prod_{b=1}^{a-1}\sin^2\theta_b, \\
R^{\theta_a}_{t\theta_a t} &= \frac{Ae^{2\Omega}}{2r}\left[\frac{\partial A}{\partial r} + 2A\left(\frac{\partial\Omega}{\partial r}\right)\right], \\
R^r_{\theta_a r\theta_a} &= -\frac{r}{2}\left(\frac{\partial A}{\partial r}\right)\prod_{1}^{a-1}\sin^2\theta_k, \\
R^{\theta_a}_{r\theta_a r} &= -\frac{1}{2rA}\left(\frac{\partial A}{\partial r}\right), \\
R^{\theta_b}_{\theta_a\theta_b\theta_a} &= (1-A)\prod_{1}^{a-1}\sin^2\theta_a.
\end{aligned}$$

From these, the nonvanishing components of Ricci tensor are

$$\begin{aligned}
R_{tt} &= \frac{Ae^{2\Omega}}{r}\left\{(d-2)\left[\frac{\partial A}{\partial r} + 2A\left(\frac{\partial A}{\partial r}\right)\right]\right\} \\
&+ \frac{Ae^{2\Omega}}{r}\left\{r\left[3\left(\frac{\partial A}{\partial r}\right)\left(\frac{\partial\Omega}{\partial r}\right) + 2A\left(\frac{\partial\Omega}{\partial r}\right)^2 + \frac{\partial^2 A}{\partial r} + 2A\left(\frac{\partial^2\Omega}{\partial r^2}\right)\right]\right\}, \\
R_{rr} &= -\frac{1}{2Ar}\left\{(d-2)\left(\frac{\partial A}{\partial r}\right) + r\left[3\left(\frac{\partial A}{\partial r}\right)\left(\frac{\partial\Omega}{\partial r}\right) + 2A\left(\frac{\partial\Omega}{\partial r}\right)^2 + \frac{\partial^2 A}{\partial r^2} + 2A\left(\frac{\partial^2\Omega}{\partial r^2}\right)\right]\right\}, \\
R_{\theta_a\theta_a} &= \frac{r^2}{d-2}\left\{\frac{d-2}{r^{d-2}}\frac{\partial}{\partial r}\left[r^{d-3}(1-A)\right] - A\left(\frac{d-2}{r}\right)\frac{\partial\Omega}{\partial r}\right\}\prod_{1}^{a-1}\sin^2\theta_b.
\end{aligned}$$

Contracting further, we arrive at the Ricci scalar which is given in section 2.2.4.

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