SELF-FORCE IN CURVED SPACETIME

Project work
submitted in partial fulfillment of the requirements
for the award of the degree of
Bachelor of Technology
in
Electrical Engineering
by

Pranay Gorantla
under the guidance of
Dr. Dawood Kothawala

Department of Physics
Indian Institute of Technology Madras
Chennai 600036, India

May 16, 2015

CERTIFICATE

This is to certify that the project titled **SELF-FORCE IN CURVED SPACETIME** is a bona fide record of work done by **Pranay Gorantla** towards the partial fulfillment of the requirements of the Bachelor of Technology degree in Electrical Engineering at the Indian Institute of Technology, Madras, Chennai 600036, India.

(Dawood Kothawala, Project supervisor)

(Deepa Venkitesh, Project co-supervisor)

ACKNOWLEDGEMENTS

I would take this opportunity to thank IIT Madras for providing us with this great opportunity. I am grateful to Dr. Dawood Kothawala for his constant guidance and motivation. Without his inspiration, this work would not have been possible. I am thankful to my faculty advisor and co-supervisor, Dr. Deepa Venkitesh, who has always supported me in making my decisions throughout my academic career at IIT Madras. I would also like to thank S. Sunil Kumar and Santhoshini V. for countless valuable discussions and helping me in preparing this report. I am grateful to my parents for always encouraging me and to my friends whose support has provided me with an enjoyable environment to work in.

ABSTRACT

Self-force has been a widely studied yet not fully understood phenomenon in Physics. From the first example of self-force, that is the motion of a point electric charge in its own electromagnetic field, it has been a mostly unexplained and controversial subject. In this project, we study the notion of self-force of a point electric charge in flat spacetime and arrive at *Lorentz-Dirac equation*. We go on to study some controversies and their resolutions in the case of uniformly accelerating point electric charge. We then develop new mathematical tools called *bi-tensors* which are useful in extending the notion of self-force to curved spacetime. Finally, we discuss the motion of a point scalar charge in the presence of its own field, and we make a note on the presence of *tail term* which does not appear in flat spacetime and is entirely a manifestation of curvature of spacetime.

Contents

Chapter 1

Introduction to self-force

1.1 Introduction to self-force - various theories

The self-force of a charged particle has been a topic of great debate in the theoretical physics community. On one hand, it leads to infinite self-energies for point particles. On the other hand, it is necessary to incorporate it to explain various phenomenon observed in nature such as radiation reaction. Here we give a small introduction to the theory of self-force and discuss two historical theories which have been suggested to explain this phenomenon without facing difficulties.

The self-force of a charged particle, which is defined as the force experienced by a charged particle due to its own electromagnetic field, is given by

$$F_{ret} = \alpha . \frac{e^2}{ac^2}\ddot{x} - \frac{2}{3}\frac{e^2}{c^3}\dddot{x} + \gamma . \frac{e^2 a}{c^4}\ddddot{x} + \dots \tag{1.1.1}$$

Here, a is the radius of the particle and α and γ are some constants which depend on the distribution of charge assumed for the particle - for example, spherically distributed charge with uniform charge density. Notice that the second term doesn't depend on the charge distribution. For a point particle, when we take the limit $a \to 0$, we immediately see that the first term goes to infinity, which is the main problem here, the *infinite self-energy*. The second term is the experimentally observed *radiation reaction* and remains the same for any charge distribution assumed as an approximation for the particle. This term is very much needed and can not be neglected by any theory where as the first term is the problematic one. All other terms go to zero and cause no problem. Hence, we need a theory which gets rid of the first term while retaining the second one. In this introduction, we discuss two such theories which try to achieve this. Most of the results in this section have been referred from [1].

1.1.1 Absorber Theory

Dirac had an ingenious idea to get the required force with minimum changes except for an arbitrary assumption. He considered the advanced version of the self-force,

$$F_{adv} = \alpha . \frac{e^2}{ac^2}\ddot{x} + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} + \gamma . \frac{e^2 a}{c^4}\ddddot{x} + \dots \tag{1.1.2}$$

Notice the sign change in the second term. Now he considered half the difference of F_{ret} and F_{adv}. That is

$$F = \frac{1}{2} \times (F_{ret} - F_{adv}) = -\frac{2}{3}\frac{e^2}{c^3}\dddot{x} + \ldots \tag{1.1.3}$$

We can immediately observe the simplicity of the idea. Dirac proposed that the point particles, here electrons, not only interact with the retarded potentials but also with their advanced counterparts, and he assumed that the interaction is half the difference between retarded and advanced potentials. Though this yields the result we need, it is based on an arbitrary assumption without any physical basis. But Feynman and Wheeler explained the physical nature of this idea in their *absorber theory.* Their idea is that any charged particle does not interact with itself electromagnetically, it can only interact with other charged particles with both advanced and retarded fields. Let an electron generate a field at time t which reaches another particle at a distance r at time $t' = t + \frac{r}{c}$. This particle, apart from reflecting the wave back which is the usual retarded wave, sends an advanced wave which reaches the electron at the time $t'' = t' - \frac{r}{c} = t$. Notice the minus sign because the advanced wave travels into the past. Hence, this combination of advanced and retarded waves results in the radiation reaction.

1.1.2 Bopp's non-linear theory

It has been known for a long time that the root of the problem of self-force is in the factor $\frac{1}{r}$ in the definition of 4-petential which comes from the Green's function. The 4-potential is given by,

$$A_\mu(t, \vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int \frac{j_\mu(t - \frac{r}{c}, \vec{x'})}{r} .d^3x' \tag{1.1.4}$$

where $r = |\vec{x} - \vec{x'}|$. Bopp modified the above formula to

$$A_\mu(t, \vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int j_\mu(t - \frac{r}{c}, \vec{x'}).f(r).d^3x' \tag{1.1.5}$$

For the theory to be relativistically invariant, we demand that,

$$A_\mu(t, \vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int j_\mu(t - \frac{r}{c}, \vec{x'}).F(s^2).d^3x'dt' \tag{1.1.6}$$

where $s^2 = c^2(t - t')^2 - r^2$, is the relativistically invariant spacetime interval between source and field points. The form of the function, $F(s^2)$, ensures that the results are relativistically invariant. Now we are free to choose the function F. We only assume that it is very small everywhere except near $s = 0$. This assumption implies that F is significant for only those values of s which satisfy $s^2 \leq a^2$ for some small a, which is not to be confused with the radius of the particle defined in the previous subsection. What this means is that only particles which are *almost light-like separated* are affected by the force whereas, in the Maxwell's case, particles which are *exactly light-like separated* are affected by the force. Suppose that we are very far away from the source such that $r \gg a$, then the effect of the field is felt only by the particles within the time interval

$$\Delta t = t - t' \approx \frac{r}{c} \pm \frac{a^2}{2rc} \tag{1.1.7}$$

Since $a \ll r$, we get

$$\Delta t = t - t' \approx \frac{r}{c} \tag{1.1.8}$$

Thus, we come back to Maxwell's description as long as we are far away from the source, in the sense that we get back the retarded solutions of Maxwell's equation, which have support only on the light cone of the source - that is the interactions travel at the speed of light. In fact, if we integrate over t' between $t' \pm \Delta t$, during which most of the contribution of F comes, and if we assume F to be a constant, say K, over this interval, we get,

$$A_\mu(t, \vec{x}) = \frac{Ka^2}{c} \int \frac{j_\mu(t - \frac{r}{c}, \vec{x'})}{r} .d^3x' \tag{1.1.9}$$

which is exactly similar to (1.1.4). Comparing these equations, it is easy to see that,

$$K = \frac{1}{4\pi\epsilon_0 ca^2} \tag{1.1.10}$$

Hence, this theory is a good classical theory for point particles. It predicts a finite self energy for a point particle and includes radiation reaction. Like all other theories which include self force, this theory is valid only as long as classical electrodynamics is considered. As soon as quantum effects are considered, the problems come back.

1.2 Course of the report

In this introduction, we have explained what a self-force is and how classical theory of electromagnetism fails to explain self-force of a point particle. We have mentioned two historical theories which tried to address this problem; one which avoided the concept of self-force altogether, and another which assumed a different, non-linear form for 4-potential in contrast to Maxwell's equations.

In the following chapters, we shall be discussing self-force in detail using modern notation. Chapter 2 deals with the notion of self-force in flat spacetime, which yields the well known *Lorentz-Dirac equation.* We derive the Green's functions associated with d'Alembertian wave equation and equation of motion of a point electric charge from scratch. We discuss two approaches to arrive at the same equation of motion.

Then we move on to chapter 3 in which we study the consequences and controversies involving self-force. Especially, this discussion is focused on the issue of radiation reaction of a uniformly accelerating particle as seen by two observers - one who is an inertial observer and another who is accelerating along with the particle. We resolve some apparent paradoxes concerning conservation of energy and principle of equivalence.

In chapter 4, we introduce and define the concept of *bi-tensors.* This is a new mathematical which is very useful to make sense of physical quantities which are defined with respect to two non-local points, for example, Green's functions. As we know, tensors are locally defined objects in curved spacetime and hence, cannot be used to describe such non-local objects. The technology of *bi-tensors* makes this task very easy. We use this concept to compute the Green's functions of covariant scalar wave equation. Before doing this, we also define *covariant Taylor expansion.* All these computations are valid in a region called *normal convex region* of a point, which will be defined in due course.

Chapter 5 deals with two different coordinate systems, namely, *Fermi-normal coordinates* and *retarded coordinates.* First we define the tetrad, then the coordinates and then we get expressions for tetrad in terms of small displacements in coordinates. We then find the metric near the worldline of the particle which is itself an object worth studying, though not in present report. Then we move on to coordinate transformations between these coordinates and finish the chapter with the relations between tetrad of the two coordinate systems.

In the final chapter, we collect all the results and arrive at the equation of motion of a scalar charge influenced by its own field. We quote the equation of motion obtained for a point electric charge and compare the two equations with Lorentz-Dirac equation.

Chapter 2

Lorentz-Dirac equation in flat spacetime

2.1 Equations - of motion and of electrodynamics

The fundamental equations which form the foundation for later work are the Maxwell's equations of Electrodynamics and Lorentz force law, which is essentially the equation of motion of particles in electromagnetic field. In this section we derive both the equations from the action principle. The action for a particle in electromagnetic field is given by,

$$S = S_{EM} + S_{int} + S_{particle} = -\frac{1}{16\pi}\int F_{\alpha\beta}F^{\alpha\beta}d^4x + \int A_\alpha j^\alpha d^4x - m\int d\tau \tag{2.1.1}$$

We can observe that the above equation has three separate terms. The first term, S_{EM}, has a pure electromagnetic origin. The third term, $S_{particle}$, is the well known action for a particle of mass m. The second term, S_{int}, is the interaction term; it connects the interaction between matter and electromagnetic field. Here all the symbols have their usual meanings,

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{2.1.2}$$

$$\implies F_{\alpha\beta} = -F_{\beta\alpha} \tag{2.1.3}$$

and for a point particle of charge q we have,

$$j^\alpha(x) = q\int u^\alpha(\tau)\delta(x-z)d\tau \tag{2.1.4}$$

where, $z^\alpha(\tau)$ is the trajectory of the particle and $u^\alpha(\tau) = \frac{dz^\alpha(\tau)}{d\tau}$ is the four-velocity of the particle. Now we derive the fundamental equations from the action principle. Throughout this report, we use the Minkowski metric defined as, $\eta_{\alpha\beta} = diag(-1,1,1,1)$.

2.1.1 Maxwell's dynamical equations

The action has two independent quantities which can be varied to get equations of motion. To get Maxwell's dynamical equations we vary the action, S, with respect to the four potential, A_μ.

$$\frac{\delta S}{\delta A_\mu} = \frac{\delta S_{EM}}{\delta A_\mu} + \frac{\delta S_{int}}{\delta A_\mu} + \frac{\delta S_{particle}}{\delta A_\mu} = 0 \tag{2.1.5}$$

Since $S_{particle}$ does not depend on A_μ,

$$\frac{\delta S_{particle}}{\delta A_\mu (x)} = 0 \tag{2.1.6}$$

The electromagnetic term gives

$$\begin{aligned}\frac{\delta S_{EM}}{\delta A_\mu(x)} &= \frac{\delta}{\delta A_\mu}\left(-\frac{1}{16\pi}\int F_{\alpha\beta}F^{\alpha\beta}d^4x'\right) \\ &= \frac{1}{4\pi}\partial_\sigma F^{\sigma\mu}(x)\end{aligned} \tag{2.1.7}$$

Similarly the interaction term gives

$$\begin{aligned}\frac{\delta S_{int}}{\delta A_\mu(x)} &= \frac{\delta}{\delta A_\mu}\left(\int A_\alpha j^\alpha d^4x'\right) \\ &= j^\mu(x)\end{aligned} \tag{2.1.8}$$

Now substituting (2.1.6), (2.1.7) and (2.1.8) into (2.1.5), we get the following equations,

$$\frac{1}{4\pi}\partial_\sigma F^{\sigma\mu}(x) + j^\mu(x) + 0 = 0$$

$$\implies F^{\mu\sigma}{}_{,\sigma} = 4\pi j^\mu \tag{2.1.9}$$

These are the two Maxwell's dynamic equations which contain source. The other two equations are actually Jacobi identities,

$$F^{\alpha\beta}{}_{,\gamma} + F^{\beta\gamma}{}_{,\alpha} + F^{\gamma\alpha}{}_{,\beta} = 0 \tag{2.1.10}$$

which follow from the antisymmetric property of $F^{\alpha\beta}$, given in (2.1.3). These equations, as can be seen, do not contain any source terms and are not dynamical equations. They constrain the possible solutions of the Maxwell's dynamic solutions.

2.1.2 Lorentz force law

The Lorentz force is in fact the equation of motion of particle in the presence of an electromagnetic field. To get the equation of motion, we vary the action, S, with respect to the trajectory of the particle, $z^\mu(\tau)$.

$$\frac{\delta S}{\delta z^\mu} = \frac{\delta S_{EM}}{\delta z^\mu} + \frac{\delta S_{int}}{\delta z^\mu} + \frac{\delta S_{particle}}{\delta z^\mu} = 0 \tag{2.1.11}$$

Consider the individual terms one by one. First,

$$\frac{\delta S_{EM}}{\delta z^\mu} = 0 \tag{2.1.12}$$

because the electromagnetic field does not explicitly depend on the trajectory of the particle.

Now, consider the interaction term. Here, we should note that a variation in worldline induces a change in $A_\mu(x)$. With this in mind, we get

$$\begin{aligned} \frac{\delta S_{int}}{\delta z^\mu(\tau)} &= \frac{\delta}{\delta z^\mu}\left(\int A_\alpha j^\alpha d^4x\right) \\ &= \int \left[\left\{\frac{\delta}{\delta z^\mu}(A_\alpha) j^\alpha\right\} + \left\{A_\alpha \frac{\delta}{\delta z^\mu}(j^\alpha)\right\}\right] d^4x \\ &= q u^\alpha F_{\alpha\mu}(z) \end{aligned} \tag{2.1.13}$$

Here, integration by parts has been used between fifth line and sixth line. Now, consider the particle term,

$$\begin{aligned} \frac{\delta S_{particle}}{\delta z^\mu(\tau)} &= \frac{\delta}{\delta z^\mu}\left(-m\int d\tau'\right) \\ &= m a_\mu \end{aligned} \tag{2.1.14}$$

Substituting the above results - (2.1.12), (2.1.13) and (2.1.14) - into the equation (2.1.11), we get the following equations of motion,

$$0 + q u^\alpha F_{\alpha\mu}(z) + m a_\mu = 0$$

$$\Longrightarrow \; m a_\mu = q F_{\mu\alpha}(z) u^\alpha \tag{2.1.15}$$

The last equation is the *Lorentz force law*. We can observe that the force is evaluated at $z(\tau)$ which is the trajectory of the particle. But $F^{\alpha\beta}$ has a singularity at this point - in fact, on the whole worldline of the particle - because the force is due to the field of the particle itself. What we tried to do was to calculate the force, generated by the particle, acting on itself. Hence, (2.1.15) does not make sense. More care should be taken in calculating the self-force of a *point* particle.

2.2 Green's functions in flat spacetime

The problem with the Lorentz force above is that there is a singularity on the world line. The singularity has its origin in the 4-potential generated by the particle which in turn depends on the Green's function.

Ultimately the singularity comes from the Green's function. First, we need to know what the Green's functions in this case are and how they affect the theory. Here, we derive the Green's function for the d'Alembertian, $\Box$.

2.2.1 Derivation

We start with the following equations,

$$(2.1.2) \text{ and } (2.1.9) \Rightarrow \Box A^{\mu} = 4\pi j^{\mu} \tag{2.2.1}$$

Hence, the Green's function for the d'Alembertian, $\Box$, satisfies the differential equation

$$\Box G(x, x') = 4\pi\delta(x - x') \tag{2.2.2}$$

$$\Longrightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(x, x') = 4\pi\delta(x - x')$$

Let the Fourier transform of $G(x, x')$ be $\tilde{G}(k, x')$ - here the Fourier transform is with respect to x but not x'. We know that $x = (ct, \vec{x})$, similarly $k = (\frac{\omega}{c}, \vec{k})$. Let $k.x = k^{\mu}x_{\mu} = (\omega t - \vec{k}.\vec{x})$. The Fourier transform and its inverse are given by

$$\tilde{G}(k, x') = \frac{1}{(2\pi)^2}\int e^{ik.x}G(x, x')d^4x \tag{2.2.3}$$

$$G(x, x') = \frac{1}{(2\pi)^2}\int e^{-ik.x}\tilde{G}(k, x')d^4k \tag{2.2.4}$$

We also know that,

$$\delta(x - x') = \frac{1}{(2\pi)^4}\int e^{-ik.(x-x')}d^4k \tag{2.2.5}$$

Substituting (2.2.4) and (2.2.5) in the equation (2.2.2), we get

$$\tilde{G}(k, x') = \left(\frac{-c^2}{\pi}\right)\frac{e^{ik.x'}}{\omega^2 - c^2\vec{k}.\vec{k}} = \left(\frac{-c^2}{\pi}\right)\frac{e^{ik.x'}}{\omega^2 - c^2k^2} \tag{2.2.6}$$

Substituting this back into (2.2.4), we get

$$G(x, x') = \frac{1}{(2\pi)^2}\left(\frac{-c^2}{\pi}\right)\int \frac{e^{-ik.(x-x')}}{\omega^2 - c^2k^2}d^4k \tag{2.2.7}$$

$$= \frac{1}{(2\pi)^2}\left(\frac{-c^2}{\pi}\right)\int \frac{e^{i[\vec{k}.(\vec{x}-\vec{x}')-\omega(t-t')]}}{\omega^2 - c^2k^2}d^4k \tag{2.2.8}$$

$$= \frac{1}{(2\pi)^2}\left(\frac{-c^2}{\pi}\right)\int \frac{e^{i[\vec{k}.(\vec{x}-\vec{x}')-\omega(t-t')]}}{(\omega - ck)(\omega + ck)}d^4k \tag{2.2.9}$$

where $k = |\vec{k}|$, it is not to be confused with the four wave vector, k^μ. Conventionally, the ω integration is done first. Contour integration followed by application of Residue theorem is the best way to find the integral with respect to ω because the integration path, which is from $\omega = -\infty$ to $\omega = +\infty$, includes the poles $\omega = \pm ck$. In contour integration we treat ω as a complex variable and integrate over a contour in its complex plane. Let $\omega = \omega_R + i\omega_I$, where ω_R and ω_I are real and imaginary parts respectively. There are two possible contours for integration: one above the real axis and the other below. The integrand then has the factor $e^{\omega_I(t-t')}$ which converges only when the exponent is negative. Hence, among the possible contours, those which lie entirely in the upper half-plane must have $t < t'$ and those which lie entirely in the lower half-plane must have $t > t'$. We consider two contours: one contour which is the real axis plus a semi circle of radius $r \to \infty$ in the upper half-plane and another which is the real axis plus a semi circle of radius $r \to \infty$ in the lower half-plane.

Even after choosing a contour, as the real axis passes through two poles we need a way to get *around* them. These paths around the poles, called as *indentations*, are semicircles of radius $\epsilon \to 0$. There are four possibilities to go around the two poles: both up, both down, one up and one down and vice versa. Out of these four possibilities the last two are of no concern in this derivation. The first two are the most important ones which give two different solutions for the Green's functions; they are *retarded* and *advanced* Green's functions for the first and second cases respectively. We derive both the functions separately.

The integration itself is straight forward once a contour with appropriate indentations are fixed. The only additional calculations are the Residues of the poles. They are

$$R_1 = \frac{e^{ick(t-t')}}{-2ck} \text{ at } \omega = -ck \tag{2.2.10}$$

$$R_2 = \frac{e^{-ick(t-t')}}{2ck} \text{ at } \omega = ck \tag{2.2.11}$$

Using the above residues, for the case of both the indentations above the poles (denoted by + sign near the integral symbol), we get

$$\oint_+ \frac{e^{-i\omega(t-t')}}{(\omega-ck)(\omega+ck)} d^4k = \begin{cases} (2\pi i)\left(\frac{-i\sin[ck(t-t^{'})]}{ck}\right) & t > t^{'} \text{ i.e., upper semicircle} \\ 0 & t < t' \text{ i.e., lower semicircle} \end{cases} \tag{2.2.12}$$

or using *Heaviside step function*, we can write the above integral as

$$\begin{aligned} \oint_+ \frac{e^{-i\omega(t-t')}}{(\omega-ck)(\omega+ck)} d^4k &= (2\pi i)\left(\frac{-i\sin[ck(t-t^{'})]}{ck}\right)\Theta(t'-t) \\ &= \frac{2\pi}{ck}\sin[ck(t-t^{'})]\Theta(t'-t) \end{aligned} \tag{2.2.13}$$

Similarly, for both indentations below the poles (denoted by − sign near the integral symbol), the integral is given by

$$\oint_{-} \frac{e^{-i\omega(t-t')}}{(\omega - ck)(\omega + ck)} d^4k = \frac{-2\pi}{ck} \sin[ck(t-t')]\Theta(t-t') \tag{2.2.14}$$

The appearance of Θ functions in the above integrals suggests a notion of causality. But calculating these integrals is only one part in the total integration. There is also an integration over total momentum space which is essentially the inverse Fourier transform from momentum space to position space. This integration can be done using spherical coordinates for momentum space. We can choose the k_z-axis to be along any direction because the integral is invariant under rotations. To simplify the integration, choose k_z-axis along the vector $\vec{x} - \vec{x'}$. The integration for both advanced and retarded Green's functions is similar. The advanced Green's function has the following form

$$G_{-}(x, x') = \frac{-c}{2\pi^2} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} k^2 \sin\theta dk d\theta d\phi e^{ik|\vec{x}-\vec{x'}|\cos\theta} \frac{\sin[ck(t-t')]\Theta(t'-t)}{k}$$

Let $R = |\vec{x} - \vec{x'}|$. After doing the integration over ϕ, which is trivial, we get

$$G_{-}(x, x') = \frac{c}{\pi} \int_0^{\infty} \int_0^{\pi} k \sin\theta dk d\theta e^{ikR\cos\theta} \sin[ck(t'-t)]\Theta(t'-t)$$

The integral over θ, which is also easy to do, yields

$$\begin{aligned} G_{-}(x, x') &= \frac{c}{\pi} \int_0^{\infty} k dk \frac{2\sin kR}{kR} \sin[ck(t'-t)]\Theta(t'-t) \\ &= \frac{2c}{\pi R} \int_0^{\infty} dk \sin(kR) \sin[ck(t'-t)]\Theta(t'-t) \end{aligned}$$

The integral over k requires a few steps to get to the final answer. The first one is to write product of sines as a sum of cosines

$$G_{-}(x, x') = \frac{c}{\pi R} \int_0^{\infty} dk \left\{ \cos\left(k\left[R - c(t'-t)\right]\right) - \cos\left(k\left[R + c(t'-t)\right]\right) \right\} \Theta(t'-t)$$

Since cosine is an even function in k, this integral can be replaced with another integral from $k = -\infty$ to $k = \infty$ by multiplying a factor of half. Also, sine is an odd function and hence, can be added to the corresponding cosine terms without changing the value of the integral. Then the combination of cosine and sine functions can be replaced by e^{ikx}. The final integrals are

$$\begin{aligned} G_{-}(x, x') &= \frac{c}{2\pi R} \int_{-\infty}^{\infty} dk \left\{ e^{ik[R-c(t'-t)]} - e^{ik[R+c(t'-t)]} \right\} \Theta(t'-t) \\ &= \frac{c}{R} \left[\delta(R - c(t'-t)) - \delta(R + c(t'-t)) \right] \Theta(t'-t) \end{aligned}$$

Since $R > 0$ and Θ ensures that $t' > t$ the second δ-function is always 0 and the only contribution comes from the first δ-function. Hence, finally, the *advanced Green's function* is

$$G_-(x,x') = \frac{c}{R}\delta(R - c(t'-t)) = \frac{1}{R}\delta\left(t' - \left[\frac{R}{c} + t\right]\right) \tag{2.2.15}$$

After a similar analysis for the indentations above the poles, we get the *retarded Green's function*

$$G_+(x,x') = \frac{c}{R}\delta(R + c(t'-t)) = \frac{1}{R}\delta\left(t' - \left[t - \frac{R}{c}\right]\right) \tag{2.2.16}$$

This completes the derivation of Green's functions for d'Alembertian which are useful in the case of electrodynamics.

2.2.2 Discussion

The form of the Green's functions agrees with the names advanced and retarded. Take the case of retarded Green's function which is easy to visualize. Clearly, the function is 0 for $t > t'$ but there is much more to it. The Green's function is non zero only at $t = t' + \frac{R}{c}$. This means that the effect of source at $\vec{x'}$ which radiates a wave at t' is felt at the field point, $\vec{x}$, at t, that is after a delay of $\frac{R}{c}$ where R is the distance between source and field points. The effect doesn't take any more or any less time but exactly $\frac{R}{c}$. Even in the case of advanced Green's functions, the wave reaches the field point $\frac{R}{c}$ seconds in advance. This implies that the interaction travels *exactly* at a speed of light both into the past and into the future. This further implies that *the Green's functions have support only on the light cone of the source.* That is the electromagnetic waves radiated from the source at any instant can reach only to those points which can be reached by a light wave radiated by the source at that same instant.

One more important aspect of the above Green's functions, which is relevant to the discussion in the previous section, is the appearance of R in the denominator. This implies a singularity at $R = 0$ (both field and source points coincide). This means the 4-potential and hence, electromagnetic fields are singular on the world line of the source particle. This can not be avoided and this is the reason why the calculations of self-force using Lorentz force law in the previous section yielded meaningless results.

We have derived all the Maxwell's equations and Lorentz force in flat spacetime. We have argued why the Lorentz force law does not make sense when applied to calculate self-force of a point particle. We have derived the Green's functions and argued that the singularity in the Green's function is the main reason for infinite self-force of a point particle. In the next section, We will extend the Lorentz force law to Lorentz-Dirac force equation. We will also introduce radiation reaction which can not be avoided in any theory explaining self-force.

2.3 Slowly moving (non-relativistic) point charges

From the advanced and retarded Green's functions calculated in the previous subsection, we get the following expression for electromagnetic 4-potential, $A^\alpha_\epsilon(x)$.

$$A^\alpha_\epsilon(t,\vec{x}) = \int \frac{j^\alpha\left(t - \epsilon\left|\vec{x} - \vec{x}'\right|, \vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|} d^3x' \tag{2.3.1}$$

where

$$\epsilon = \begin{cases} +1 & \text{for retarded solution} \\ -1 & \text{for advanced solution} \end{cases}$$

2.3.1 Far zone

In the far zone, we have $|\vec{x}| \gg |\vec{x}'|$. Let $r \equiv |\vec{x}|$. Then we have

$$\left|\vec{x} - \vec{x}'\right| = r - \hat{n} \cdot \vec{x}' + O\left(r^{-1}\right)$$

Now expanding j^α in Taylor series around $w \equiv t - \epsilon r$, we obtain

$$j^\alpha\left(t - \epsilon\left|\vec{x} - \vec{x}'\right|, \vec{x}'\right) = \sum_{l=0}^{\infty} \frac{\epsilon^l}{l!}\left(\hat{n} \cdot \vec{x}'\right)^l \frac{\partial^l}{\partial w^l} j^\alpha\left(w, \vec{x}'\right)$$

Substituting this expression into (2.3.1), we get a multipole expansion for 4-potential

$$\begin{aligned} A_\epsilon^\alpha(t, \vec{x}) &= \frac{1}{r}\sum_{l=0}^{\infty} \frac{\epsilon^l}{l!} \int \left(\hat{n} \cdot \vec{x}'\right)^l \frac{\partial^l}{\partial w^l} j^\alpha\left(w, \vec{x}'\right) d^3x' \\ &= \frac{1}{r}\sum_{l=0}^{\infty} \frac{\epsilon^l}{l!} \frac{d^l}{dw^l} \int \left(\hat{n} \cdot \vec{x}'\right)^l j^\alpha\left(w, \vec{x}'\right) d^3x' \end{aligned} \tag{2.3.2}$$

To the leading order, the scalar potential is given by

$$\begin{aligned} r\Phi_\epsilon(t, \vec{x}) &= \int \rho\left(w, \vec{x}'\right) d^3x' + \epsilon \frac{d}{dw} \int \rho\left(w, \vec{x}'\right)\left(\hat{n} \cdot \vec{x}'\right) d^3x' \\ &= q + \epsilon \hat{n} \cdot \frac{d}{dw} \int \rho \vec{x}' d^3x' \\ &= q + \epsilon \hat{n} \cdot \dot{\vec{p}} \end{aligned} \tag{2.3.3}$$

where $q = \int \rho d^3x'$ and $\vec{p} = \int \rho \vec{x}' d^3x'$ and the overdot denotes differentiation with respect to w. Similarly, the vector potential is given by

$$rA_\epsilon^i(t, \vec{x}) = \int j^i\left(w, \vec{x}'\right) d^3x'$$

But, by Gauss's theorem, we know that

$$\begin{aligned}
& \int \vec{\nabla}' \cdot \left(x'^i \vec{j}\right) d^3x' &&= 0 \\
\Longrightarrow \quad & \int \left[\vec{\nabla}' \left(x'^i\right) \cdot \vec{j} + x'^i \left(\vec{\nabla}' \cdot \vec{j}\right)\right] d^3x' &&= 0 \\
\Longrightarrow \quad & \int \left[j^i + x'^i \left(-\tfrac{\partial \rho}{\partial w}\right)\right] d^3x' &&= 0 \\
\Longrightarrow \quad & \int j^i d^3x' &&= \frac{d}{dw} \int \rho x'^i d^3x' \\
\Longrightarrow \quad & \int \vec{j} d^3x' &&= \dot{\vec{p}}
\end{aligned}$$

Substituting this back into previous equation we get

$$r\vec{A}_\epsilon\left(t, \vec{x}\right) = \dot{\vec{p}} \tag{2.3.4}$$

The magnetic field due to this vector potential is given by

$$\begin{aligned}
r\vec{B}_\epsilon &= r\vec{\nabla} \times \vec{A}_\epsilon \\
&= r\vec{\nabla} \times \left(\frac{\dot{\vec{p}}}{r}\right) \\
&= r\vec{\nabla}\left(\frac{1}{r}\right) \times \dot{\vec{p}} + r\left(\frac{1}{r}\right)\vec{\nabla} \times \dot{\vec{p}} \\
&= r\left(-\frac{\hat{n}}{r^2}\right) \times \dot{\vec{p}} + \left(-\epsilon\hat{n} \times \ddot{\vec{p}}\right) \\
&= -\frac{\hat{n}}{r} \times \dot{\vec{p}} + \left(-\epsilon\hat{n} \times \ddot{\vec{p}}\right)
\end{aligned}$$

To the leading order in r, the magnetic field is given by

$$r\vec{B}_\epsilon = -\epsilon\hat{n} \times \ddot{\vec{p}} \tag{2.3.5}$$

Similarly the electric field is given by

$$\begin{aligned}
\vec{E}_\epsilon &= -\vec{\nabla}\Phi_\epsilon - \frac{\partial \vec{A}_\epsilon}{\partial t} \\
&= -\vec{\nabla}\left(\frac{q}{r} + \frac{\epsilon\hat{n} \cdot \dot{\vec{p}}}{r}\right) - \frac{\partial}{\partial t}\left(\frac{\dot{\vec{p}}}{r}\right) \\
&= -q\vec{\nabla}\left(\frac{1}{r}\right) - \epsilon\left[\vec{\nabla}\left(\frac{1}{r}\right)\hat{n} \cdot \dot{\vec{p}} + \frac{1}{r}\vec{\nabla}\left(\hat{n} \cdot \dot{\vec{p}}\right)\right] - \frac{1}{r}\frac{\partial}{\partial t}\left(\dot{\vec{p}}\right) \\
&= \frac{q}{r^2} + \frac{\epsilon}{r^2}\left[2\hat{n}\left(\hat{n} \cdot \dot{\vec{p}}\right) - \dot{\vec{p}}\right] + \frac{1}{r}\left[\hat{n}\left(\hat{n} \cdot \ddot{\vec{p}}\right) - \ddot{\vec{p}}\right]
\end{aligned}$$

To the leading order in r, the electric field is then given by

$$r\vec{E}_\epsilon = \hat{n}\left(\hat{n}\cdot\ddot{\vec{p}}\right) - \ddot{\vec{p}} \tag{2.3.6}$$

It can be observed that

$$\vec{E}_\epsilon = \epsilon\vec{B}_\epsilon \times \hat{n} \tag{2.3.7}$$

That is the electric and magnetic fields are perpendicular to each other and are perpendicular to direction of propagation $\hat{n}$. Hence, these are transverse electromagnetic waves. For $\epsilon = +1$, the waves are *outgoing* and hence, energy is removed by these waves from the source. For $\epsilon = -1$, the waves are *incoming* and hence, energy is provided by these waves to the source.

The Poynting vector, $\vec{S}$, is given by

$$\begin{aligned}
\vec{S}_\epsilon &= \frac{1}{4\pi}\vec{E}_\epsilon \times \vec{B}_\epsilon \\
&= \frac{1}{4\pi}\left(\frac{\hat{n}\left(\hat{n}\cdot\ddot{\vec{p}}\right) - \ddot{\vec{p}}}{r}\right) \times \left(\frac{-\epsilon\hat{n}\times\ddot{\vec{p}}}{r}\right) \\
&= -\frac{\epsilon}{4\pi r^2}\left[\left(\hat{n}\cdot\ddot{\vec{p}}\right)\left\{\hat{n}\times\left(\hat{n}\times\ddot{\vec{p}}\right)\right\} - \ddot{\vec{p}}\times\left(\hat{n}\times\ddot{\vec{p}}\right)\right] \\
&= -\frac{\epsilon}{4\pi r^2}\left[\left(\hat{n}\cdot\ddot{\vec{p}}\right)\left\{\hat{n}\left(\hat{n}\cdot\ddot{\vec{p}}\right) - \ddot{\vec{p}}(\hat{n}\cdot\hat{n})\right\} - \left\{\hat{n}\left(\ddot{\vec{p}}.\ddot{\vec{p}}\right) - \ddot{\vec{p}}\left(\ddot{\vec{p}}\cdot\hat{n}\right)\right\}\right] \\
&= -\frac{\epsilon}{4\pi r^2}\left[\left(\hat{n}\cdot\ddot{\vec{p}}\right)^2\hat{n} - \left|\ddot{\vec{p}}\right|^2\hat{n}\right] \\
&= \frac{\epsilon}{4\pi r^2}\left(\left|\ddot{\vec{p}}\right|^2\sin^2(\theta)\right)\hat{n}
\end{aligned} \tag{2.3.8}$$

where θ is the angle between the vectors $\hat{n}$ and $\ddot{\vec{p}}$.

The rate of energy flowing out of a sphere of radius r is given by

$$\begin{aligned}
\frac{dE}{dw} &= \int \vec{S}\cdot d\vec{A} \\
&= \int_0^\pi\int_0^{2\pi}\frac{\epsilon}{4\pi r^2}\left(\left|\ddot{\vec{p}}\right|^2\sin^2(\theta)\right)r^2\sin(\theta)\,d\theta d\phi \\
&= \int_0^\pi 2\pi\frac{\epsilon}{4\pi}\left|\ddot{\vec{p}}\right|^2\sin^3(\theta)\,d\theta \\
&= \frac{\epsilon}{2}\left|\ddot{\vec{p}}\right|^2\int_0^\pi\sin^3(\theta)\,d\theta \\
&= \epsilon\frac{2}{3}\left|\ddot{\vec{p}}\right|^2
\end{aligned} \tag{2.3.9}$$

Thus, we can confirm that energy flux is *outward* for $\epsilon = +1$ and *inward* for $\epsilon = -1$. For a point charge, since $\vec{p} = q\vec{z}$, we have $\frac{dE}{dw} = \epsilon\frac{2}{3}q^2a^2$, where a is the acceleration of the point charge. This is *Larmor formula.*

2.3.2 Near zone

Near the source, we have $|\vec{x}-\vec{x}'| \ll t$. Hence, expanding j^α in Taylor series about t, treating $|\vec{x}-\vec{x}'|$ as a small quantity, we get

$$j^\alpha\left(t-\epsilon\left|\vec{x}-\vec{x}'\right|,\vec{x}'\right)=\sum_{l=0}^{\infty}\frac{(-\epsilon)^l}{l!}\left|\vec{x}-\vec{x}'\right|^l\frac{\partial^l}{\partial t^l}j^\alpha\left(t,\vec{x}'\right)$$

Substituting this into (2.3.1), we obtain

$$\begin{aligned}A^\alpha_\epsilon\left(t,\vec{x}\right)&=\int\frac{\left[\sum_{l=0}^{\infty}\frac{(-\epsilon)^l}{l!}\left|\vec{x}-\vec{x}'\right|^l\frac{\partial^l}{\partial t^l}j^\alpha\left(t,\vec{x}'\right)\right]}{\left|\vec{x}-\vec{x}'\right|}d^3x'\\&=\sum_{l=0}^{\infty}\frac{(-\epsilon)^l}{l!}\int\left|\vec{x}-\vec{x}'\right|^{l-1}\frac{\partial^l}{\partial t^l}j^\alpha\left(t,\vec{x}'\right)d^3x'\\&=\left[\sum_{l\ even}\frac{1}{l!}\frac{\partial^l}{\partial t^l}\int\left|\vec{x}-\vec{x}'\right|^{l-1}j^\alpha\left(t,\vec{x}'\right)d^3x'\right]-\epsilon\left[\sum_{l\ odd}\frac{1}{l!}\frac{\partial^l}{\partial t^l}\int\left|\vec{x}-\vec{x}'\right|^{l-1}j^\alpha\left(t,\vec{x}'\right)d^3x'\right]\end{aligned}\tag{2.3.10}$$

It can be observed that the first sum is independent of ϵ and is same irrespective of radiation at infinity. Hence it is not responsible for radiation reaction. On the other hand, the second sum changes sign with ϵ and hence, corresponds to radiation reaction force. This can be looked at in another way. The first sum can be written as $\frac{1}{2}\left(A^\alpha_{ret}+A^\alpha_{adv}\right)$, the *coloumb part*, and the second sum as $\frac{1}{2}\left(A^\alpha_{ret}-A^\alpha_{adv}\right)$. Hence, the *radiation reaction potential* is given by

$$A^\alpha_{rr}\left(t,\vec{x}\right)=\frac{1}{2}\left(A^\alpha_{ret}-A^\alpha_{adv}\right)=-\sum_{l\ odd}\frac{1}{l!}\frac{\partial^l}{\partial t^l}\int\left|\vec{x}-\vec{x}'\right|^{l-1}j^\alpha\left(t,\vec{x}'\right)d^3x'\tag{2.3.11}$$

The leading term for scalar potential comes from $l=3$ because $l=1$ term vanishes by virtue of charge conservation.

$$\Phi_{rr}\left(t,\vec{x}\right)=-\frac{1}{3!}\frac{\partial^3}{\partial t^3}\int\rho\left(t,\vec{x}'\right)\left|\vec{x}-\vec{x}'\right|^2d^3x'$$

The leading term for vector potential comes from $l=1$.

$$\vec{A}_{rr}\left(t,\vec{x}\right)=-\frac{\partial}{\partial t}\int\vec{j}\left(t,\vec{x}'\right)d^3x'$$

In the special case of a point charge, we have $\rho\left(t,\vec{x}'\right)=q\delta\left(\vec{x}'-\vec{z}\left(t\right)\right)$ and $\vec{j}\left(t,\vec{x}'\right)=q\vec{v}\delta\left(\vec{x}'-\vec{z}\left(t\right)\right)$.

The scalar potential is

$$\begin{aligned}\Phi_{rr}(t,\vec{x}) &= -\frac{q}{3!}\frac{d^3}{dt^3}\left|\vec{x}-\vec{z}(t)\right|^2 \\ &= -\frac{q}{3!}\frac{d^2}{dt^2}\left[2\left(\vec{x}-\vec{z}\right)\cdot\left(\vec{v}\right)\right] \\ &= -\frac{q}{3!}\frac{d}{dt}\left[2\left(-\vec{v}\right)^2+2\left(\vec{x}-\vec{z}\right)\cdot\left(-\vec{a}\right)\right] \\ &= -\frac{q}{3!}\left[4\vec{v}\cdot\vec{a}-2\left(\vec{x}-\vec{z}\right)\cdot\dot{\vec{a}}+2\vec{v}.\vec{a}\right] \\ &= \frac{q}{3}\left(\vec{x}-\vec{z}\right)\cdot\dot{\vec{a}}-q\vec{v}\cdot\vec{a} \end{aligned} \tag{2.3.12}$$

Similarly, the vector potential is given by

$$\begin{aligned}\vec{A}_{rr}(t,\vec{x}) &= -q\frac{d}{dt}\vec{v} \\ &= -q\vec{a}\end{aligned} \tag{2.3.13}$$

The magnetic field associated with this radiation reaction potential is then given by

$$\vec{B}_{rr}(t,\vec{x}) = \vec{\nabla}\times\vec{A}_{rr} = 0 \tag{2.3.14}$$

Similarly, the electric field is given by

$$\begin{aligned}\vec{E}_{rr}(t,\vec{x}) &= -\vec{\nabla}\Phi_{rr}-\frac{\partial}{\partial t}\vec{A}_{rr} \\ &= -\vec{\nabla}\left[\frac{q}{3}\left(\vec{x}-\vec{z}\right)\cdot\dot{\vec{a}}-q\vec{v}\cdot\vec{a}\right]-\frac{\partial}{\partial t}\left(-q\vec{a}\right) \\ &= -\frac{q}{3}\vec{\nabla}\left(\vec{x}\cdot\dot{\vec{a}}\right)+q\frac{\partial}{\partial t}\left(\vec{a}\right) \\ &= -\frac{q}{3}\dot{\vec{a}}+q\dot{\vec{a}} \\ &= \frac{2}{3}q\dot{\vec{a}}\end{aligned} \tag{2.3.15}$$

Finally, the radiation reaction force is given by

$$\vec{F}_{rr} = q\left(\vec{E}_{rr}+\vec{v}\times\vec{B}_{rr}\right) = \frac{2}{3}q^2\dot{\vec{a}} \tag{2.3.16}$$

The rate of work done by this force is given by

$$\dot{W} = \vec{F}_{rr}\cdot\vec{v} = \frac{2}{3}q^2\dot{\vec{a}}\cdot\vec{v} = \frac{2}{3}\left[\frac{d}{dt}\left(\vec{a}\cdot\vec{v}\right)-\left|\vec{a}\right|^2\right]$$

Averaging over time and assuming that motion is either periodic or unaccelerated at early and late times, the first term in the above expression vanishes and we get

$$\left\langle\dot{W}\right\rangle = -\frac{2}{3}q^2\left|\vec{a}\right|^2 \tag{2.3.17}$$

This quantity is negative and is exactly equal in magnitude to the rate of energy radiated by the point charge given by *Larmor formula*. Hence, there is an energy balance on average.

2.4 Covariant form of Lorentz-Dirac equation

In this section, the covariant form of Lorentz-Dirac equation is derived. We start with the light-cone mapping. In this subsection we define several quantities which will be useful in the derivation that follows. The definitions and results in this subsection can be found in [2].

2.4.1 Light-cone mapping

For the retarded wave, the trajectory, $z^{\alpha}(\tau)$, of the particle intersects the light cone of the field point, x^{α}, on the past cone.

$$\sigma(x,u) \equiv \frac{1}{2}\eta_{\alpha\beta}\left(x^{\alpha}-z^{\alpha}(u)\right)\left(x^{\beta}-z^{\beta}(u)\right)=0 \tag{2.4.1}$$

From the above equation, we can get $u(x)$, the proper time of the particle when $z(\tau)$ intersects the *past light cone*. We shall call u as the *retarded time*.

We shall also need an invariant measure of distance between x and $z(u)$. The scalar quantity

$$r(x) = -\eta_{\alpha\beta}\left(x^{\alpha}-z^{\alpha}(u)\right)u^{\beta}(u) \tag{2.4.2}$$

satisfies the required property. We can confirm the same by observing that in momentarily comoving Lorentz frame, $r = t - z^{0}(u)$, and since the speed of light is assumed to be unity, it is also the spatial separation between x and $z(u)$. Hence, $r(x)$ may be referred to as *retarded distance*.

Now we define a new vector, $k^{\alpha}(x)$, in the direction of $x^{\alpha}-z^{\alpha}(u)$, which is a null vector.

$$k^{\alpha}(x) = \frac{\left(x^{\alpha}-z^{\alpha}(u)\right)}{r} \tag{2.4.3}$$

It can be easily observed that the vector $k^{\alpha}(x)$ satisfies

$$k^{\alpha}(x)\,k_{\alpha}(x) = 0 \text{ and } k^{\alpha}(x)\,u_{\alpha}(u) = -1 \tag{2.4.4}$$

where the second equation provides convenient normalization for the rescaled null vector.

It is easy to see that a change in x results in a change in u, unless that change is along the null geodesic linking x and z. Let the field point be displaced to a new point $x+\delta x$. The corresponding intersection point is $z(u+\delta u)$. Using 2.4.1, we can obtain a relation between δu and δx.

$$\begin{aligned}&\sigma(x+\delta x, u+\delta u) - \sigma(x,u) = d\sigma(x,u) = 0\\ \Longrightarrow\ & u_{,\beta} = -k_{\beta}\end{aligned} \tag{2.4.5}$$

The above relation simplifies the process of differentiation of a function $f(x)$ with implicit reference to u. Now a function $f(x) = F(x,u)$ can be differentiated with respect to x and the dependence on

u can be removed using the relation (2.4.1). That is, we have

$$\begin{aligned}\frac{\partial f}{\partial x^\alpha} &= \left(\frac{\partial F}{\partial x^\alpha}\right)_u + \frac{\partial u}{\partial x^\alpha}\left(\frac{\partial F}{\partial u}\right)_x \\ &= \left(\frac{\partial F}{\partial x^\alpha}\right)_u - k_\alpha\left(\frac{\partial F}{\partial u}\right)_x \end{aligned} \tag{2.4.6}$$

This is the differentiation rule under light-cone mapping.

Using the above rule we can find that

$$\begin{aligned} r_\alpha \equiv r_{,\alpha} &= \frac{\partial r}{\partial x^\alpha} - k_\alpha \frac{\partial r}{\partial u} \\ &= -u_\alpha + k_\alpha\left(1 + r a_k\right) \end{aligned} \tag{2.4.7}$$

where $a_k \equiv a_\beta k^\beta$. Here, it is understood that all worldline quantities such as u^α and a^α are to be evaluated at $\tau = u$. From (2.4.7), using the relations in (2.4.4), we obtain

$$k^\alpha r_\alpha = -k^\alpha u_\alpha + k^\alpha k_\alpha\left(r a_k + 1\right) = 1 \tag{2.4.8}$$

Another example of differentiation, using the rule (2.4.6), which will be useful in subsequent sections, is

$$\begin{aligned} k_{\alpha,\beta} &= \left(\frac{\partial k_\alpha}{\partial x^\beta}\right)_u - k_\beta\left(\frac{\partial k_\alpha}{\partial u}\right)_x \\ &= \frac{1}{r}\left(\eta_{\alpha\beta} + k_\alpha u_\beta + k_\beta u_\alpha - k_\alpha k_\beta\right) - a_k k_\alpha k_\beta \end{aligned} \tag{2.4.9}$$

From the above equation it is clear that

$$k_{\alpha,\beta}k^\beta = \frac{1}{r}\left(\eta_{\alpha\beta}k^\beta + k_\alpha k^\beta u_\beta + k^\beta k_\beta u_\alpha - k_\alpha k_\beta k^\beta\right) - a_k k_\alpha k_\beta k^\beta = \frac{1}{r}\left(k_\alpha - k_\alpha + 0 - 0\right) - 0 = 0 \tag{2.4.10}$$

which is the geodesic equation of null vector. We also have

$$k^\alpha{}_{,\alpha} = \frac{\delta^\alpha_\alpha}{r} + \frac{k^\alpha u_\alpha + u^\alpha k_\alpha}{r} - k_\alpha k^\alpha\left(a_k + \frac{1}{r}\right) = \frac{4}{r} - \frac{2}{r} - 0 = \frac{2}{r} \tag{2.4.11}$$

2.4.2 Electromagnetic Field of a relativistic point charge

The current density of a relativistic point charge is given by

$$j^\alpha\left(x'\right) = q\int u^\alpha\left(\tau\right)\delta\left(x' - z\left(\tau\right)\right)d\tau$$

The elctromagnetic 4-potential is given by

$$\begin{aligned}A_\epsilon^\alpha(x) &= \int G_\epsilon(x,x')\, j^\alpha(x')\, d^4x' \\ &= \int G_\epsilon(x,x')\left(q\int u^\alpha(\tau)\,\delta\left(x'-z(\tau)\right)d\tau\right)d^4x' \\ &= q\int\left(\int G_\epsilon(x,x')\,\delta\left(x'-z(\tau)\right)d^4x'\right)u^\alpha(\tau)\,d\tau \\ &= q\int G_\epsilon(x,z)\,u^\alpha(\tau)\,d\tau \end{aligned}\tag{2.4.12}$$

For the retarded wave, we know that the Green's function is given by

$$G_{ret}(x,z) = \frac{\delta\left(z^0-(t-|\vec{x}-\vec{z}|)\right)}{|\vec{x}-\vec{z}|} = \Theta\left(t-z^0\right)\delta(\sigma)$$

Substituting this result in (2.4.12) gives

$$\begin{aligned}A_{ret}^\alpha(x) &= q\int d\tau u^\alpha(\tau)\,\Theta\left(t-z^0\right)\delta(\sigma) \\ &= q\int d\tau u^\alpha(\tau)\,\Theta\left(t-z^0\right)\frac{\delta\left(\tau-\tau|_{\sigma=0}\right)}{\sum\left|\frac{\partial\sigma}{\partial\tau}\right|_{\sigma=0}} \\ &= q\frac{u^\alpha(u)}{r(x)}\end{aligned}\tag{2.4.13}$$

From the above equation we get

$$\begin{aligned}A_{\alpha,\beta}^{ret}(x) &= q\left[\frac{u_{\alpha,\beta}}{r}+u_\alpha\left(\frac{-r_\beta}{r^2}\right)\right] \\ &= q\left[\frac{-k_\beta a_\alpha}{r}-\frac{u_\alpha}{r^2}\left(-u_\beta+k_\beta r a_k+k_\beta\right)\right] \\ &= q\left[\frac{-k_\beta a_\alpha-k_\beta u_\alpha a_k}{r}+\frac{u_\alpha u_\beta-u_\alpha k_\beta}{r^2}\right]\end{aligned}\tag{2.4.14}$$

Using this relation, the electromagnetic field tensor is given by

$$\begin{aligned}F_{\alpha\beta}(x) &= A_{\beta,\alpha}-A_{\alpha,\beta} \\ &= \frac{2q}{r}\left(a_{[\alpha}k_{\beta]}+a_k u_{[\alpha}k_{\beta]}\right)+\frac{2q}{r^2}u_{[\alpha}k_{\beta]}\end{aligned}\tag{2.4.15}$$

The first term is the *radiative part*, which depends on the acceleration of the point charge, and the second term, the *Coloumb part*, which does not depend on acceleration of the particle, falls off more rapidly than radiative part and hence, is also called *bound part*.

After a lengthy algebra, using the equations derived in subsection 2.4.1, it can be shown that

$$F^{\alpha\beta}{}_{,\beta} = 0 \tag{2.4.16}$$

away from the world line. These are the vacuum Maxwell's equations.

We can now calculate electromagnetic field's stress-energy tensor, $T_{em}^{\alpha\beta}$, given by

$$T_{em}^{\alpha\beta} = \frac{1}{4\pi}\left[F^{\alpha\mu}F^{\beta}_{\ \mu} - \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right] \tag{2.4.17}$$

Using the expression for $F^{\alpha\beta}$ in (2.4.15) in the above definition gives

$$T_{em}^{\alpha\beta} = \left[\frac{q^2}{4\pi r^2}\left(a^2 - a_k^2\right)k^\alpha k^\beta\right] + \left[\frac{q^2}{2\pi r^3}\left(a^{(\alpha}k^{\beta)} + a_k\left(u^{(\alpha}k^{\beta)} - k^\alpha k^\beta\right)\right)\right] + \left[\frac{q^2}{4\pi r^4}\left(2u^{(\alpha}k^{\beta)} - k^\alpha k^\beta + \frac{\eta^{\alpha\beta}}{2}\right)\right] \tag{2.4.18}$$

In the above expression, the first term is referred to as *radiative component*, $T_{rad}^{\alpha\beta}$, and the second and third terms combined is referred to as *bound or Coloumb component*, $T_{bnd}^{\alpha\beta}$. Therefore

$$T_{rad}^{\alpha\beta} = \left[\frac{q^2}{4\pi r^2}\left(a^2 - a_k^2\right)k^\alpha k^\beta\right]$$

and

$$T_{bnd}^{\alpha\beta} = \left[\frac{q^2}{2\pi r^3}\left(a^{(\alpha}k^{\beta)} + a_k\left(u^{(\alpha}k^{\beta)} - k^\alpha k^\beta\right)\right)\right] + \left[\frac{q^2}{4\pi r^4}\left(2u^{(\alpha}k^{\beta)} - k^\alpha k^\beta + \frac{\eta^{\alpha\beta}}{2}\right)\right]$$

Note that $a^2 = a^\alpha a_\alpha$. The above decomposition into radiative and bound parts is meaningful because each component is conserved separately, that is

$$\partial_\beta T_{rad}^{\alpha\beta} = 0, \quad \partial_\beta T_{bnd}^{\alpha\beta} = 0 \quad (r \neq 0) \tag{2.4.19}$$

These equations can be confirmed easily after a lengthy algebra using the equations derived in section 2.1. The motivation behind the name radiative is that it scales as r^{-2} and is proportional to $k^\alpha k^\beta$.

2.4.3 Radiation-Reaction force

After doing calculation similar to that of the retarded 4-potential, we get the advanced 4-potential

$$A_{adv}^{\alpha}(x) = q\frac{u^\alpha(v)}{r_{adv}(x)} \tag{2.4.20}$$

where $v(x)$ is the proper time of the particle when $z(\tau)$ intersects the *future light cone*. We shall call v as the *advanced time*. Similar to $r(x)$, $r_{adv}(x)$ is the *advanced distance* given by

$$r_{adv}(x) = -\eta_{\alpha\beta}\left[z^\alpha(v) - x^\alpha\right]u^\beta(v) \tag{2.4.21}$$

Before going further, we need to express all the advanced quantities in terms of retarded quantities like r. This is helpful because our point of interest is the behaviour of the field very close to the world-line. Also, for such close points, since r is small, we can express $\Delta\tau \equiv v - u$ and $r_{adv}(x)$ as a Taylor expansion in powers of r. It can be noticed that $\Delta\tau$ and r are of the same order of smallness.

We have

$$\begin{aligned} z^\alpha(v) &= z^\alpha(u) + \left.\frac{\partial z^\alpha(\tau)}{\partial\tau}\right|_{\tau=u}\Delta\tau + \frac{1}{2}\left.\frac{\partial^2 z^\alpha(\tau)}{\partial\tau^2}\right|_{\tau=u}\Delta\tau^2 + \frac{1}{6}\left.\frac{\partial^3 z^\alpha(\tau)}{\partial\tau^3}\right|_{\tau=u}\Delta\tau^3 \\ &+ \frac{1}{24}\left.\frac{\partial^4 z^\alpha(\tau)}{\partial\tau^4}\right|_{\tau=u}\Delta\tau^4 + O\left(\Delta\tau^5\right) \\ &= z^\alpha + u^\alpha\Delta\tau + \frac{1}{2}a^\alpha\Delta\tau^2 + \frac{1}{6}\dot{a}^\alpha\Delta\tau^3 + \frac{1}{24}\ddot{a}^\alpha\Delta\tau^4 + O\left(\Delta\tau^5\right) \end{aligned}$$

in which the terms on the right hand side are evaluated at the retarded time, u.

Substituting this in the relation $\sigma(x, z(v)) = 0$, and using $\sigma(x, z(u)) = 0$, gives the following

$$\begin{aligned} & \sigma(x, z(v)) &= 0 \\ \implies \quad & 2r\Delta\tau - [1 + ra_k]\Delta\tau^2 - \tfrac{r\dot{a}_k}{3}\Delta\tau^3 - \tfrac{1}{12}\left[a^2 + r\ddot{a}_k\right]\Delta\tau^4 + O\left(\Delta\tau^5\right) &= 0 \end{aligned}$$

where $\dot{a}_k = \dot{a}^\alpha k_\alpha$ and $\ddot{a}_k = \ddot{a}^\alpha k_\alpha$.

Assume that

$$\Delta\tau = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + O\left(r^4\right)$$

Substituting this into the previous relation and equating coefficients of different powers of r to 0, we get

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 2 \\ a_2 &= -2a_k \\ a_3 &= 2\left(a_k^2 - \frac{a^2}{3} - \frac{2}{3}\dot{a}_k\right) \end{aligned}$$

Hence,

$$\Delta\tau = 2r\left[1 - a_k r + \left(a_k^2 - \frac{a^2}{3} - \frac{2}{3}\dot{a}_k\right)r^2 + O\left(r^3\right)\right] \tag{2.4.22}$$

We substitute (2.4.22) into the expansion of $z^\alpha(v)$, and also into a similar expansion of $u^\alpha(v)$,

$$u^\alpha(v) = u^\alpha + a^\alpha\Delta\tau + \frac{1}{2}\dot{a}^\alpha\Delta\tau^2 + \frac{1}{6}\ddot{a}^\alpha\Delta\tau^3 + O\left(\Delta\tau^4\right)$$

and then we substitute the above expansions into (2.4.21),

$$\begin{aligned} r_{adv}(x) &= \eta_{\alpha\beta}\left[z^\alpha(v) - x^\alpha\right]u^\beta(v) \\ &= r + \frac{2}{3}\left(a^2 + \dot{a}_k\right)r^3 + O\left(r^4\right) \end{aligned} \tag{2.4.23}$$

Similarly we can express $u^\alpha(v)$ in terms of powers of r, we get

$$u^\alpha(v) = u^\alpha + 2a^\alpha r + 2\left[\dot{a}^\alpha - a_k a^\alpha\right]r^2 + O\left(r^3\right) \tag{2.4.24}$$

The advanced 4-potential is obtained by substituting (2.4.23) and (2.4.24) into (2.4.20)

$$\begin{aligned} A^{\alpha}_{adv}(x) &= q\frac{u^{\alpha}(v)}{r_{adv}(x)} \\ &= \frac{qu^{\alpha}}{r} + 2qa^{\alpha} + 2q\left[\dot{a}^{\alpha} - a_k a^{\alpha} - \frac{1}{3}\left(a^2 + \dot{a}_k\right)u^{\alpha}\right] r + O\left(r^2\right) \end{aligned} \tag{2.4.25}$$

The radiation-reaction potential is given by

$$\begin{aligned} A^{\alpha}_{rr}(x) &= \frac{1}{2}\left[A^{\alpha}_{ret}(x) - A^{\alpha}_{adv}(x)\right] \\ &= -qa^{\alpha} - q\left[\dot{a}^{\alpha} - a_k a^{\alpha} - \frac{1}{3}\left(a^2 + \dot{a}_k\right)u^{\alpha}\right] r + O\left(r^2\right) \end{aligned} \tag{2.4.26}$$

here, we consider only terms of order less than 2 because the higher order terms, after differentiation, give rise to terms which become 0 on the world-line.

After a lengthy but straight-forward algebra using relations derived in section 2.1, on the world-line, where $x^{\alpha} = z^{\alpha}$ or equivalently $r = 0$, we get

$$A^{rr}_{\alpha,\beta}(z) = \frac{q}{3}u_{\alpha}\dot{a}_{\beta} + q\dot{a}_{\alpha}u_{\beta}$$

and hence,

$$\begin{aligned} F^{rr}_{\alpha\beta}(z) &= A^{rr}_{\beta,\alpha}(z) - A^{rr}_{\alpha,\beta}(z) \\ &= -\frac{2}{3}q\left(\dot{a}_{\alpha}u_{\beta} - \dot{a}_{\beta}u_{\alpha}\right) \end{aligned} \tag{2.4.27}$$

The radiation reaction force is given by

$$\begin{aligned} F^{\alpha}_{rr} &= qF^{\alpha\beta}_{rr}(z)\,u_{\beta} \\ &= -\frac{2}{3}q^2 u_{\beta}\left(\dot{a}^{\alpha}u^{\beta} - \dot{a}^{\beta}u^{\alpha}\right) \\ &= \frac{2}{3}q^2\left(\dot{a}^{\alpha} + a^2 u^{\alpha}\right) \end{aligned}$$

or equivalently

$$F^{\alpha}_{rr} = \frac{2}{3}q^2\left(\delta^{\alpha}_{\ \beta} + u^{\alpha}u_{\beta}\right)\dot{a}^{\beta} \tag{2.4.28}$$

The equation of motion of the charged particle is therefore

$$ma^{\alpha} = F^{\alpha}_{ext} + F^{\alpha}_{rr} = F^{\alpha}_{ext} + \frac{2}{3}q^2\left(\delta^{\alpha}_{\ \beta} + u^{\alpha}u_{\beta}\right)\dot{a}^{\beta} \tag{2.4.29}$$

This is in fact the *Lorentz-Dirac equation*. Thus, the half-retarded minus half-advanced potential is indeed responsible for the radiation reaction because it gives correct covariant form of Lorentz-Dirac equation.

Define $\Omega_{\alpha\beta} \equiv a_{[\alpha}u_{\beta]}$, then

$$\dot{\Omega}_{\alpha\beta} = \dot{a}_{[\alpha}u_{\beta]} = \frac{1}{2}\left(\dot{a}_{\alpha}u_{\beta} - \dot{a}_{\beta}u_{\alpha}\right)$$

Therefore, the radiation-reaction field becomes

$$F^{rr}_{\alpha\beta}(z) = -\frac{4}{3}q\dot{\Omega}_{\alpha\beta}$$

2.5 Discussion

We can verify that the covariant form of radiation-reaction force indeed gives the correct expression for the force on slowly moving particles derived in the first section. For slowly moving particles we know that $u^\alpha \approx (1,0,0,0)$, which then implies

$$F^0_{rr} = \frac{2}{3}q^2\left(\delta^0{}_\beta + u^0 u_\beta\right)\dot{a}^\beta = \frac{2}{3}q^2\left(\dot{a}^0 - \dot{a}^0\right) = 0$$

and

$$\begin{aligned} F^i_{rr} &= \frac{2}{3}q^2\left(\delta^i{}_\beta + u^i u_\beta\right)\dot{a}^\beta = \frac{2}{3}q^2\left(\dot{a}^i - 0\right) = \frac{2}{3}q^2\dot{a}^i \\ \Longrightarrow \quad \vec{F}_{rr} &= \frac{2}{3}q^2\dot{\vec{a}} \end{aligned}$$

which is the radiation reaction force on slowly moving charged particle.

The correct form of Lorentz-Dirac equation also confirms that the radiation reaction force is indeed due to the half the difference of retarded and advanced potentials.

In this part, we have derived the radiation-reaction force in both non-relativistic and relativistic cases. We have compared the equations in both the cases and proved that Lorentz-Dirac equation reduces to the former case in the non-relativistic limit. The next step is literature survey of radiation reaction in the case of a uniformly accelerating charge and its consequences.

Chapter 3

Literature survey

3.1 Rohrlich's papers

This paper addresses a few questions about the applicability and consequences of self-force and radiation reaction in the special case of a uniformly accelerating charged particle. The questions raised are as follows:

1. There have been claims that uniformly accelerated charges do not radiate. But, from the radiation rate formula, it is widely known that accelerated charges *do* radiate. Is this formula always true?

2. Maxwell's equations are known to be conformal invariant. Conformal transformations is a more general class of transformations of which Lorentz transformations is only a subgroup. Conformal group also consists of transformation from an inertial frame to a uniformly accelerated frame. Since there is no radiation by a uniformly moving charge, by conformal invariance, it is thought that the charge does not radiate even in uniformly accelerated frames. If the charge does radiate where does the argument of conformal invariance breakdown?

3. The radiation reaction of a uniformly accelerating charge is zero. Does this imply that the charge does not radiate? If the charge does radiate, and since the radiation reaction force is zero, where does it get the necessary energy from? Does this radiation violate conservation of energy?

4. In case of gravitation, do neutral and charged particles, which are identical in every other aspect, undergo same motion in presence of a uniform gravitational field? If the charged particle does radiate, this can be used as a test to differentiate gravity-free regions from regions with gravitational fields. Does this violate principle of equivalence?

We discuss the problems stated above in detail and develop answers to all the questions raised.

3.1.1 Fields of a uniformly accelerating charge

The electromagnetic fields at $(t, \vec{r})$ of a uniformly accelerating charge at $(t', \vec{r}')$, with the magnitude of acceleration $\frac{1}{\alpha}$, are given by (Ref [3])

$$\begin{aligned} E_\phi^B &= 0 = H_\rho^B = H_z^B \\ E_z^B &= -4e\alpha^2 \frac{\left(\alpha^2 + t^2 + \rho^2 - z^2\right)}{\xi^3} \\ E_\rho^B &= 8e\alpha^2 \frac{\rho z}{\xi^3} \\ H_\phi^B &= 8e\alpha^2 \frac{\rho t}{\xi^3} \end{aligned} \tag{3.1.1}$$

where $\xi \equiv \left[\left(\alpha^2 + t^2 - \rho^2 - z^2\right)^2 + 4\alpha^2\rho^2\right]^{\frac{1}{2}}$. Here, we have used cylindrical coordinates ρ, ϕ, z. These fields were derived by Born (and hence the superscript B), who assumed that they held everywhere in the spacetime. But Schott pointed out that the equations held only in that region of spacetime where $z + t > 0$. This condition is a consequence of causality. The actual condition for causality requires

$$t - t' = R \equiv \left|\vec{r} - \vec{r}'\right| > 0.$$

Since the charge undergoes hyperbolic motion, we have, $z' = \sqrt{\alpha^2 + t'^2}$. Now, requiring that $R > 0$ implies that $z + t > 0$. Hence, the Schott solution is equivalent to Born solution except for the restriction to the spacetime region, $z + t > 0$.

As an aside, Bondi and Gold gave modifies expressions for the fields which extended their applicability to entire spacetime region. The expressions remain the same for the region $z + t > 0$ and zero for $z + t < 0$. The fields attain infinite values on the region $z + t = 0$ because these fields have been emitted by charges moving at the speed of light at $t = -\infty$. The final expressions which satisfies all these qualifications are as follows

$$\begin{aligned} E_\phi &= 0 = H_\rho = H_z \\ E_z &= -4e\alpha^2 \frac{\left(\alpha^2 + t^2 + \rho^2 - z^2\right)}{\xi^3} .\theta\left(z + t\right) \\ E_\rho &= 8e\alpha^2 \frac{\rho z}{\xi^3} .\theta\left(z + t\right) + \frac{2e\rho}{\rho^2 + \alpha^2} .\delta\left(z + t\right) \\ H_\phi &= 8e\alpha^2 \frac{\rho t}{\xi^3} .\theta\left(z + t\right) - \frac{2e\rho}{\rho^2 + \alpha^2} .\delta\left(z + t\right) \end{aligned} \tag{3.1.2}$$

These fields satisfy Maxwell's equations everywhere and contain the causality condition within them.

It is easy to see that the complete hyperbolic motion of the particle is invariant under Lorentz transformations. This implies that the fields have same form (form invariant) in all inertial frames. This form invariance is a consequence of conformal invariance of Born solution. But the causality condition restricts the symmetry of the solution of Maxwell's field equations to Lorentz transformations. In other words, Born solution is a sum of advanced and retarded fields which is conformal invariant but Schott solution is restricted to retarded fields which is invariant under only a subgroup of full conformal group, namely Lorentz group. Hence, conformal invariance is not a physical symmetry of the solution and any physical arguments based on conformal invariance do not yield sensible conclusions. This answers the second of the questions raised above.

3.1.2 Radiation rate of uniformly accelerating charge

The intensity of radiation emitted at time t' in the (space-like) direction n^μ which is orthogonal to the velocity of the charge at that time $v^\mu(\tau)$ is given by

$$I = T^{\mu\nu} v_\mu n_\nu \tag{3.1.3}$$

where τ is the propertime corresponding to t'. We define radiation rate as the total rate of radiation energy emitted by the charge at time t'. This is obtained by integrating I invariantly over a light sphere in the limit of infinite radius $R = t - t'$. That is

$$\mathcal{R} = \lim_{R\to\infty} \int T^{\mu\nu} v_\mu n_\nu d^2\sigma, \text{ for a fixed } t'$$

Observe that $R \to \infty$ and t' being fixed $\implies t \to \infty$ because the causality condition must still be valid. The expression for $\mathcal{R}$, in a most general motion, is given by

$$\mathcal{R} = \frac{2}{3} e^2 a_\mu a^\mu \tag{3.1.4}$$

where $a^\mu(\tau)$ is the four-acceleration of the particle. It can be observed that $\mathcal{R}$ is an invariant, which is expected since the integration is carried out over light sphere which is itself an invariant. In general, $\mathcal{R}$ is a function of source point, that is of the propertime τ. However, in the case of hyperbolic motion, it is evidently a constant and is independent of when and where the radiation has been emitted.

Using the fields (3.1.1) given in the previous subsection, along with causality condition, the intensity and radiation rate of a charged particle in hyperbolic motion are given by

$$\mathcal{R} = \frac{2}{3} \frac{e^2}{\alpha^2} \tag{3.1.5}$$

$$I = \frac{e^2 \alpha^4}{4\pi R^2} \frac{\sin^2\theta}{\left(\sqrt{\alpha^2 + t'^2} - t' \cos\theta\right)^6} \tag{3.1.6}$$

where $\cos\theta = \hat{z} \cdot \hat{n}$. Therefore, it can be seen that a uniformly accelerating point charge *does* radiate at a constant rate given by (3.1.5), with a radiation pattern given by (3.1.6). This answers the first question raised previously.

Since our definition of radiation is invariant under Lorentz transformations, every inertial observer sees the same radiation rate and intensity.

The radiation rate computed above holds for all values of t' such that $z + t > 0$ and this result is independent of the value of the fields in the region $z + t \leq 0$. Hence, the argument by Bondi and Gold that all the radiation is emitted at $t' = -\infty$ is unphysical and unnecessary.

3.1.3 Conservation of energy

The equation of motion of a charged particle with the inclusion of radiation reaction is given by

$$ma^{\mu} = F^{\mu}_{ext} + \Gamma^{\mu} \tag{3.1.7}$$

where

$$\Gamma^{\mu} = \frac{2}{3}e^2 \left(\frac{da^{\mu}}{d\tau} - v^{\mu} a_{\nu} a^{\nu} \right) \tag{3.1.8}$$

For a uniformly accelerating charge, it is easy to show that $\Gamma^{\mu} = 0$. This yields the equation of motion

$$ma^{\mu} = F^{\mu}_{ext}. \tag{3.1.9}$$

We know that the kinetic energy is given by $T = m(\gamma - 1)$. Hence the zeroth component of (3.1.9) implies

$$\frac{dT}{d\tau} = F^{0}_{ext} = \frac{dW_{ext}}{d\tau} \tag{3.1.10}$$

Hence, the change in kinetic energy of the particle is due to the work done by the external force and is independent of the charge of the particle. But we have seen in the previous subsection that the charge does radiate. This might seem to be in direct contradiction with the principle of conservation of energy. The resolution to this apparent contradiction is very simple.

Let us consider a general motion. In this case, the zeroth component of (3.1.7), after noting that $\frac{dt}{d\tau} = \gamma$, gives

$$\frac{dT}{dt} - \left(\frac{dQ}{dt} - \mathcal{R} \right) = \frac{dW_{ext}}{dt} \tag{3.1.11}$$

since $\Gamma^0 = \frac{2}{3}e^2 \left(\frac{da^0}{d\tau} - \gamma a^2 \right) = \left(\frac{dQ}{dt} - \mathcal{R} \right)$, where $Q = \frac{2}{3}e^2 a^0$. This equation tells that the work done by the external force equals the increase in kinetic energy of the particle minus the work done by radiation reaction, which has two parts - one which is always positive ($\mathcal{R} > 0$) and the other which can be either positive or negative or zero. In a general motion, the work done by radiation reaction force need not be zero. Part of this work goes out as radiation emitted by the charged particle and the remaining part goes into changing the value of Q. Hence, Q can be interpreted as a form of internal energy of the particle like kinetic energy. It has been termed as *acceleration energy* by Schott.

For periodic motions, averaging over long periods of time, the term $\frac{dQ}{dt}$ vanishes. Hence, on an average all the work done by radiation reaction goes out as radiation emitted by the charged particle. It might now be tempting to say that there is no radiation when radiation reaction is zero. However, this is not true because hyperbolic motion is a very special case. the work done by radiation reaction itself is zero. But $\frac{dQ}{dt}$ does not vanish as in periodic or bound motion. Hence, we have

$$\mathcal{R} = \frac{dQ}{dt} > 0$$

Considering $m - Q$ as the internal energy of the particle, we can interpret the above equation in an intuitive way. That is, the radiation emitted originates from the internal energy of the particle. This reduction in internal energy of accelerating charge does not change the rest mass of the particle as can be confirmed from the rest frame of the particle. In the rest frame, $a^0 = \gamma \vec{v} \cdot \vec{a} = 0$. Hence, $Q = 0$ and this is consistent with the physical picture. This answers the third question raised.

3.1.4 Principle of equivalence

It has been shown by Rohrlich (Ref [4]) that a charged particle falling freely in a homogeneous constant gravitational field undergoes hyperbolic motion. We have also seen in the previous subsections that charged and neutral particles follow same trajectories in the presence of a homogeneous constant gravitational field. But a charged particle in hyperbolic motion does radiate and hence, this radiation can be used to identify whether the particle is in an field-free region or in a gravitational field. This clearly seems to violate the principle of equivalence. However, by the definition of radiation rate given above, an observer can measure the radiation only at a very large distance from the source, that is, at $R \to \infty$, whereas principle of equivalence is a locally valid principle. Hence, an observer who tries to test the validity of principle of equivalence has to do so locally and hence cannot use radiation emitted by the source as a strategy. This clarifies the fourth question raised above.

3.2 Teitelboim's and Hirayama's papers

In the previous section, radiation has been defined at $R \to \infty$, also called the *wave zone*. But later, Rohrlich and Teitelboim derived the same radiation rate formula using another approach. In this approach, radiation can be identified at any arbitrary distance from the source. Radiation can then be pictured as something which tends to exist immediately after the emission of the fields by the source.

The important idea of this approach is identification of *bound* and *radiative* parts of electromagnetic energy momentum tensor. This splitting is well defined in the sense that both the parts are conserved separately everywhere off the worldline of the particle. There exists such a splitting even in Rindler frame which can be used to find the radiation rate formula by an observer fixed in a Rindler frame.

The bound part has a property that for a specific surface, the flux of the bound part across this surface is zero. This property was first identified and used by Rohrlich [5], who gave a description of this surface, and then developed by Teitelboim [6]. This property, along with a few other properties of the radiative part such as its flux being independent of direction of the hypersurface and the distance from the source, can be exploited to compute radiation rate at any arbitrary distance from the source.

3.2.1 Classical radiation in Lorentzian frame

In the previous section, the quantity $Q = \frac{2}{3}e^2 a^0$, which has been called acceleration energy, has been identified as a part of internal energy. Teitelboim [6], in his paper, has refined the idea of this term using the concept of *bound* and *radiative* parts of electromagnetic energy momentum tensor. After a rigorous derivation he arrives at the following equation of motion for a charged particle

$$ma^\mu - \frac{2}{3}e^2 \dot{a}^\mu = -\frac{2}{3}e^2 a^2 v^\mu + F^\mu_{ext}.$$

The interpretation of this equation goes as follows. The mass m consists of both *bare* and *electromagnetic* masses, the latter of which is a divergent term for point charge. The left hand side of the equation is identified as the rate of change of four-momentum, which is given by

$$p^\mu = mv^\mu - \frac{2}{3}e^2 a^\mu,$$

and the right hand side is the sum of all the forces - in this case, the radiation reaction force plus external force.

In the case of hyperbolic motion, $\dot{a}^\mu = a^2 v^\mu$. Hence, all the radiated energy is supplied by the bound electromagnetic energy, previously called acceleration energy by Schott. Also, all the work done by the external force goes into changing the mechanical energy of the particle in both charged and uncharged cases. This fact is clearly specific to the case of hyperbolic motion. In general, both mechanical and bound electromagnetic energies contribute to the radiation.

3.2.2 Classical radiation in Rindler frame

Hirayama [7] extended the work of Teitelboim to calculate the energy radiated by a uniformly charged particle as seen by an observer who is accelerating uniformly at a different rate. This is a very difficult task because there is no standard convention of defining acceleration of a body relative to a non-inertial observer. Hirayama obtained the formula

$$\mathcal{R}_{Rindler} = \frac{2}{3}e^2 \alpha^\mu \alpha_\mu \left(-u \cdot v\right)$$

where

$$\begin{aligned}
\alpha^\mu &= (\delta^\mu_\nu + v^\mu v_\nu)\left(a^\nu - g^\nu - (g \cdot g)^{\frac{1}{2}} u^\nu\right) \\
a^\mu &= \text{acceleration of the particle} \\
v^\mu &= \text{velocity of the particle} \\
g^\mu &= \text{acceleration of the observer fixed in the Rindler frame} \\
u^\mu &= \text{velocity of the observer fixed in the Rindler frame}
\end{aligned}$$

In the above formula, when the particle is instantaneously at rest in the Rindler frame, that is when $v^\mu = u^\mu$, then $\alpha^\mu = a^\mu - g^\mu$. Hence, α^μ can be interpreted as the relative acceleration of the particle relative to the observer fixed in the Rindler frame.

Observe that $\mathcal{R}_{Rindler}$ is identical to $\mathcal{R}$ when $g^\mu = 0$ and $v^\mu = u^\mu$, that is when the observer is an inertial observer comoving with the charge instantaneously. Also, $\mathcal{R}_{Rindler} = 0$ when $\alpha^\mu = 0$, that is when the observer coaccelerates with the charge he does not observe any radiation. This implies that an observer fixed in a static homogeneous gravitational field does not observe radiation from a charge fixed in the same field.

3.2.3 Principle of equivalence - revisited

Previously we argued that radiation cannot be used to test the validity of principle of equivalence because radiation was defined to be a wave zone phenomenon which is measured far away from the source whereas principle of equivalence is a locally valid principle. But with the new approach it might be argued that radiation can be measured at arbitrary distances from the source and hence can be used to test the validity of principle of equivalence.

The paradox is that for a charge fixed in a static homogeneous gravitational field, an observer falling freely in this field observes radiation from the charge whereas another observer fixed in this field observes no radiation. Hence, it can be argued that charged test particles can be used to differentiate field-free regions from real gravitational fields which is a clear violation of principle of equivalence.

This paradox can be resolved by noting that all the radiation observed by freely falling observer goes into that region of spacetime which is inaccessible to the co-accelerating observer. However, for this resolution to be valid, it is necessary, as pointed out by Rohrlich, that radiation be only Lorentz invariant instead of generally invariant.

Chapter 4

Bi-tensors - Green's functions in curved spacetime

4.1 Bi-tensors

4.1.1 Basic definitions

The definition of a vector is applicable only locally, in the sense that, we cannot compare two vectors which are defined at two points separated by a finite distance unless one of the vectors is parallel transported to the other point. But, frequently in physics, there appear quantities which are functions of two points and hence cannot be defined in the conventional way. Such quantities, whose arguments consist of n points, are called *n-tensors*. A special case of n-tensor is *bi-tensor* where the quantity is a function of two points in the spacetime. There are many examples of bi-tensors such as geodesic distance between two points, Green's function which is a function of source point and field point, Dirac delta function, etc. The local definition of vectors is sufficient to describe these quantities in flat spacetime. However, we need to be careful in defining these quantities in a curved spacetime. Most of the below discussed concepts on bi-tensors can be found in [8].

The simplest example of a bi-tensor is product of two vectors defined at two points, that is

$$C^{\alpha}_{\alpha'}\left(x,x'\right) = A^{\alpha}\left(x\right) B_{\alpha'}\left(x'\right).$$

Observe that vectors defined at x have Greeks indices without primes and vectors defined at x' have Greek indices with primes. This notation will be followed throughout this report.

The coordinate transformation law for the bi-tensor is given by

$$\bar{C}^{\alpha}_{\alpha'}\left(\bar{x},\bar{x}'\right) = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} \frac{\partial x'^{\beta'}}{\partial \bar{x}'^{\alpha'}} C^{\beta}_{\beta'}\left(x,x'\right),$$

which can be extended to any bi-tensor of higher or lower rank. Similarly, contraction of indices can be done provided it is performed over indices referring to same point. Covariant derivative with respect to a variable can also be defined in the usual sense, where indices corresponding to other variables should be ignored. Following the usual convention, we have

$$C^{\alpha}_{\alpha';\beta} = C^{\alpha}_{\alpha',\beta} + \Gamma_{\gamma\beta}{}^{\alpha}\, C^{\gamma}_{\alpha'}$$

and

$$C^{\alpha}_{\alpha';\beta'} = C^{\alpha}_{\alpha',\beta'} - \Gamma_{\alpha'\beta'}{}^{\gamma'}\, C^{\alpha}_{\gamma'}$$

where it is understood that prime or no-prime on the indices denote the point where a quantity is being evaluated. Also, one can note that indices referring to different points commute. Hence, successive covariant derivatives at different points commute whereas successive covariant derivatives at the same point involve Riemann tensor in their commutation relations.

4.1.2 Bi-scalar of geodetic interval

One of the fundamental bi-scalars in the study of nonlocal behaviour of spacetime is the *bi-scalar of geodetic interval* denoted by $s(x, x')$ (A bi-scalar is a bi-tensor with no indices). It is the magnitude of invariant spacetime distance between x and x' along a geodesic joining them. It is defined by the following equations

$$g^{\alpha\beta} s_{;\alpha} s_{;\beta} = g^{\alpha'\beta'} s_{;\alpha'} s_{;\beta'} = \pm 1 \tag{4.1.1}$$

and

$$\lim_{x \to x'} s = 0 \tag{4.1.2}$$

where the $+$ sign holds when the separation between x and x' is space-like and $-$ sign holds when it is time-like. All points x for which $s = 0$ constitute the light cone of x'. There can be more than one geodesic which connect the two points x and x' and hence s is usually a multi-valued function. But when we confine ourselves to a region close enough to x', all the points x within this region have a unique geodesic connecting them to x'. Such a region is called *convex neighbourhood of* x', denoted by $\mathcal{N}(x')$. Throughout this report, we study the behaviour of quantities in this region.

4.1.3 Synge's world function

In most problems in physics, it is more convenient to work with half the square of the invariant distance between two spacetime points. This object is called *Synge's world function,* denoted by $\sigma(x, x')$. It is given by

$$\sigma(x, x') = \pm \frac{1}{2} s^2$$

and is positive for space-like intervals and negative for time-like intervals. From (4.1.1) and (4.1.2), we have

$$g^{\alpha\beta} \sigma_{;\alpha} \sigma_{;\beta} = g^{\alpha'\beta'} \sigma_{;\alpha'} \sigma_{;\beta'} = 2\sigma \tag{4.1.3}$$

and

$$\lim_{x \to x'} \sigma = 0 \tag{4.1.4}$$

Hereafter, we define the notation $\sigma_\alpha \equiv \sigma_{;\alpha}$ and so on. Also, we re-express the equation (4.1.4) as

$$[\sigma] = 0$$

where the square brackets denote the limit $x \to x'$. Such a limit is called coincidence limit.

4.1.4 Coincidence limits

In order to define a covariant Taylor expansion of a general bi-tensor around a point x we need higher derivatives of σ. Applying repeated differentiation of (4.1.3), we get

$$\sigma_\gamma = g^{\alpha\beta}\sigma_\alpha\sigma_{\beta\gamma} \tag{4.1.5}$$

$$\sigma_{\gamma\delta} = g^{\alpha\beta}\left(\sigma_{\alpha\delta}\sigma_{\beta\gamma} + \sigma_\alpha\sigma_{\beta\gamma\delta}\right) \tag{4.1.6}$$

$$\sigma_{\gamma\delta\epsilon} = g^{\alpha\beta}\left(\sigma_{\alpha\delta\epsilon}\sigma_{\beta\gamma} + \sigma_{\alpha\delta}\sigma_{\beta\gamma\epsilon} + \sigma_{\alpha\epsilon}\sigma_{\beta\gamma\delta} + \sigma_\alpha\sigma_{\beta\gamma\delta\epsilon}\right) \tag{4.1.7}$$

$$\begin{aligned}\sigma_{\gamma\delta\epsilon\zeta} = g^{\alpha\beta}\big(&\sigma_{\alpha\delta\epsilon\zeta}\sigma_{\beta\gamma} + \sigma_{\alpha\delta\epsilon}\sigma_{\beta\gamma\zeta} + \sigma_{\alpha\delta\zeta}\sigma_{\beta\gamma\epsilon} + \sigma_{\alpha\delta}\sigma_{\beta\gamma\epsilon\zeta} \\ &+\sigma_{\alpha\epsilon\zeta}\sigma_{\beta\gamma\delta} + \sigma_{\alpha\zeta}\sigma_{\beta\gamma\delta\epsilon} + \sigma_\alpha\sigma_{\beta\gamma\delta\epsilon\zeta}\big)\end{aligned} \tag{4.1.8}$$

From (4.1.3), we can see that

$$[\sigma_\alpha] = 0 \tag{4.1.9}$$

Using (4.1.5), we get

$$\sigma^\beta\left(g_{\beta\gamma} - \sigma_{\beta\gamma}\right) = 0 \implies [\sigma_{\alpha\beta}] = g_{\alpha'\beta'} \tag{4.1.10}$$

From (4.1.7), using (4.1.9) and (4.1.10), taking the coincidence limit, we get

$$[\sigma_{\gamma\delta\epsilon}] = [\sigma_{\gamma\delta\epsilon}] + [\sigma_{\delta\gamma\epsilon}] + [\sigma_{\epsilon\gamma\delta}]\,.$$

Using the result $\sigma_{\delta\gamma\epsilon} + \sigma_{\epsilon\gamma\delta} = 2\sigma_{\gamma\delta\epsilon} - R_{\epsilon\delta\gamma}{}^{\zeta}\,\sigma_\zeta$, we obtain, after taking coincidence limit,

$$[\sigma_{\alpha\beta\gamma}] = 0 \tag{4.1.11}$$

From (4.1.8), in coincidence limit, using previous results, we have,

$$[\sigma_{\gamma\delta\epsilon\zeta}] = [\sigma_{\gamma\delta\epsilon\zeta}] + [\sigma_{\delta\gamma\epsilon\zeta}] + [\sigma_{\epsilon\gamma\delta\zeta}] + [\sigma_{\zeta\gamma\delta\epsilon}]$$

Using the result $\sigma_{\delta\gamma\epsilon\zeta} + \sigma_{\epsilon\gamma\delta\zeta} + \sigma_{\zeta\gamma\delta\epsilon} = 3\sigma_{\gamma\delta\epsilon\zeta} - \left(R_{\epsilon\gamma\delta}{}^{\eta}\,\sigma_\eta\right)_{;\zeta} - \left(R_{\zeta\gamma\delta}{}^{\eta}\,\sigma_\eta\right)_{;\epsilon} - R_{\zeta\epsilon\gamma}{}^{\eta}\,\sigma_{\eta\delta} - R_{\zeta\epsilon\delta}{}^{\eta}\,\sigma_{\gamma\eta}$, we get

$$[\sigma_{\alpha\beta\delta\zeta}] = -\frac{1}{3}\left(R_{\alpha'\gamma'\beta'\delta'} + R_{\alpha'\delta'\beta'\gamma'}\right) \tag{4.1.12}$$

The other combinations of derivatives such as $\left[\sigma_{\alpha\beta\gamma'\delta'}\right]$ can be similarly obtained. These can be found in (*poisson review)

4.1.5 Covariant Taylor expansion

A bi-tensor, $T_{\alpha'\beta'}$, whose indices all refer to the same point and which is sufficiently differentiable, can be expanded around the point x' in the following way (for a more rigorous definition, refer Section 1 of [8])

$$T_{\alpha'\beta'} = A_{\alpha'\beta'} + A_{\alpha'\beta'}{}^{\gamma'}\,\sigma_{\gamma'} + \frac{1}{2}\,A_{\alpha'\beta'}{}^{\gamma'\delta'}\,\sigma_{\gamma'}\sigma_{\delta'} + O\left(s^3\right) \tag{4.1.13}$$

Using the results derived in the previous subsection, we have

$$\begin{aligned} A_{\alpha'\beta'} &= \left[T_{\alpha'\beta'}\right] \\ A_{\alpha'\beta'\gamma'} &= \left[T_{\alpha'\beta';\gamma'}\right] - A_{\alpha'\beta';\gamma'} \\ A_{\alpha'\beta'\gamma'\delta'} &= \left[T_{\alpha'\beta';\gamma'\delta'}\right] - A_{\alpha'\beta';\gamma'\delta'} - A_{\alpha'\beta'\gamma';\delta'} - A_{\alpha'\beta'\delta';\gamma'} \end{aligned}$$

Applying these results in the expansion of $\sigma_{\alpha'\beta'}$, we get the following expressions

$$\sigma_{\alpha'\beta'} = g_{\alpha'\beta'} - \frac{1}{3}\;R_{\alpha'}{}^{\gamma'}{}_{\beta'}{}^{\delta'}\;\sigma_{\gamma'}\sigma_{\delta'} + O\left(s^3\right) \tag{4.1.14}$$

$$\sigma_{\alpha'\beta'\gamma'} = -\frac{1}{3}\left(R_{\alpha'\gamma'\beta'}{}^{\delta'} +\; R_{\alpha'}{}^{\delta'}{}_{\beta'\gamma'}\right)\sigma_{\delta'} + O\left(s^2\right) \tag{4.1.15}$$

$$\sigma_{\alpha'\beta'\gamma'\delta'} = -\frac{1}{3}\left(R_{\alpha'\gamma'\beta'\delta'} + R_{\alpha'\delta'\beta'\gamma'}\right) + O\left(s\right) \tag{4.1.16}$$

This procedure of expanding a tensor around a point x' in terms of first derivatives of σ is applicable only to tensors whose indices all refer to the same point. For bi-tensors, we need a way to *bring* all the indices to same point before expanding. We define a new bi-tensor from the old bi-tensor whose indices all refer to same point. Then we can use above results.

4.1.6 Bi-vector of geodetic parallel displacement

The new bi-tensor with all indices referring to same point is obtained by *changing* the indices which correspond to different point to this same point. This can be achieved in a most natural way using an object of the form $g^{\alpha}{}_{\alpha'}\left(x, x'\right)$. This object is called *bi-vector of geodetic parallel displacement.* It is determined by the following defining equations

$$g_{\alpha\alpha';\beta}g^{\beta\gamma}\sigma_{\gamma} = 0, \quad g_{\alpha\alpha';\beta'}g^{\beta'\gamma'}\sigma_{\gamma'} = 0 \text{ and } \left[g^{\alpha}{}_{\beta'}\right] = \delta^{\alpha}{}_{\beta} \tag{4.1.17}$$

By definition, applying $g^{\alpha}{}_{\alpha'}$ to a vector $A_{\alpha}\left(x\right)$ outputs the vector $A_{\alpha'}\left(x'\right)$ which is the vector obtained by parallel transporting $A_{\alpha}\left(x\right)$ along the geodesic joining the two point x and x'. Therefore, we have the following relations,

$$g^{\alpha}{}_{\alpha'}\;g^{\beta}{}_{\beta'}\,g_{\alpha\beta} = g_{\alpha'\beta'}, \quad g_{\alpha}{}^{\alpha'}\;g_{\beta}{}^{\beta'}\;g_{\alpha'\beta'} = g_{\alpha\beta}, \tag{4.1.18}$$

$$g^{\alpha}{}_{\alpha'}\,\sigma_{\alpha} = -\sigma_{\alpha'}, \quad g_{\alpha}{}^{\alpha'}\,\sigma_{\alpha'} = -\sigma_{\alpha} \tag{4.1.19}$$

and

$$g_{\alpha\alpha'}g^{\alpha\beta'} = \delta^{\beta'}_{\alpha'}, \quad g_{\alpha\alpha'}g^{\beta\alpha'} = \delta^{\beta}_{\alpha} \tag{4.1.20}$$

The derivatives of $g^{\alpha}{}_{\alpha'}$ will be useful in expanding an arbitrary bi-tensor around the point x'. Differentiating (4.1.17) successively with respect to the point x' gives

$$0 = g_{\alpha\alpha';\beta'\delta'} g^{\beta'\gamma'} \sigma_{\gamma'} + g_{\alpha\alpha';\beta'} g^{\beta'\gamma'} \sigma_{\gamma'\delta'} \tag{4.1.21}$$

$$0 = g_{\alpha\alpha';\beta'\delta'\epsilon'} g^{\beta'\gamma'} \sigma_{\gamma'} + g_{\alpha\alpha';\beta'\delta'} g^{\beta'\gamma'} \sigma_{\gamma'\epsilon'} + g_{\alpha\alpha';\beta'\epsilon'} g^{\beta'\gamma'} \sigma_{\gamma'\delta'} + g_{\alpha\alpha';\beta'} g^{\beta'\gamma'} \sigma_{\gamma'\delta'\epsilon'} \tag{4.1.22}$$

In the coincidence limit, using $g_{\alpha\alpha';\epsilon'\delta'} = g_{\alpha\alpha';\delta'\epsilon'} - R_{\epsilon'\delta'\alpha'}{}^{\gamma'} g_{\alpha\gamma'}$, we get

$$\left[g_{\alpha\beta';\gamma'}\right] = 0 \tag{4.1.23}$$

$$\left[g_{\alpha\beta';\gamma'\delta'}\right] = \frac{1}{2} R_{\alpha'\beta'\gamma'\delta'} \tag{4.1.24}$$

4.1.7 Covariant Taylor expansion (continued)

Consider a bi-tensor with two indices referring to point x and x'. It can be denoted as $T_{\alpha\beta'}$. Define a new bi-tensor

$$\bar{T}_{\alpha'\beta'} = g^{\alpha}{}_{\alpha'} T_{\alpha\beta'}.$$

This is a tensor since both the indices refer to the same point. Hence, we can use the results of expansion give in previous subsection. Differentiating this equation, applying the coincidence limits and using equations (4.1.23) and (4.1.24) gives

$$\left[\bar{T}_{\alpha'\beta';\gamma'}\right] = \left[g^{\alpha}{}_{\alpha'} T_{\alpha\beta';\gamma'}\right] \tag{4.1.25}$$

$$\left[\bar{T}_{\alpha'\beta';\gamma'\delta'}\right] = \left[g^{\alpha}{}_{\alpha'} T_{\alpha\beta';\gamma'\delta'}\right] + \left[\frac{1}{2} R_{\gamma'\delta'\alpha'}{}^{\epsilon'} T_{\epsilon'\beta'}\right] \tag{4.1.26}$$

4.1.8 van Vleck determinant

The van Vleck determinant is defined as

$$\Delta\left(x, x'\right) = \det\left[\Delta^{\alpha'}{}_{\beta'}\left(x, x'\right)\right], \tag{4.1.27}$$

$$\Delta^{\alpha'}{}_{\beta'}\left(x, x'\right) = -\, g^{\alpha'}{}_{\alpha}\left(x, x'\right) \sigma^{\alpha}{}_{\beta'}\left(x, x'\right) \tag{4.1.28}$$

It can be shown that

$$\Delta\left(x, x'\right) = -\frac{\det\left[-\sigma_{\alpha\beta'}\left(x, x'\right)\right]}{\sqrt{-g}\sqrt{-g'}}$$

where g and g' are the determinants of the metric tensor at x and x' respectively. Using (4.1.10) and (4.1.17), it can also be shown that in the coincidence limit

$$\left[\Delta^{\alpha'}{}_{\beta'}\right] = \delta^{\alpha'}{}_{\beta'}, \quad [\Delta] = 1$$

Using the covariant Taylor expansion defined above, we get, near the coincidence,

$$\Delta^{\alpha'}{}_{\beta'} = \delta^{\alpha'}{}_{\beta'} + \frac{1}{6} R^{\alpha'}{}_{\gamma'\beta'\delta'} \sigma^{\gamma'} \sigma^{\delta'} + O\left(s^3\right) \tag{4.1.29}$$

and hence,

$$\Delta = 1 + \frac{1}{6} R_{\gamma'\delta'} \sigma^{\gamma'} \sigma^{\delta'} + O\left(s^3\right) \tag{4.1.30}$$

where we have used the fact that for a "small" matrix, $\mathbf{a}$, we have $\det\left(\mathbf{1}+\mathbf{a}\right) = 1 + \mathrm{tr}\left(\mathbf{a}\right) + O\left(\mathbf{a}^2\right)$.

Differentiating (4.1.3) twice, we get

$$\sigma_{\alpha\beta'} = \sigma^{\gamma}{}_{\alpha}\, \sigma_{\gamma\beta'} + \sigma^{\gamma} \sigma_{\gamma\alpha\beta'}.$$

Multiplying by $-g^{\alpha'\alpha}$, we obtain

$$\Delta^{\alpha'}{}_{\beta'} = g^{\alpha'}{}_{\alpha}\, g^{\gamma'}{}_{\gamma}\, \sigma^{\alpha}{}_{\gamma}\, \Delta^{\gamma'}{}_{\beta'} + \Delta^{\alpha'}{}_{\beta';\gamma}\, \sigma^{\gamma}$$

After multiplying both by the inverse of van Vleck bi-tensor, we find that

$$\delta^{\alpha'}{}_{\beta'} = g^{\alpha'}{}_{\alpha}\, g^{\beta}{}_{\beta'}\, \sigma^{\alpha}{}_{\beta} + \left(\Delta^{-1}\right)^{\gamma'}{}_{\beta'}\, \Delta^{\alpha'}{}_{\gamma';\gamma}\, \sigma^{\gamma}$$

whose trace yields the differential equation

$$4 = \sigma^{\alpha}{}_{\alpha} + \left(\ln \Delta\right)_{,\alpha} \sigma^{\alpha} \tag{4.1.31}$$

where we have used the fact that $\delta \ln\left(\det\left(\mathbf{M}\right)\right) = \mathrm{tr}\left(\mathbf{M}^{-1}\delta\mathbf{M}\right)$. This equation can also be written as

$$\Delta^{-1}\left(\Delta\sigma^{\alpha}\right)_{;\alpha} = 4 \tag{4.1.32}$$

4.2 Green's functions in curved spacetime

4.2.1 Elementary solution using Hadamard approach

In this section, following Hadamard's method, we find an *elementary solution* to homogeneous covariant scalar wave equation, which will then be used to obtain Green's functions for inhomogeneous covariant scalar wave equation (refer Section 2 of [8]). The covariant scalar wave equation is given by

$$g^{\alpha\beta}\phi_{;\alpha\beta} = 0 \tag{4.2.1}$$

Using Hadamard's approach, we try to find an "elementary solution" to the above equation, which is a bi-scalar of the form

$$G^{(1)} = \frac{1}{(2\pi)^2}\left(\frac{U}{\sigma} + V \ln|\sigma| + W\right) \tag{4.2.2}$$

where U, V and W are bi-scalars which are regular everywhere. Also, U satisfies a normalization condition given by

$$[U] = 1 \tag{4.2.3}$$

Substituting $G^{(1)}$ into the wave equation, using (4.1.3) and (4.1.32), we get

$$\begin{aligned}(2\pi)^2 g^{\alpha\beta} G^{(1)}_{;\alpha\beta} = 0 = & -\frac{1}{\sigma^2} g^{\alpha\beta}\left(2U_{;\alpha} - U\Delta^{-1}\Delta_{;\alpha}\right)\sigma_\beta \\ & + \frac{1}{\sigma}\left[2V + g^{\alpha\beta}\left(2V_{;\alpha} - V\Delta^{-1}\Delta_{;\alpha}\right)\sigma_\beta + g^{\alpha\beta}U_{;\alpha\beta}\right] \\ & + g^{\alpha\beta}V_{;\alpha\beta}\ln|\sigma| + g^{\alpha\beta}W_{;\alpha\beta}\end{aligned} \tag{4.2.4}$$

In order for this expression to vanish everywhere, the coefficient of the logarithmic term must vanish everywhere, and coefficieents of the singular terms must vanish at least on the light cone while the last term can take care off the light cone. This implies

$$g^{\alpha\beta}\left(2U_{;\alpha} - U\Delta^{-1}\Delta_{;\alpha}\right)\sigma_\beta = 0 \tag{4.2.5}$$

$$g^{\alpha\beta}V_{;\alpha\beta} = 0 \tag{4.2.6}$$

We assume a power series expansion for v and w in terms of σ

$$V = \sum_{n=0}^{\infty} v_n \sigma^n, \quad W = \sum_{n=0}^{\infty} w_n \sigma^n \tag{4.2.7}$$

Substituting the power series for v into (4.2.6) and using (4.1.3) and (4.1.32), we arrive the following recurrence relation for v_n

$$v_0 + g^{\alpha\beta}\left(v_{0;\alpha} - \frac{1}{2}v_0\Delta^{-1}\Delta_{;\alpha}\right)\sigma_\beta = -\frac{1}{2}g^{\alpha\beta}U_{;\alpha\beta} \tag{4.2.8}$$

$$v_n + \frac{1}{n+1}g^{\alpha\beta}\left(v_{n;\alpha} - \frac{1}{2}v_n\Delta^{-1}\Delta_{;\alpha}\right)\sigma_\beta = -\frac{1}{2n(n+1)}g^{\alpha\beta}v_{n-1;\alpha\beta} \tag{4.2.9}$$

Similarly, substituting the power series for w into

$$2V + g^{\alpha\beta}\left(2V_{;\alpha} - V\Delta^{-1}\Delta_{;\alpha}\right)\sigma_\beta + g^{\alpha\beta}U_{;\alpha\beta} + \sigma g^{\alpha\beta}W_{;\alpha\beta} = 0 \tag{4.2.10}$$

which is obtained from (4.2.4), we get the following recurrence relation for w_n

$$\begin{aligned} w_n + \frac{1}{n+1}g^{\alpha\beta}\left(w_{n;\alpha} - \frac{1}{2}w_n\Delta^{-1}\Delta_{;\alpha}\right)\sigma_\beta = & -\frac{1}{2n(n+1)}g^{\alpha\beta}w_{n-1;\alpha\beta} - \frac{1}{n+1}v_n \\ & + \frac{1}{2n^2(n+1)}g^{\alpha\beta}v_{n-1;\alpha\beta}\end{aligned} \tag{4.2.11}$$

These equations hold for all $n = 1, 2, \ldots, n$. All the v_n can be found uniquely by integrating the above equations but there is an arbitrariness in w_n's since w_0 still remains arbitrary. This arbitrariness is expected since $G^{(1)}$ is not a unique solution to the homogeneous wave equation and any "singularity-free" solution can be added to it.

The equation (4.2.5) holds everywhere and on every geodesic emanating from x'. Hence, that equation is equivalent to the following equation

$$U^{-1}U_{;\alpha} = \frac{1}{2}\Delta^{-1}\Delta_{;\alpha}$$

which on integrating, using the initial condition (4.2.3), gives the following solution

$$U = \Delta^{\frac{1}{2}} \tag{4.2.12}$$

Substituting this solution into the equation (4.2.8) and using the expansion (4.1.30), we find that

$$[V] = [v_0] = \frac{1}{12}R \tag{4.2.13}$$

4.2.2 Green's function for inhomogeneous wave equation

Now we introduce *Feynman propagator*, and move the elementary solution to the complex plane. Thus, we have (refer Section 2 of [8]),

$$G^F = G^{(1)} - 2iG$$

where G is the sought after Green's function for the inhomogeneous wave equation.. Using the well known identities, which can be obtained by assuming a limiting Cauchy distribution for δ-function,

$$\frac{1}{\sigma + i\epsilon} = \mathcal{P}\left(\frac{1}{\sigma}\right) - i\pi\delta\left(\sigma\right)$$

and

$$\ln\left(\sigma + i\epsilon\right) = \ln|\sigma| + i\pi\theta\left(-\sigma\right)$$

we can obtain the *symmetric* Green's function

$$G = \frac{1}{8\pi}\left(\Delta^{\frac{1}{2}}\delta\left(\sigma\right) - V\theta\left(-\sigma\right)\right) \tag{4.2.14}$$

One can immediately see that G is independent of W and is hence unique unlike $G^{(1)}$. Secondly, G vanishes for space-like separation of x and x'. Finally, it has support on the light-cone like the Green's functions of flat spacetime. However, it is non-zero even inside the light-cone which is not the case in flat spacetime. The bi-scalar V hence represents the *tail* term of the Green's function.

4.2.3 Advanced distributional methods

In this subsection, we introduce some advanced distributional methods which will be useful in verifying that G is indeed a solution to inhomogeneous wave equation. Let $\theta_+\left(x,\Sigma\right)$ be a generalized step function, defined to be one when x is in the future of the spacelike hypersurface Σ and zero otherwise. Similarly, define $\theta_-\left(x,\Sigma\right) = 1 - \theta_+\left(x,\Sigma\right)$ to be one when x is in the past of the hypersurface Σ and zero otherwise (for a more intuitive discussion on the *generalized step function*, refer Section 2 of [8] and Part III, Section 12.5 of [9]). Now define *light-cone step functions,*

$$\theta_{\pm}\left(-\sigma\right)=\theta_{\pm}\left(x,\Sigma\right)\theta\left(-\sigma\right),\ \ x'\in\Sigma \tag{4.2.15}$$

where $\theta_{+}\left(-\sigma\right)$ is one when x is in chronological future of x' and zero otherwise. Similarly, the other step function. Define similarly, *light-cone Dirac delta fucntionals,*

$$\delta_{\pm}\left(-\sigma\right)=\theta_{\pm}\left(x,\Sigma\right)\delta\left(-\sigma\right),\ \ x'\in\Sigma \tag{4.2.16}$$

These light-cone functions cannot be differentiated at $x=x'$. Hence, we shift σ by a small positive quantity ϵ. Note that the equation $\sigma+\epsilon=0$ defines two hyperboloids just inside the light-cone of x', one in past cone and another in future cone. The light-cone step functions can now be differentiated without any pathologies because each of the signs, $\pm$, *selects* a hyperboloid in future and past of x' respectively making the limiting process smooth. Hence, we have

$$\theta'_{\pm}\left(-\sigma-\epsilon\right)=\theta_{\pm}\left(x,\Sigma\right)\theta'\left(-\sigma-\epsilon\right)=-\theta_{\pm}\left(x,\Sigma\right)\delta\left(\sigma+\epsilon\right)=-\delta_{\pm}\left(\sigma+\epsilon\right) \tag{4.2.17}$$

We also need derivatives of $\delta_{\pm}\left(\sigma\right)$ in order to proceed further. For this purpose we shall rely on the distributional identity

$$\lim_{\epsilon\to0^{+}}\frac{\epsilon}{R^{5}}=\frac{2\pi}{3}\delta^{(3)}\left(\mathbf{x}\right) \tag{4.2.18}$$

where $R=\sqrt{r^{2}+2\epsilon}$ and $r=|\mathbf{x}|$. This follows from another identity $\nabla^{2}\left(\frac{1}{r}\right)=-4\pi\delta^{(3)}\left(\mathbf{x}\right)$, where we replace $\nabla^{2}\left(\frac{1}{r}\right)$ by $\lim_{\epsilon\to0^{+}}\nabla^{2}\left(\frac{1}{R}\right)=-\lim_{\epsilon\to0^{+}}\frac{6\epsilon}{R^{5}}$.

Using $2\left(\sigma+\epsilon\right)=-t^{2}+r^{2}+2\epsilon=-\left(t+R\right)\left(t-R\right)$, we get

$$\delta_{\pm}\left(\sigma+\epsilon\right)=\frac{\delta\left(t\mp R\right)}{R}$$

Using this result, we consider the following functionals

$$\begin{aligned}A_{\pm}\left[f\right]&=\lim_{\epsilon\to0^{+}}\int\epsilon\delta_{\pm}\left(\sigma+\epsilon\right)f\left(x\right)d^{4}x\\&=\lim_{\epsilon\to0^{+}}\int\epsilon\frac{f\left(\pm R,\mathbf{x}\right)}{R}d^{3}x\\&=\frac{2\pi}{3}\int\delta^{(3)}\left(\mathbf{x}\right)r^{4}f\left(\pm r,\mathbf{x}\right)d^{3}x\\&=0\end{aligned}$$

$$\begin{aligned} B_{\pm}[f] &= \lim_{\epsilon\to 0^+} \int \epsilon\delta'_{\pm}(\sigma+\epsilon) f(x)\, d^4x \\ &= \lim_{\epsilon\to 0^+} \epsilon \frac{d}{d\epsilon} \int \frac{f(\pm R, \mathbf{x})}{R} d^3x \\ &= \lim_{\epsilon\to 0^+} \epsilon \int \left(\pm \frac{\dot{f}}{R^2} - \frac{f}{R^3} \right) d^3x \\ &= \frac{2\pi}{3} \int \delta^{(3)}(\mathbf{x}) \left(\pm r^3 \dot{f} - r^2 f \right) d^3x \\ &= 0 \end{aligned}$$

$$\begin{aligned} C_{\pm}[f] &= \lim_{\epsilon\to 0^+} \int \epsilon\delta''_{\pm}(\sigma+\epsilon) f(x)\, d^4x \\ &= \lim_{\epsilon\to 0^+} \epsilon \frac{d^2}{d\epsilon^2} \int \frac{f(\pm R, \mathbf{x})}{R} d^3x \\ &= \lim_{\epsilon\to 0^+} \epsilon \int \left(\frac{\ddot{f}}{R^3} \mp 3\frac{\dot{f}}{R^4} - 3\frac{f}{R^5} \right) d^3x \\ &= 2\pi \int \delta^{(3)}(\mathbf{x}) \left(\frac{1}{3} r^2 \ddot{f} \pm r\dot{f} + r^2 f \right) d^3x \\ &= 2\pi f(0, \mathbf{0}) \end{aligned}$$

Using these results we can establish the following distributional identities

$$\lim_{\epsilon\to 0^+} \epsilon\delta_{\pm}(\sigma+\epsilon) = 0 \tag{4.2.19}$$

$$\lim_{\epsilon\to 0^+} \epsilon\delta'_{\pm}(\sigma+\epsilon) = 0 \tag{4.2.20}$$

$$\lim_{\epsilon\to 0^+} \epsilon\delta''_{\pm}(\sigma+\epsilon) = 2\pi\delta^{(4)}(x-x') \tag{4.2.21}$$

Notice that these results hold only in flat spacetime.

4.2.4 Invariant Dirac distribution

The *invariant Dirac distribution* is defined by the following equations

$$\int f(x)\, \delta_4(x,x') \sqrt{-g} d^4x = f(x') \quad \text{and} \quad \int f(x')\, \delta_4(x,x') \sqrt{-g'} d^4x' = f(x)$$

It is easy to see that this distribution is symmetric and can be expressed as

$$\delta_4(x,x') = \frac{\delta^{(4)}(x-x')}{\sqrt{-g}} = \frac{\delta^{(4)}(x-x')}{\sqrt{-g'}} = (gg')^{-\frac{1}{4}} \delta^{(4)}(x-x') \tag{4.2.22}$$

Using exactly same method as in the previous subsection, it can be shown that

$$\lim_{\epsilon\to 0^+} \epsilon\delta_\pm(\sigma+\epsilon) = 0 \tag{4.2.23}$$

$$\lim_{\epsilon\to 0^+} \epsilon\delta'_\pm(\sigma+\epsilon) = 0 \tag{4.2.24}$$

$$\lim_{\epsilon\to 0^+} \epsilon\delta''_\pm(\sigma+\epsilon) = 2\pi\delta_4(x,x') \tag{4.2.25}$$

To prove these relations we argue that the above equations are scalar equations and hence should hold in any coordinate system. Hence, in curved spacetime, doing the analysis in *Riemann normal coordinates* is preferred since working in these coordinates is equivalent to working in flat spacetime. Since we get same equations as (4.2.19) to (4.2.21), going back to general coordinates from frame components gives the desired result.

4.2.5 Green's function for inhomogeneous wave equation (continued)

In this subsection we define *retarded* and *advanced* Green's functions. Then we shall use the distributional identities derived in the previous subsection to prove that the Green's function G satisfies inhomogeneous wave equation. The retarded and advanced Green's functions are defined respectively as follows

$$G_\pm(x,x') = \frac{1}{4\pi}\left[U\delta_\pm(\sigma) + V\theta_\pm(-\sigma)\right] \tag{4.2.26}$$

Before going further we see the limiting process introduced in the previous subsection, that $\epsilon\to 0^+$. The above equation then becomes

$$G^\epsilon_\pm(x,x') = \frac{1}{4\pi}\left[U\delta_\pm(\sigma+\epsilon) + V\theta_\pm(-\sigma-\epsilon)\right]$$

where $G_\pm(x,x') = \lim_{\epsilon\to 0^+} G^\epsilon_\pm(x,x')$. Using the distributional identities derived above along with $\sigma\delta_\pm(\sigma+\epsilon) = -\epsilon\delta_\pm(\sigma+\epsilon)$, $\sigma\delta'_\pm(\sigma+\epsilon) = -\delta_\pm(\sigma+\epsilon) - \epsilon\delta'_\pm(\sigma+\epsilon)$, and $\sigma\delta''_\pm(\sigma+\epsilon) = -2\delta'_\pm(\sigma+\epsilon) - \epsilon\delta''_\pm(\sigma+\epsilon)$, we get

$$g^{\alpha\beta}G^\epsilon_{\pm;\alpha\beta} = \frac{1}{4\pi}\left[-2\epsilon\delta''_\pm(\sigma+\epsilon)U + 2\epsilon\delta'_\pm(\sigma+\epsilon)V + \delta'_\pm(\sigma+\epsilon)\left\{2U_{;\alpha}\sigma^\alpha + \left(\sigma^\alpha{}_\alpha - 4\right)U\right\}\right.$$
$$\left. + \delta_\pm(\sigma+\epsilon)\left\{-2V_{;\alpha}\sigma^\alpha + \left(2-\sigma^\alpha{}_\alpha\right)V + g^{\alpha\beta}U_{;\alpha\beta}\right\} + \theta^\epsilon_\pm(-\sigma-\epsilon)\left\{g^{\alpha\beta}V_{;\alpha\beta}\right\}\right]$$

After taking the limit $\epsilon\to 0^+$, and using the equations (4.2.5), (4.2.6) and (4.2.10), we get

$$g^{\alpha\beta}G_{\pm;\alpha\beta} = -\delta_4(x,x')$$

This proves that $G_\pm$ satisfies inhomogeneous covariant scalar wave equation. Therefore

$$G = \frac{1}{2}\left(G_+ + G_-\right)$$

also satisfies inhomogeneous covariant scalar wave equation.

4.2.6 Discussion

In this part, we have introduced bi-tensors, highlighting some useful bi-tensors and their properties. The we have computed Green's functions using the results on bi-tensors. We have used Hadamard's approach to find these Green's functions and developed some advanced distributional methods to verify that the solutions obtained do indeed satisfy the inhomogeneous covariant scalar wave equation. In the next part, we define two important coordinate systems and derive the transformations between these coordinates which will be useful in understanding the field of a point scalar charge and hence, its equation of motion.

Chapter 5

Coordinate systems

5.1 Conventions

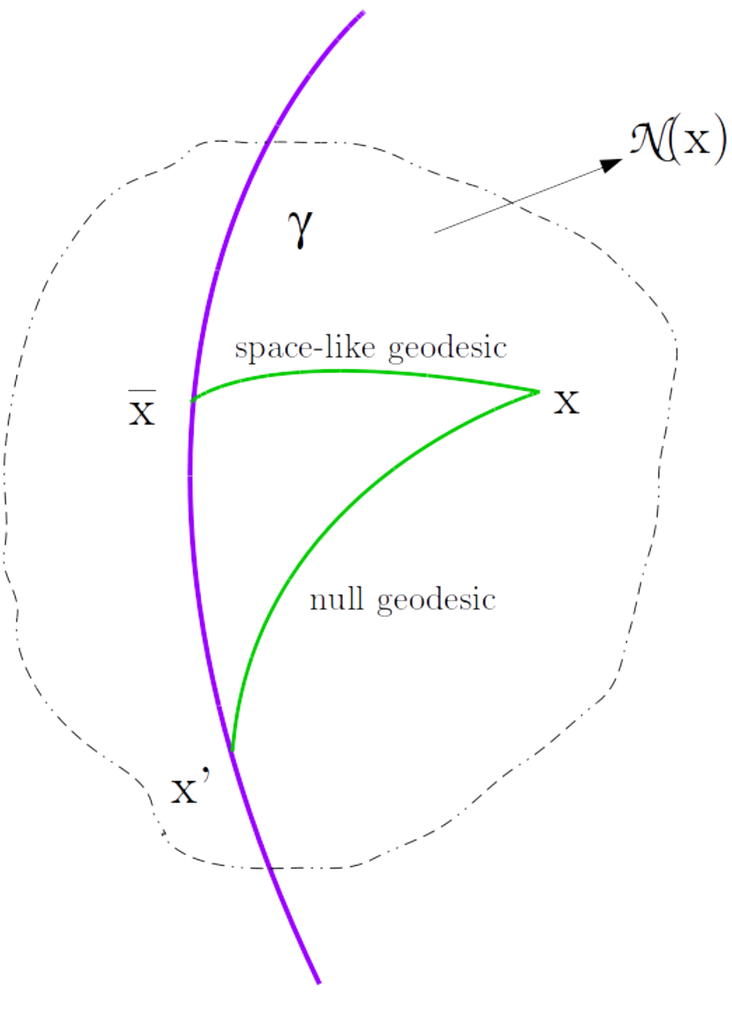

Figure 5.1.1: Geometrical meaning of the points x, x' and $\bar{x}$.

In the foregoing discussion, we shall follow the convention given below.

$$\begin{aligned}
\gamma &: \text{timelike curve representing the worldline of the particle} \\
z^{\mu}(\tau) &: \text{a point on } \gamma \text{ at proper time } \tau \\
u^{\mu} &: \frac{dz^{\mu}(\tau)}{d\tau}, \text{ velocity at that point} \\
a^{\mu} &: \frac{Du^{\mu}(\tau)}{d\tau}, \text{ acceleration at that point} \\
x &: \text{a point in the convex neighbourhood of } \gamma \\
x' &: z(u), \text{ the point of intersection of a null geodesic from } \gamma \text{to } x, \text{ at proper time } u, \\
&\qquad \text{such that } x \text{is on the future cone of } x' \\
\bar{x} &: z(t), \text{ the point of intersection of spacelike geodesic from } \gamma \text{to } x, \\
&\qquad \text{which is orthogonal to } \gamma \text{at } \bar{x}, \text{ at proper time } t
\end{aligned}$$

Here, the *convex neighbourhood* of a point x is defined as the spacetime region around it where all the points inside this region are simply connected to x, that is, for a point x' in this region, there is a *unique* geodesic *entirely inside* the region which connects x' to x. All the results derived below hold only in this region.

5.2 Fermi normal coordinates

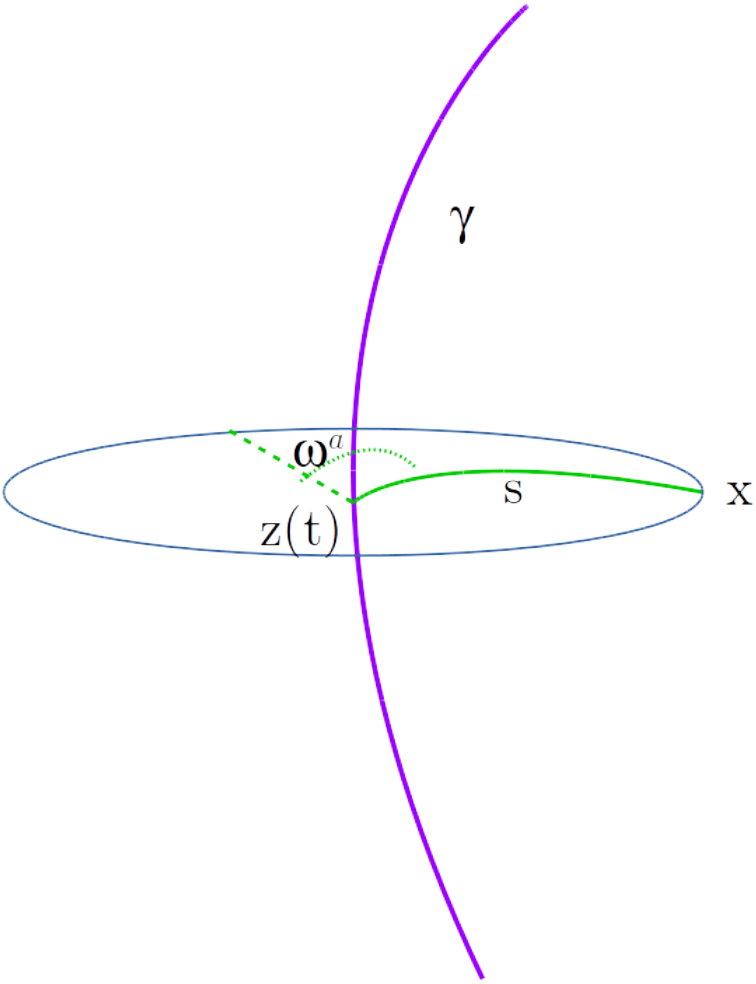

Figure 5.2.1: Geometrical interpretation of Fermi Normal coordinates.

5.2.1 Fermi-Walker transport

A vector field v^μ is said to be *Fermi-Walker transported* on γ, if it is a solution of the equation

$$\frac{Dv^\mu}{d\tau} = (v_\nu a^\nu)\, u^\mu - (v_\nu u^\nu)\, a^\mu \tag{5.2.1}$$

The properties of FW transport are that u^μ automatically satisfies the above equation and if two vector fields v^μ and w^μ satisfy this equation then their inner product is a constant on the curve γ.

5.2.2 Tetrad and dual tetrad on γ

We can erect an orthonormal tetrad (u^μ, e^μ_a) on curve γ at an arbitrary point, which is then FW transported along the curve to get the tetrad at each point on γ, so that they remain orthonormal (for a more rigorous way of defining the tetrad, refer [9]). The tetrad satisfy

$$\frac{De_a^\mu}{d\tau} = (a_\nu e_a^\nu)\, u^\mu, \; g_{\mu\nu} u^\mu u^\nu = -1, \; g_{\mu\nu} e_a^\mu u^\nu = 0, \; g_{\mu\nu} e_a^\mu e_b^\nu = \delta_{ab} \tag{5.2.2}$$

From the tetrad, we define the dual tetrad $\left(e_\mu^0, e_\mu^a\right)$, as follows

$$e_\mu^0 = -u_\mu, \; e_\mu^a = \delta^{ab} g_{\mu\nu} e_b^\nu \tag{5.2.3}$$

which is also FW transported on γ. The tetrad and its dual give rise to the completeness relations

$$g^{\mu\nu} = -u^\mu u^\nu + \delta^{ab} e_a^\mu e_b^\nu, \quad g_{\mu\nu} = -e_\mu^0 e_\nu^0 + \delta_{ab} e_\mu^a e_\nu^b \tag{5.2.4}$$

5.2.3 Fermi normal coordinates

The Fermi normal coordinates of x are defined as

$$\hat{x}^0 = t, \; \hat{x}^a = -e_{\bar\alpha}^a\left(\bar{x}\right) \sigma^{\bar\alpha}\left(x, \bar{x}\right), \; \sigma_{\bar\alpha}\left(x, \bar{x}\right) u^{\bar\alpha}\left(\bar{x}\right) = 0 \tag{5.2.5}$$

where the third relation defines the point $\bar{x}$, from the requirement that geodesic connecting x and $\bar{x}$ and γ are orthogonal. Using this definition and from (5.2.4), it can be shown that

$$s^2 = \delta_{ab} \hat{x}^a \hat{x}^b = 2\sigma\left(x, \bar{x}\right) \tag{5.2.6}$$

where s is the spatial distance between x and $\bar{x}$ along the geodesic connecting them. This s should not be confused with the one introduced in subsection 4.1.2. We can see that

$$\sigma_{\bar\alpha} = -\hat{x}^a e_a^{\bar\alpha} = -s\omega^a e_a^{\bar\alpha} \tag{5.2.7}$$

A change in the point x to $x+\delta x$, induces a change in the point $\bar{x}$ to $\bar{x}+\delta\bar{x}$ due to the change in the geodesic connecting them. The new point $\bar{x}+\delta\bar{x}$ is determined using $\sigma_{\bar\alpha}\left(x+\delta x, \bar{x}+\delta\bar{x}\right) u^{\bar\alpha}\left(\bar{x}+\delta\bar{x}\right) = 0$ and the fact that $\delta\bar{x}^{\bar\alpha} = u^{\bar\alpha}\delta t$. The change in FNC are given by

$$dt = \mu \sigma_{\bar\alpha\beta} u^{\bar\alpha} dx^\beta, \; d\hat{x}^a = -e_{\bar\alpha}^a \left(\sigma^{\bar\alpha}{}_\beta + \mu\, \sigma^{\bar\alpha}{}_{\bar\beta}\, u^{\bar\beta} \sigma_{\beta\bar\gamma} u^{\bar\gamma}\right) dx^\beta \tag{5.2.8}$$

where $\mu^{-1} = -\left(\sigma_{\bar\alpha\bar\beta} u^{\bar\alpha} u^{\bar\beta} + \sigma_{\bar\alpha} a^{\bar\alpha}\right)$.

5.2.4 Coordinate displacements near γ

The above equation for coordinate displacements can be expressed in term of powers of s. For this we need (4.1.14) and also the relations $\sigma^{\bar\alpha} = -e_a^{\bar\alpha}\hat{x}^a$ and $g^{\bar\alpha}{}_\alpha = u^{\bar\alpha}\bar{e}_\alpha^0 + e_a^{\bar\alpha}\bar{e}_\alpha^a$ where $\left(\bar{e}_\alpha^0, \bar{e}_\alpha^a\right)$ is dual tetrad at x. After some algebra we get,

$$\mu^{-1} = 1 + a_a \hat{x}^a + \frac{1}{3} R_{0c0d} \hat{x}^c \hat{x}^d + O\left(s^3\right)$$

where $a_a(t) = a_{\bar{\alpha}} e_a^{\bar{\alpha}}$ and $R_{0c0d}(t) = R_{\bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\delta}} u^{\bar{\alpha}} e_c^{\bar{\gamma}} u^{\bar{\beta}} e_d^{\bar{\delta}}$. Inverting the above equation gives,

$$\mu = 1 - a_a \hat{x}^a + (a_a \hat{x}^a)^2 - \frac{1}{3} R_{0c0d} \hat{x}^c \hat{x}^d + O\left(s^3\right)$$

Using similar expansions and substituting the above relation in equation (5.2.8), we get

$$dt = \left[1 - a_a \hat{x}^a + (a_a \hat{x}^a)^2 - \frac{1}{3} R_{0c0d} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] \left(\bar{e}_\beta^0 dx^\beta\right) + \left[-\frac{1}{6} R_{0cbd} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] \left(\bar{e}_\beta^b dx^\beta\right) \tag{5.2.9}$$

$$d\hat{x}^a = \left[\frac{1}{2} R^a{}_{c0d} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] \left(\bar{e}_\beta^0 dx^\beta\right) + \left[\delta^a{}_b + \frac{1}{6} R^a{}_{cbd} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] \left(\bar{e}_\beta^b dx^\beta\right) \tag{5.2.10}$$

where $R_{ac0d}(t) = R_{\bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\delta}} e_a^{\bar{\alpha}} e_c^{\bar{\gamma}} u^{\bar{\beta}} e_d^{\bar{\delta}}$ and $R_{acbd}(t) = R_{\bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\delta}} e_a^{\bar{\alpha}} e_c^{\bar{\gamma}} e_b^{\bar{\beta}} e_d^{\bar{\delta}}$.

5.2.5 Metric near γ

Inverting equations (5.2.9) and (5.2.10), gives the following relations for the tetrad,

$$\bar{e}_\alpha^0 dx^\alpha = \left[1 + a_a \hat{x}^a + \frac{1}{2} R_{0c0d} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] dt + \left[\frac{1}{6} R_{0cbd} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] d\hat{x}^b \tag{5.2.11}$$

$$\bar{e}_\alpha^a dx^\alpha = \left[\delta^a{}_b - \frac{1}{6} R^a{}_{cbd} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] d\hat{x}^b + \left[-\frac{1}{2} R^a{}_{c0d} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] dt \tag{5.2.12}$$

These relations give the components of tetrad at x in terms of FNC.

Invoking the completeness relation (5.2.4), the metric at x is given by,

$$g_{tt} = -\left[1 + 2a_a \hat{x}^a + (a_d \hat{x}^a)^2 + R_{0c0d} \hat{x}^c \hat{x}^d + O\left(s^3\right)\right] \tag{5.2.13}$$

$$g_{ta} = -\frac{2}{3} R_{0cad} \hat{x}^c \hat{x}^d + O\left(s^3\right) \tag{5.2.14}$$

$$g_{ab} = \delta_{ab} - \frac{1}{3} R_{acbd} \hat{x}^c \hat{x}^d + O\left(s^3\right) \tag{5.2.15}$$

This is the metric near γ in FNC.

5.3 Retarded coordinates

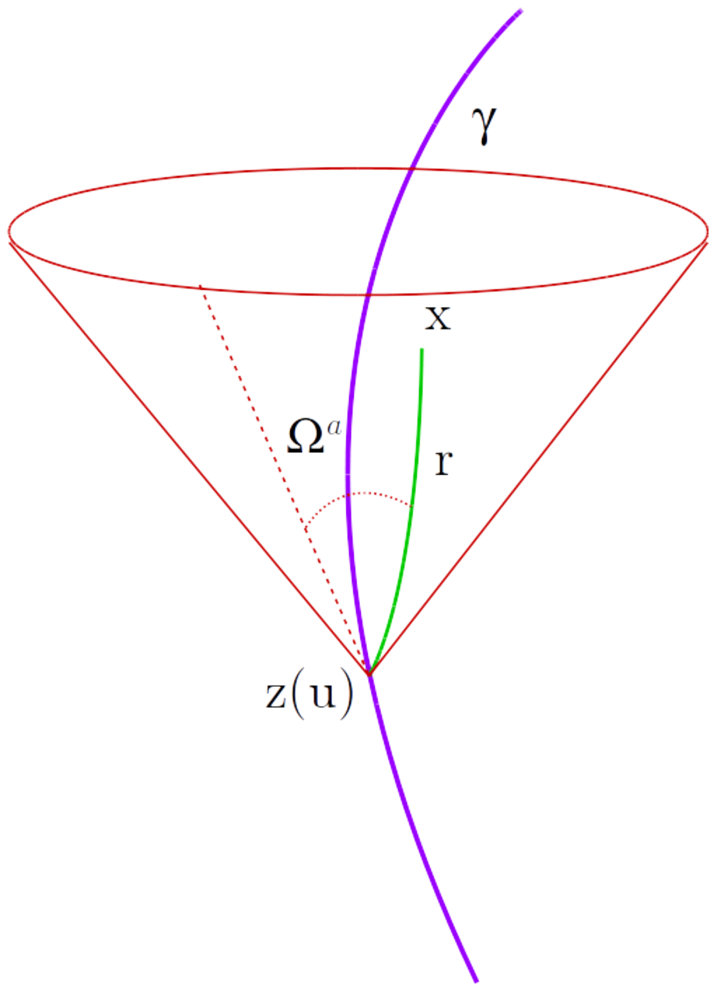

Figure 5.3.1: Geometrical interpretation of Retarded coordinates

5.3.1 Tetrad and dual tetrad on γ

Similar to FNC, we can build an orthonormal tetrad (u^μ, e_a^μ) which is FW transported on world line according to

$$\frac{De_a^\mu}{d\tau} = a_a u^\mu \tag{5.3.1}$$

where $a_a(\tau) = a_\mu e_a^\mu$. We also have the dual tetrad given by $e_\mu^0 = -u_\mu$ and $e_\mu^a = \delta^{ab} g_{\mu\nu} e_b^\nu$. The tetrad and its dual give rise to the following completeness relations

$$g^{\mu\nu} = -u^\mu u^\nu + \delta^{ab} e_a^\mu e_b^\nu, \;\; g_{\mu\nu} = -e_\mu^0 e_\nu^0 + \delta_{ab} e_\mu^a e_\nu^b \tag{5.3.2}$$

By parallel transporting the tetrad at the point x' to the point x, we can obtain the tetrad at x, (e_0^α, e_a^α), and also the dual tetrad at x, given by $e_\alpha^0 = -g_{\alpha\beta} e_0^\beta$ and $e_\alpha^a = \delta^{ab} g_{\alpha\beta} e_b^\beta$. The metric at x can then be written as

$$g_{\alpha\beta} = -e^0_\alpha e^0_\beta + \delta_{ab} e^a_\alpha e^b_\beta \tag{5.3.3}$$

and the parallel propagator from x' to x is given by

$$g^\alpha{}_{\alpha'}\left(x, x'\right) = -e^\alpha_0 u_{\alpha'} + e^\alpha_a e^a_{\alpha'}, \quad g^{\alpha'}{}_\alpha\left(x', x\right) = u^{\alpha'} e^0_\alpha + e^{\alpha'}_a e^a_\alpha \tag{5.3.4}$$

5.3.2 Retarded coordinates

The retarded coordinates are defined by

$$\hat{x}^0 = u, \quad \hat{x}^a = -e^a_{\alpha'}\left(x'\right) \sigma^{\alpha'}\left(x, x'\right), \quad \sigma(x, x') = 0 \tag{5.3.5}$$

where the third relation indicates that x and x' are connected by a null geodesic. Since $\sigma^{\alpha'}$ is a null vector, we get

$$r = \left(\delta_{ab} \hat{x}^a \hat{x}^b\right)^{\frac{1}{2}} = u_{\alpha'} \sigma^{\alpha'} \tag{5.3.6}$$

where r is a positive quantity since the geodesic connecting x and x' is future directed and hence, $\sigma^{\alpha'}$ is past directed. In flat spacetime r is equal to the spatial distance between the points x and x'. Hence, in curved spacetime, r can still be called *retarded distance* between x and x'.

From (5.3.5), we have

$$\sigma^{\alpha'} = -r\left(u^{\alpha'} + \Omega^a e^{\alpha'}_a\right) \tag{5.3.7}$$

where $\Omega^a = \frac{\hat{x}^a}{r}$ is a unit spatial vector which satisfies $\delta_{ab}\Omega^a\Omega^b = 1$.

Similar to the case of FNC, a displacement in the point x induces a displacement in the point x' and hence, th retarded coordinates change according to the following relations

$$du = -k_\alpha dx^\alpha, \quad d\hat{x}^a = -\left(r a^a + e^a_{\alpha'}\,\sigma^{\alpha'}{}_{\beta'}\,u^{\beta'}\right) du - e^a_{\alpha'}\,\sigma^{\alpha'}{}_\beta\, dx^\beta \tag{5.3.8}$$

where $k_\alpha = \frac{\sigma_\alpha}{r}$ the displacement in x' is found using the fact that $x+\delta x$ and $x'+\delta x'$ are still connected by a null geodesic, that is, $\sigma(x, x') = 0 = \sigma(x+\delta x, x'+\delta x')$, and the relation $\delta x'^{\alpha'} = u^{\alpha'}\delta u$.

5.3.3 The scalar field $r\left(x\right)$ and the vector field $k_\alpha\left(x\right)$

As long as x and x' are linked by the relation $\sigma(x, x') = 0$, the quantity $r\left(x\right) = \sigma_{\alpha'}(x, x')u^{\alpha'}(x')$ can be considered to be a scalar field. The gradient of r, remembering the induced displacement in x', is given by

$$\partial_\beta r = -\left(\sigma_{\alpha'} a^{\alpha'} + \sigma_{\alpha'\beta'} u^{\alpha'} u^{\beta'}\right) k_\beta + \sigma_{\alpha'\beta} u^{\alpha'} \tag{5.3.9}$$

Similarly, $k^\alpha\left(x\right) = \frac{\sigma^\alpha(x,x')}{r(x)}$ can be viewed as an ordinary vector field defined at the point x. Using the equation (4.1.3), it is easy to see that k_α satisfies the following relations

$$\sigma_{\alpha\beta} k^\beta = k_\alpha, \quad \sigma_{\alpha'\beta} k^\beta = \frac{\sigma_{\alpha'}}{r} \tag{5.3.10}$$

from which it can be seen that $\sigma_{\alpha'\beta}u^{\alpha'}k^{\beta} = 1$. Substituting this into (5.3.9), gives

$$k^{\alpha}\partial_{\alpha}r = 1 \tag{5.3.11}$$

Combining the relation $\sigma^{\alpha} = g^{\alpha}{}_{\alpha'}\sigma^{\alpha'}$ with (5.3.7), gives

$$k^{\alpha} = g^{\alpha}{}_{\alpha'}\left(u^{\alpha'} + \Omega^{a}e^{\alpha'}_{a}\right) \tag{5.3.12}$$

which can alternatively be written as

$$k^{\alpha} = e^{\alpha}_{0} + \Omega^{a}e^{\alpha}_{a} \tag{5.3.13}$$

which is the vector k^{α} at the point x.

Using the results (5.3.9) and derivative of σ_{α} under a displacement of x, we find the covariant derivative of rk_{α}, which is easier to compute. This then immediately gives the covariant derivative of k_{α},

$$rk_{\alpha;\beta} = \sigma_{\alpha\beta} - k_{\alpha}\sigma_{\beta\gamma'}u^{\gamma'} - k_{\beta}\sigma_{\alpha\gamma'}u^{\gamma'} + \left(\sigma_{\alpha'}a^{\alpha'} + \sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'}\right)k_{\alpha}k_{\beta}$$

From this, using the results previously established in this subsection, we can infer that k^{α} satisfies the geodesic equation in affine parameter form, that is, $k^{\alpha}{}_{;\beta}\,k^{\beta} = 0$, and from (5.3.11), we can also infer that the affine parameter is r.

5.3.4 Coordinate displacements near γ

Substituting the expansion for $\sigma_{\alpha'\beta'}$ from (4.1.14), using (5.3.4) in the expansion of $\sigma_{\alpha'\beta}$, and substituting these expansions into the equations (5.3.8), along with (5.3.13), gives

$$du = \left(e^{0}_{\alpha}dx^{\alpha}\right) - \Omega_{\alpha}\left(e^{a}_{\alpha}dx^{\alpha}\right) \tag{5.3.14}$$

$$d\hat{x}^{a} = -\left[ra^{a} + \frac{1}{2}r^{2}S^{a} + O\left(r^{3}\right)\right]\left(e^{0}_{\alpha}dx^{\alpha}\right) + \left[\delta^{a}{}_{b} + \left(ra^{a} + \frac{1}{3}r^{2}S^{a}\right)\Omega_{b} + \frac{1}{6}r^{2}\,S^{a}{}_{b} + O\left(r^{3}\right)\right]\left(e^{b}_{\alpha}dx^{\alpha}\right) \tag{5.3.15}$$

which can also be expressed in gradient form as

$$\partial_{\alpha}u = e^{0}_{\alpha} - \Omega_{a}e^{a}_{\alpha} \tag{5.3.16}$$

$$\partial_{\alpha}\hat{x}^{a} = -\left[ra^{a} + \frac{1}{2}r^{2}S^{a} + O\left(r^{3}\right)\right]e^{0}_{\alpha} + \left[\delta^{a}{}_{b} + \left(ra^{a} + \frac{1}{3}r^{2}S^{a}\right)\Omega_{b} + \frac{1}{6}r^{2}\,S^{a}{}_{b} + O\left(r^{3}\right)\right]e^{b}_{\alpha} \tag{5.3.17}$$

From the last equation, using the identity $\partial_{\alpha}r = \Omega_{a}\partial_{\alpha}\hat{x}^{a}$, we get

$$\partial_{\alpha}r = -\left[ra^{a} + \frac{1}{2}r^{2}S^{a} + O\left(r^{3}\right)\right]e^{0}_{\alpha} + \left[\left(1 + ra_{b}\Omega^{b} + \frac{1}{3}r^{2}S\right)\Omega_{a} + \frac{1}{6}r^{2}S_{a} + O\left(r^{3}\right)\right]e^{a}_{\alpha} \tag{5.3.18}$$

We can arrive at the same expansion using (5.3.9).

In the above expansions, we have defined the following quantities,

$$S_{ab} = R_{a0b0} + R_{a0bc}\Omega^{c} + R_{b0ac}\Omega^{c} + R_{acbd}\Omega^{c}\Omega^{d} = S_{ba}$$

$$S_a = S_{ab}\Omega^b = R_{a0b0}\Omega^b - R_{ab0c}\Omega^b\Omega^c$$

$$S = S_a\Omega^a = R_{a0b0}\Omega^a\Omega^b$$

5.3.5 Metric near γ

Inverting the relations (5.3.14) and (5.3.15), gives the following relations of tetrad with coordinate displacements,

$$e^0_\alpha dx^\alpha = \left[1 + ra^a + \frac{1}{2}r^2S^a + O\left(r^3\right)\right]du + \left[\left(1 + \frac{1}{6}r^2S\right)\Omega_a - \frac{1}{6}r^2S_a + O\left(r^3\right)\right]d\hat{x}^a \tag{5.3.19}$$

$$e^a_\alpha dx^\alpha = \left[ra^a + \frac{1}{2}r^2S^a + O\left(r^3\right)\right]du + \left[\delta^a{}_b - \frac{1}{6}r^2\ S^a{}_b + \frac{1}{6}r^2S^a\Omega_b\right]d\hat{x}^b \tag{5.3.20}$$

These relations, along with the equation (5.3.3), can be used to obtain the components of metric near γ in retarded coordinates,

$$g_{uu} = -\left(1 + ra_a\Omega^a\right)^2 + r^2a^2 - r^2S + O\left(r^3\right) \tag{5.3.21}$$

$$g_{ua} = -\left(1 + ra_b\Omega^b + \frac{2}{3}r^2S\right)\Omega_\alpha + ra_a + \frac{2}{3}r^2S_a + O\left(r^3\right) \tag{5.3.22}$$

$$g_{ab} = \delta_{ab} - \left(1 + \frac{1}{3}r^2S\right)\Omega_a\Omega_b - \frac{1}{3}r^2S_{ab} + \frac{1}{3}r^2\left(S_a\Omega_b + \Omega_aS_b\right) + O\left(r^3\right) \tag{5.3.23}$$

where $a^2 = \delta_{ab}a^aa^b$.

5.4 Transformation from retarded to Fermi normal coordinates

Define the quantity $\Delta = t - u$, which is not to be confused with van Vleck determinant. Also, define the function

$$p(\tau) = \sigma_\mu\left(x, z(\tau)\right)u^\mu(\tau)$$

in which x is always kept fixed and $z\left(\tau\right)$ is an arbitrary point on the world line. We know that $p\left(u\right) = r$ and $p\left(t\right) = 0$. Taylor expansion of $p\left(t\right)$ around u gives,

$$p(t) = p(u) + \dot{p}(u)\Delta + \frac{1}{2}\ddot{p}(u)\Delta^2 + \frac{1}{6}p^{(3)}(u)\Delta^3 + O(\Delta^4)$$

where,

$$\begin{aligned}
\dot{p}(u) &= \sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'} + \sigma_{\alpha'}a^{\alpha'} \\
\ddot{p}(u) &= \sigma_{\alpha'\beta'\gamma'}u^{\alpha'}u^{\beta'}u^{\gamma'} + 3\sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'} + \sigma_{\alpha'}\dot{a}^{\alpha'} \\
p^{(3)}(u) &= \sigma_{\alpha'\beta'\gamma'\delta'}u^{\alpha'}u^{\beta'}u^{\gamma'}u^{\delta'} + \sigma_{\alpha'\beta'\gamma'}(5u^{\alpha'}u^{\beta'}u^{\gamma'} + u^{\alpha'}u^{\beta'}a^{\gamma'}) \\
&\quad + \sigma_{\alpha'\beta'}(3u^{\alpha'}u^{\beta'} + 4u^{\alpha'}\dot{a}^{\beta'}) + \sigma_{\alpha'}\ddot{a}^{\alpha'}
\end{aligned}$$

Using the expansions (4.1.14), (4.1.15), (4.1.16) and the equation (5.3.7), we get

$$\begin{aligned}
\dot{p}(u) &= -\left[1 + ra_a\Omega^a + \frac{1}{3}r^2 S + O(r^3)\right] \\
\ddot{p}(u) &= -r(\dot{a}_0 + \dot{a}_a\Omega^a) + O(r^2) \\
p^{(3)}(u) &= \dot{a}_o + O(r)
\end{aligned}$$

where $\dot{a}_0 = \dot{a}_{\alpha'}u^{\alpha'}$, $\dot{a}_a = \dot{a}_{\alpha'}e^{\alpha'}_a$ and we used $a^2 + \dot{a}_{\alpha'}u^{\alpha'} = 0$.

Substituting these results in expansion of $p\,(t)$, we obtain

$$r = \left[1 + ra_a\Omega^a + \frac{1}{3}r^2 S + O(r^3)\right]\Delta + \frac{1}{2}r\left[\dot{a}_o + \dot{a}_a\Omega^a + O(r)\right]\Delta^2 - \frac{1}{6}\left[\dot{a}_o + O(r)\right]\Delta^3 + O(\Delta^4) \quad (5.4.1)$$

Inverting the above power series gives Δ in terms of powers of r

$$\Delta = t - u = r\left\{1 - ra_a(u)\Omega^a + r^2\left[a_a(u)\Omega^a\right]^2 - \frac{1}{3}r^2\dot{a}_o(u) - \frac{1}{2}r^2\dot{a}_a(u)\Omega^a - \frac{1}{2}r^3 R_{a0b0}(u)\Omega^a\Omega^b + O(r^3)\right\} \quad (5.4.2)$$

This gives the relation between time coordinates.

Now, consider the following function

$$p_a(\tau) = -\sigma_\mu(x, z(\tau))e^\mu_a(\tau)$$

in which, again, x is fixed and $z\,(\tau)$ is arbitrary. We know that $p_a\,(t) = s\omega^a = \hat{x}^a = p_a\,(u+\Delta)$ and $p_a\,(u) = r\Omega^a$. Expanding $p_a\,(t)$ around u gives

$$s\omega^a = p^a(u) + \dot{p}^a(u)\Delta + \frac{1}{2}\ddot{p}^a(u)\Delta^2 + \frac{1}{6}p^{a(3)}(u)\Delta^3 + O(\Delta^4)$$

where

$$
\begin{aligned}
\dot{p}_a(u) &= -\sigma_{\alpha'\beta'}e^{\alpha'}_a u^{\beta'} - (\sigma_{\alpha'}u^{\alpha'})(a_{\beta'}e^{\beta'}_a) \\
&= -ra_a - \frac{1}{3}r^2 S_a + O\left(r^3\right) \\
\ddot{p}_a(u) &= -\sigma_{\alpha'\beta'\gamma'}e^{\alpha'}_a u^{\beta'}u^{\gamma'} - (2\sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'} + \sigma_{\alpha'}a^{\alpha'})(a_{\gamma'}e^{\gamma'}_a) - \sigma_{\alpha'\beta'}e^{\alpha'}_a a^{\beta'} - (\sigma_{\alpha'}u^{\alpha'})(\dot{a}_{\beta'}e^{\beta'}_a) \\
&= (1 + ra_b\Omega^b)a_a - r\dot{a}_a + \frac{1}{3}rR_{a0b0}\Omega^b + O(r^2) \\
p^{(3)}_a(u) &= -\sigma_{\alpha'\beta'\gamma'\delta'}e^{\alpha'}_a u^{\beta'}u^{\gamma'}u^{\delta'} - (3\sigma_{\alpha'\beta'\gamma'}u^{\alpha'}u^{\beta'}u^{\gamma'} + 6\sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'} + \sigma_{\alpha'}\dot{a}^{\alpha'} + \sigma_{\alpha'}u^{\alpha'}\dot{a}_{\beta'}u^{\beta'})(a_{\delta'}e^{\delta'}_a) \\
&\quad - \sigma_{\alpha'\beta'\gamma'\delta'}e^{\alpha'}_a(2a^{\beta'}u^{\gamma'} + u^{\beta'}a^{\gamma'}) - (3\sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'} + 2\sigma_{\alpha'}a^{\alpha'})(\dot{a}_{\gamma'}e^{\gamma'}_a) - \sigma_{\alpha'\beta'}e^{\alpha'}_a\dot{a}^{\beta'} - (\sigma_{\alpha'}u^{\alpha'})(\ddot{a}_{\beta'}e^{\beta'}_a) \\
&= 2\ddot{a}_a + O(r)
\end{aligned}
$$

and we used the same expansions as in the previous derivation.

Combining all these results, we get

$$
s\omega^a = r\Omega^a - r\left[a^a + \frac{1}{3}S^a + O(r^2)\right]\Delta + \frac{1}{2}\left[(1 + ra_b\Omega^b)a^a - r\dot{a}^a + \frac{1}{3}rR^a{}_{0b0}\Omega^b + O(r^2)\right]\Delta^2 + \frac{1}{3}\left[\dot{a}^a + O(r)\right]\Delta^3 + O(\Delta^4)
$$

which, after substituting (5.4.2), yields

$$
s\omega^a = r\left\{\Omega^a - \frac{1}{2}r\left[1 - ra_b(u)\Omega^b\right]a^a(u) - \frac{1}{6}r^2\dot{a}^a(u) - \frac{1}{6}r^2R^a{}_{0b0}(u)\Omega^b + \frac{1}{3}r^2R^a{}_{b0c}(u)\Omega^b\Omega^c + O(r^3)\right\} \tag{5.4.3}
$$

Using the identity $\delta_{ab}\omega^a\omega^b = 1$ after squaring the above expression gives

$$
s = r\left\{1 - \frac{1}{2}ra_a(u)\Omega^a + \frac{3}{8}r^2\left[a_a(u)\Omega^a\right]^2 - \frac{1}{8}r^2\dot{a}_0(u) - \frac{1}{6}r^2\dot{a}_a(u)\Omega^a - \frac{1}{6}r^2R_{a0b0}(u)\Omega^a\Omega^b + O(r^3)\right\} \tag{5.4.4}
$$

5.5 Transformation from Fermi normal to retarded coordinates

Define the function

$$
\sigma(\tau) = \sigma(x, z(\tau))
$$

which only depends on τ since x is held fixed. We have $\sigma(u) = 0$ and $\sigma(t) = \frac{s^2}{2}$ and we define $\Delta = t - u$. Also, $\dot{\sigma}(\tau) = p(\tau)$. Expanding $\sigma(u)$ around t gives

$$
\sigma(u) = \sigma(t) - p(t)\Delta + \frac{1}{2}\dot{p}(t)\Delta^2 - \frac{1}{6}\ddot{p}(t)\Delta^3 + \frac{1}{24}p^{(3)}(t)\Delta^4 + O(\Delta^5)
$$

where, using previous expressions for derivatives of $p(\tau)$, now evaluated at $\tau = t$, along with (5.2.7), we have

$$\begin{aligned}
\dot{p}(t) &= -\left[1 + s a_a \omega^a + \frac{1}{3} s^2 R_{a0b0} \omega^a \omega^b + O(s^3)\right] \\
\ddot{p}(t) &= -s \dot{a}_a \omega^a + O(s^2) \\
p^{(3)}(t) &= \dot{a}_0 + O(s)
\end{aligned}$$

Substituting these expansions into expansion of $\sigma(u)$, we get

$$s^2 = \left[1 + s a_a \omega^a + \frac{1}{3} s^2 R_{a0b0} \omega^a \omega^b + O(s^3)\right] \Delta^2 - \frac{1}{3} s \left[\dot{a}_a \omega^a + O(s)\right] \Delta^3 - \frac{1}{12} \left[\dot{a}_0 + O(s)\right] \Delta^4 + O(\Delta^5) \tag{5.5.1}$$

Inverting the above equation gives Δ as an expansion in powers of s

$$\Delta = t - u = s \left\{1 - \frac{1}{2} s a_a(t) \omega^a + \frac{3}{8} s^2 [a_a(t) \omega^a]^2 + \frac{1}{24} s^2 \dot{a}_0(t) + \frac{1}{6} s^2 \dot{a}_a(t) \omega^a - \frac{1}{6} s^2 R_{a0b0}(t) \omega^a \omega^b + O(s^3)\right\} \tag{5.5.2}$$

Noting that $r = p(u)$, expanding this around t gives

$$r = -\dot{p}(t) \Delta + \frac{1}{2} \ddot{p}(t) \Delta^2 - \frac{1}{6} p^{(3)}(t) \Delta^3 + O(\Delta^4)$$

Substituting (5.5.2) gives

$$r = s \left\{1 + \frac{1}{2} s a_a(t) \omega^a - \frac{1}{8} s^2 [a_a(t) \omega^a]^2 - \frac{1}{8} s^2 \dot{a}_0(t) - \frac{1}{3} s^2 \dot{a}_a(t) \omega^a + \frac{1}{6} s^2 R_{a0b0}(t) \omega^a \omega^b + O(s^3)\right\} \tag{5.5.3}$$

Expanding $r\Omega^a = p^a(u)$ around t gives

$$r\Omega^a = p^a(t) - \dot{p}^a(t) \Delta + \frac{1}{2} \ddot{p}^a(t) \Delta^2 - \frac{1}{6} p^{a(3)}(t) \Delta^3 + O(\triangle^4)$$

where, using previous expressions for derivatives of $p_a(\tau)$, now evaluated at $\tau = t$, we have

$$\begin{aligned}
\dot{p}_a(t) &= \frac{1}{3} s^2 R_{ab0c} \omega^b \omega^c + O(s^3) \\
\ddot{p}_a(t) &= (1 + s a_b \omega^b) a_a + \frac{1}{3} s R_{a0b0} \omega^b + O(s^2) \\
p_a^{(3)}(t) &= 2 \dot{a}_a(t) + O(s)
\end{aligned}$$

Substituting these results including (5.5.2), we obtain

$$r\Omega^a = s \left\{\omega^a + \frac{1}{2} s a^a(t) - \frac{1}{3} s^2 \dot{a}^a(t) - \frac{1}{3} s^2 R^a_{b0c}(t) \omega^b \omega^c + \frac{1}{6} s^2 R^a_{0b0}(t) \omega^b + O(s^3)\right\} \tag{5.5.4}$$

5.6 Transformation of tetrads at x

The tetrad at x can be obtained by parallel transporting the tetrad at x' along the null geodesic joining x and x' or the tetrad at $\bar{x}$ along the space-like geodesic joining x and $\bar{x}$, orthogonal to γ at $\bar{x}$. In this section, we derive the relation between these two tetrads so obtained.

Consider the functions

$$p^{\alpha}(\tau) = g^{\alpha}_{\ \mu}\left(x, z(\tau)\right) u^{\mu}(\tau), \qquad p^{\alpha}_{a}(\tau) = g^{\alpha}_{\ \mu}\left(x, z(\tau)\right) e^{\mu}_{a}(\tau)$$

where x is a fixed point, $z\left(\tau\right)$ is an arbitrary point on γ, and $g^{\alpha}_{\ \mu}$ is a parallel propagator on the unique geodesic connecting x and z. We know that $\bar{e}^{\alpha}_{0} = p^{\alpha}\left(t\right),\ \bar{e}^{\alpha}_{a} = p^{\alpha}_{a}\left(t\right),\ e^{\alpha}_{0} = p^{\alpha}\left(u\right),$ and $e^{\alpha}_{a} = p^{\alpha}_{a}\left(u\right)$.

Expanding $p^{\alpha}\left(t\right)$ around u gives

$$\bar{e}^{\alpha}_{0} = p^{\alpha}(u) + \dot{p}^{\alpha}(u)\Delta + \frac{1}{2}\ddot{p}^{\alpha}(u)\Delta^{2} + O(\Delta^{3})$$

where

$$\begin{aligned}
\dot{p}^{\alpha}(u) &= g^{\alpha}_{\ \alpha';\beta'} u^{\alpha'} u^{\beta'} + g^{\alpha}_{\ \alpha'} a^{\alpha'} \\
&= \left[a^{a} + \frac{1}{2} r R^{a}_{\ 0b0}\Omega^{b} + O(r^{2})\right] e^{\alpha}_{a} \\
\ddot{p}^{\alpha}(u) &= g^{\alpha}_{\ \alpha';\beta'\gamma'} u^{\alpha'} u^{\beta'} u^{\gamma'} + g^{\alpha}_{\ \alpha';\beta'}(2a^{\alpha'}u^{\beta'} + u^{\alpha'}a^{\beta'}) + g^{\alpha}_{\ \alpha'}\dot{a}^{\alpha'} \\
&= \left[-\dot{a}_{0} + O(r)\right] e^{\alpha}_{0} + \left[\dot{a}^{a} + O(r)\right] e^{\alpha}_{a}
\end{aligned}$$

To obtain these expansions we have used the expansions (4.1.21), (4.1.22) and (5.3.4). Substituting these expansions and (5.4.2) in the previous equation gives

$$\begin{aligned}
\bar{e}^{\alpha}_{0} = &\left[1 - \frac{1}{2} r^{2}\dot{a}_{0}(u) + O(r^{3})\right] e^{\alpha}_{0} \\
&+ \left[r(1 - r a_{b}\Omega^{b}) a^{a}(u) + \frac{1}{2} r^{2}\dot{a}^{a}(u) + \frac{1}{2} r^{2} R^{a}_{\ 0b0}(u)\Omega^{b} + O(r^{3})\right] e^{\alpha}_{a}
\end{aligned} \tag{5.6.1}$$

Expanding $p^{\alpha}_{a}\left(t\right)$ around u gives

$$\bar{e}^{\alpha}_{a} = p^{\alpha}_{a}(u) + \dot{p}^{\alpha}_{a}(u)\Delta + \frac{1}{2}\ddot{p}^{\alpha}_{a}(u)\Delta^{2} + O(\Delta^{3})$$

where, again using previously discussed expansions, we have

$$\begin{aligned}
\dot{p}^\alpha_a(u) &= g^\alpha{}_{\alpha';\beta'} e^{\alpha'}_a u^{\beta'} + (g^\alpha{}_{\alpha'} u^{\alpha'})(a_{\beta'} e^{\beta'}_a) \\
&= \left[a_a + \frac{1}{2} r R_{a0b0} \Omega^b + O(r^2)\right] e^\alpha_0 + \left[-\frac{1}{2} r R^b{}_{a0c} \Omega^c + O(r^2)\right] e^\alpha_b \\
\ddot{p}^\alpha_a(u) &= g^\alpha{}_{\alpha';\beta'\gamma'} e^{\alpha'}_a u^{\beta'} u^{\gamma'} + g^\alpha{}_{\alpha';\beta'}(2u^{\alpha'} u^{\beta'} a_{\gamma'} e^{\gamma'}_a + e^{\alpha'}_a a^{\beta'}) + (g^\alpha{}_{\alpha'} a^{\alpha'})(a_{\beta'} e^{\beta'}_a) + (g^\alpha{}_{\alpha'} u^{\alpha'})(\dot{a}_{\beta'} e^{\beta'}_a) \\
&= [\dot{a}_a + O(r)]\, e^\alpha_0 + \left[a_a a^b + O(r)\right] e^\alpha_b
\end{aligned}$$

Substituting these expansions and (5.4.2) into the previous equation gives

$$\begin{aligned}
\bar{e}^\alpha_a = &\left[\delta^b{}_a + \frac{1}{2} r^2 a^b(u) a_a(u) - \frac{1}{2} r^2 R^b{}_{a0c}(u) \Omega^c + O(r^3)\right] e^\alpha_b \\
&+ \left[r(1 - r a_b \Omega^b) a_a(u) + \frac{1}{2} r^2 \dot{a}_a(u) + \frac{1}{2} r^2 R_{a0b0}(u) \Omega^b + O(r^3)\right] e^\alpha_0
\end{aligned} \tag{5.6.2}$$

Expanding $p^\alpha(u)$ around t gives

$$e^\alpha_0 = p^\alpha(t) - \dot{p}^\alpha(t)\Delta + \frac{1}{2}\ddot{p}^\alpha(t)\Delta^2 + O(\Delta^3)$$

where using similar expansions as in the above case, we have

$$\begin{aligned}
\dot{p}^\alpha(t) &= \left[a^a + \frac{1}{2} s R^a{}_{0b0} \omega^b + O(s^2)\right] \bar{e}^\alpha_a \\
\ddot{p}^\alpha(t) &= [-\dot{a} + O(s)]\, \bar{e}^\alpha_0 + [\dot{a}^a + O(s)]\, \bar{e}^\alpha_a
\end{aligned}$$

which, along with (5.5.2), gives

$$\begin{aligned}
e^\alpha_0 = &\left[1 - \frac{1}{2} s^2 \dot{a}_0(t) + O(s^3)\right] \bar{e}^\alpha_0 \\
&+ \left[-s\left(1 - \frac{1}{2} s a_b \omega^b\right) a^a(t) + \frac{1}{2} s^2 \dot{a}^a(t) - \frac{1}{2} s^2 R^a{}_{0b0}(t) \omega^b + O(s^3)\right] \bar{e}^\alpha_a
\end{aligned} \tag{5.6.3}$$

Similarly, expanding $p^\alpha_a(u)$ around t gives

$$e^\alpha_a = p^\alpha_a(t) - \dot{p}^\alpha_a(t)\Delta + \frac{1}{2}\ddot{p}^\alpha_a(t)\Delta^2 + O(\Delta^3)$$

where using similar expansions as in the above case, we have

$$\begin{aligned}
\dot{p}^\alpha_a(t) &= \left[a_a + \frac{1}{2} s R_{a0b0} \omega^b + O(s^2)\right] \bar{e}^\alpha_0 + \left[-\frac{1}{2} s R^b{}_{a0c} \omega^c + O(s^2)\right] \bar{e}^\alpha_b \\
\ddot{p}^\alpha_a(t) &= [\dot{a}_a + O(s)]\, \bar{e}^\alpha_0 + \left[a_a a^b + O(s)\right] \bar{e}^\alpha_b
\end{aligned}$$

which, along with (5.5.2), gives

$$\begin{aligned} e_a^\alpha = &\left[\delta^b{}_a + \frac{1}{2}s^2 a^b(t)a_a(t) + \frac{1}{2}R^b{}_{a0c}(t)\omega^c + O(s^3)\right]\bar{e}_b^\alpha \\ &+ \left[-s\left(1 - \frac{1}{2}sa_b\omega^b\right)a_a(t) + \frac{1}{2}s^2\dot{a}_a(t) - \frac{1}{2}s^2 R_{a0b0}(u)\omega^b + O(s^3)\right]\bar{e}_0^\alpha \end{aligned} \tag{5.6.4}$$

Chapter 6

Motion of a point particle

6.1 Dynamics of a point scalar charge

A point particle carries charge q and moves on the world line γ described by the relations $z^{\mu}(\lambda)$. The particle generates a scalar potential $\Phi(x)$ and a field $\Phi_{\alpha}(x) = \nabla_{\alpha}\Phi(x)$. The dynamics of this system is governed by the action

$$S = S_{field} + S_{particle} + S_{interaction}$$

The free field action is given by

$$S_{field} = -\frac{1}{8\pi}\int \left(g^{\alpha\beta}\Phi_{\alpha}\Phi_{\beta} + \xi R\Phi^{2}\right)\sqrt{-g}d^{4}x \tag{6.1.1}$$

where the field is coupled to Ricci scalar R through an arbitrary constant ξ.

The free particle action is given by

$$S_{particle} = -m_{0}\int_{\gamma} d\tau \tag{6.1.2}$$

where m_0 is the bare mass of the particle and $d\tau = \sqrt{g_{\mu\nu}(z)\dot{z}^{\mu}\dot{z}^{\nu}}d\lambda$. The overdot denotes differentiation with respect to λ.

Finally, the interaction term is given by

$$S_{interaction} = q\int_{\gamma}\Phi(z)\,d\tau = q\int \Phi(x)\,\delta_{4}(x,z)\sqrt{-g}d^{4}xd\tau \tag{6.1.3}$$

The stationary action principle under the variation of field $\delta\Phi(x)$, gives the wave equation

$$(\Box - \xi R)\,\Phi(x) = -4\pi\mu(x) \tag{6.1.4}$$

where

$$\mu(x) = q\int_{\gamma}\delta_{4}(x,z)\,d\tau \tag{6.1.5}$$

is the charge density. These equations determine $\Phi_\alpha$once the motion of scalar charge is specified.

The stationary action principle under the variation of the worldline $\delta z^\mu(\lambda)$, yields the equation of motion

$$m(\tau)\frac{Du^\mu}{d\tau} = q\left(g^{\mu\nu} + u^\mu u^\nu\right)\Phi_\nu(z) \tag{6.1.6}$$

for the scalar charge, where $m(\tau) = m_0 - q\Phi(z)$. This expression for $m(\tau)$ can also be written as a differential equation

$$\frac{dm}{d\tau} = -q\Phi_\mu(z)\,u^\mu \tag{6.1.7}$$

It is advised to note that the above equation of motion for the particle is only valid formally since the potential $\Phi(x)$, which is a solution of the wave equation, diverges on the world line.

6.2 Retarded potential near the world line

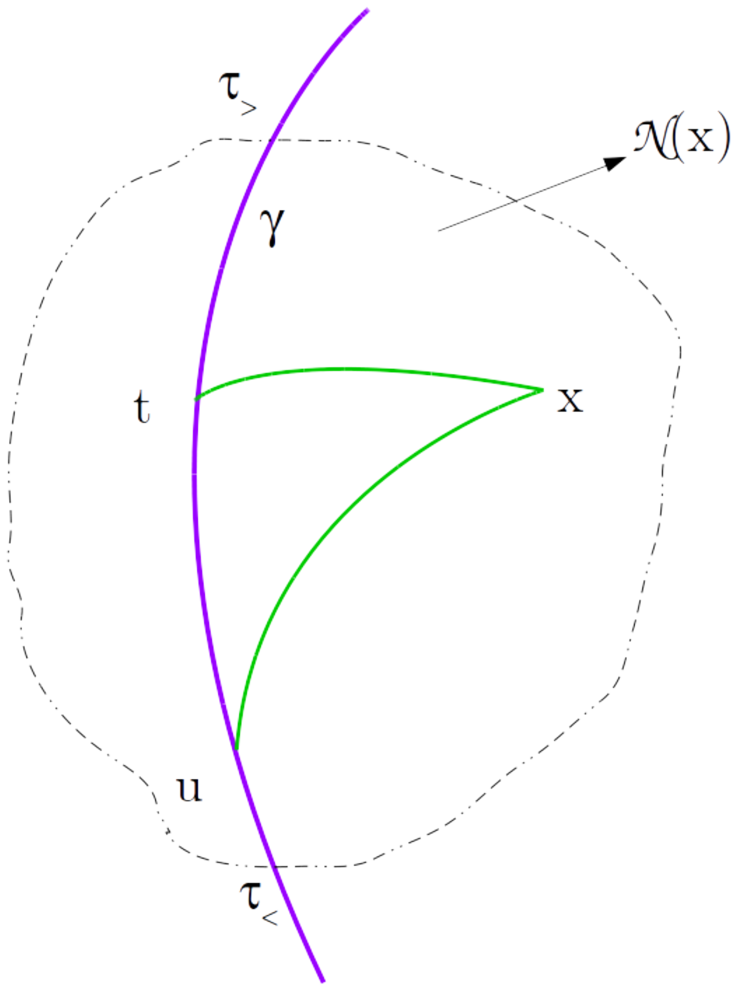

Figure 6.2.1: Normal Convex neighbourhood of x

The retarded solution to (6.1.4), after boundary conditions are suitably taken care of, is given by

$$\Phi(x) = \int G_+(x,x')\,\mu(x')\,\sqrt{g'}d^4x' \tag{6.2.1}$$

where $G_+(x,x')$ is the retarded Green's function given by the equation (4.2.26). Substituting (6.1.5), gives

$$\Phi(x) = q\int_\gamma G_+(x,z)\,d\tau \tag{6.2.2}$$

Now the spacetime is divided into two regions, one is the normal convex neighbourhood of x, denoted by $\mathcal{N}(x)$, and the other is the region outside $\mathcal{N}(x)$. Assuming that the worldline goes through $\mathcal{N}(x)$, we can divide the above integral into three parts (for a more detailed discussion of this, refer [9])

$$\Phi(x) = q\int_{-\infty}^{\tau_<} G_+(x,z)\,d\tau + q\int_{\tau_<}^{\tau_>} G_+(x,z)\,d\tau + q\int_{\tau_>}^{\infty} G_+(x,z)\,d\tau$$

where $\tau_<$ and $\tau_>$are the proper-time values when γ enters and leaves $\mathcal{N}(x)$. The last term vanishes since, $G_+ = 0$ in that region. In the second term, we can use Hadamard's solution derived in previous part on Green's functions in curved spacetime,

$$\int_{\tau_<}^{\tau_>} G_+(x,z)\,d\tau = \int_{\tau_<}^{\tau_>} U(x,z)\,\delta_+(\sigma)\,d\tau + \int_{\tau_<}^{\tau_>} V(x,z)\,\theta_+(-\sigma)\,d\tau$$

The first integration in the last equation can be computed easily by changing the variables from τ to σ. This action yields proper results because we are in the normal convex region of x and hence, there is only one point on the worldline where $\sigma = 0$. This point is in the past of x, which is nothing but $x' = z(u)$. Also, σ increases as $z(\tau)$ passes through x'. Hence, $\sigma_{\alpha'}$ is non-zero at this point. Using the result $r = \sigma_{\alpha'}u^{\alpha'}$, and noting that $d\sigma = \sigma_\mu u^\mu d\tau$, we get,

$$\Phi(x) = \frac{q}{r}U(x,x') + q\int_{\tau_<}^{u} V(x,z)\,d\tau + q\int_{-\infty}^{\tau_<} G_+(x,z)\,d\tau \tag{6.2.3}$$

6.3 Field of a scalar charge in retarded coordinates

While differentiating the above potential, we must be careful to note that a displacement in x induces a displacement in x' because the two points are always connected by a null geodesic. Hence, $\delta U = U_{;\alpha}\delta x^\alpha + U_{;\alpha'}u^{\alpha'}\delta u$. Thus, the gradient of scalar potential is given by

$$\Phi_\alpha(x) = -\frac{q}{r^2}U(x,x')\,\partial_\alpha r + \frac{q}{r}U_{;\alpha}(x,x') + \frac{q}{r}U_{;\alpha'}(x,x')\,u^{\alpha'}\partial_\alpha u + qV(x,x')\partial_\alpha u + \Phi_\alpha^{tail}(x) \tag{6.3.1}$$

where the *tail term* is defined by

$$\begin{aligned}\Phi_\alpha^{tail}(x) &= q\int_{\tau_<}^{u} \nabla_\alpha V(x,z)\,d\tau + q\int_{-\infty}^{\tau_<} \nabla_\alpha G_+(x,z)\,d\tau \\ &= q\int_{-\infty}^{u^-} \nabla_\alpha G_+(x,z)\,d\tau\end{aligned} \tag{6.3.2}$$

In the last equation, the integration is done only up to $\tau = u^- = u - 0^+$, because the retarded Green's function is singular at $\sigma = 0$.

The scalar field $\Phi_\alpha(x)$ is now expanded in powers of r and the results are expressed in retarded coordinates. It is convenient to work with frame components of$\Phi_\alpha(x)$ by decomposing it in the tetrad (e_0^α, e_a^α) defined in section 14. For this purpose, we shall need (5.3.16), (5.3.18) and

$$U(x,x') = 1 + \frac{1}{12} r^2 \left(R_{00} + 2R_{0a}\Omega^a + R_{ab}\Omega^a\Omega^b \right) + O(r^3) \tag{6.3.3}$$

which is obtained from the equations (4.2.12), (4.1.30) and (5.3.7). The frame components of Ricci tensor are defined as follows

$$R_{00}(u) = R_{\alpha'\beta'} u^{\alpha'} u^{\beta'}$$

$$R_{0a}(u) = R_{\alpha'\beta'} u^{\alpha'} e_a^{\beta'}$$

$$R_{ab}(u) = R_{\alpha'\beta'} e_a^{\alpha'} e_b^{\beta'}$$

We shall also need the following expansions,

$$U_{;\alpha}(x,x') = \frac{1}{6} r\, g^{\alpha'}{}_{\alpha} \left(R_{\alpha'0} + R_{\alpha'b}\Omega^b \right) + O(r^2) \tag{6.3.4}$$

and

$$U_{;\alpha'}(x,x')\, u^{\alpha'} = -\frac{1}{6} r \left(R_{00} + R_{0b}\Omega^b \right) + O(r^2) \tag{6.3.5}$$

Finally, we shall need

$$V(x,x') = \frac{1}{12}(1-6\xi) R + O(r) \tag{6.3.6}$$

which was first derived in the equation (4.2.13). Here, $R = R(u)$.

Using all these results, we finally obtain the equations

$$\begin{aligned}
\Phi_0(u,r,\Omega^a) &= \Phi_\alpha(x)\, e_0^\alpha(x) \\
&= \frac{q}{r} a_a \Omega^a + \frac{1}{2} q R_{a0b0}\Omega^a\Omega^b + \frac{1}{12}(1-6\xi) q R + \Phi_0^{tail} + O(r) \qquad (6.3.7) \\
\Phi_a(u,r,\Omega^a) &= \Phi_\alpha(x)\, e_a^\alpha(x) \\
&= -\frac{q}{r^2}\Omega_a - \frac{q}{r} a_b \Omega^b \Omega_a - \frac{1}{3} q R_{b0c0}\Omega^b\Omega^c\Omega_a - \frac{1}{6} q \left(R_{a0b0}\Omega^b - R_{ab0c}\Omega^b\Omega^c \right) \\
&\quad + \frac{1}{12} q \left[R_{00} - R_{bc}\Omega^b\Omega^c - (1-6\xi) R \right] \Omega_a + \frac{1}{6} q \left(R_{a0} + R_{ab}\Omega^b \right) + \Phi_a^{tail} + O(r) \qquad (6.3.8)
\end{aligned}$$

where all the frame components have their usual meanings and they are evaluated at $\tau = u$.

The above equations clearly show that the field $\Phi_\alpha(x)$ is singular on the worldline. There are two reasons for this - one is that the field diverges as r^{-2} as $r \to 0$, and the other is that many terms depend on Ω^a and therefore possess directional ambiguity at $r = 0$.

6.4 Field of scalar charge in Fermi normal coordinates

First find the frame components of $\Phi_\alpha(x)$ in the tetrad of FNC, $(\bar{e}^\alpha_0, \bar{e}^\alpha_a)$. This can be done using (5.6.1) and (5.6.2). We obtain

$$\begin{aligned}\bar{\Phi}_0 &= \Phi_\alpha \bar{e}^\alpha_0 \\ &= \left[1 + O\left(r^2\right)\right]\Phi_0 + \left[r(1 - r a_b \Omega^b) a^a(u) + \frac{1}{2} r^2 \dot{a}^a(u) + \frac{1}{2} r^2 R^a{}_{0b0}(u)\Omega^b + O(r^3)\right]\Phi_a \\ &= -\frac{1}{2} q \dot{a}_a \Omega^a + \frac{1}{12}(1 - 6\xi) qR + \bar{\Phi}^{tail}_0 + O(r)\end{aligned}$$

and

$$\begin{aligned}\bar{\Phi}_a &= \Phi_\alpha \bar{e}^\alpha_a \\ &= \left[\delta^b{}_a + \frac{1}{2} r^2 a^b(u) a_a(u) - \frac{1}{2} r^2 R^b{}_{a0c}(u)\Omega^c + O(r^3)\right]\Phi_b + \left[r a_a + O\left(r^2\right)\right]\Phi_0 \\ &= -\frac{q}{r^2}\Omega_a - \frac{q}{r} a_b \Omega^b \Omega_a + \frac{1}{2} q a_b \Omega^b a_a - \frac{1}{3} q R_{b0c0}\Omega^b \Omega^c \Omega_a - \frac{1}{6} q R_{a0b0}\Omega^b - \frac{1}{3} q R_{ab0c}\Omega^b \Omega^c \\ &\quad + \frac{1}{12} q\left[R_{00} - R_{bc}\Omega^b\Omega^c - (1 - 6\xi) R\right]\Omega_a + \frac{1}{6} q\left(R_{a0} + R_{ab}\Omega^b\right) + \bar{\Phi}^{tail}_a + O(r)\end{aligned}$$

Note that the components are still functions of retarded coordinates which are evaluated at x', except $\bar{\Phi}^{tail}_0$ and $\bar{\Phi}^{tail}_a$ which are evaluated at $\bar{x}$.

Using the results (5.5.2), (5.5.3) and (5.5.4), we have

$$\begin{aligned}\frac{1}{r^2}\Omega_a &= \frac{1}{s^2}\omega_a + \frac{1}{2s} a_b \omega^b \omega_a - \frac{3}{2s} a_b \omega^b a_a + \frac{15}{8}\left(a_b \omega^b\right)^2 \omega_a + \frac{3}{8}\dot{a}_0 \omega_a - \frac{1}{3}\dot{a}_0 \\ &\quad + \dot{a}_b \omega^b \omega_a + \frac{1}{6} R_{a0b0}\omega^b - \frac{1}{2} R_{b0c0}\omega^b \omega^c \omega_a - \frac{1}{3} R_{ab0c}\omega^b \omega^c + O(s)\end{aligned}$$

and

$$\frac{1}{r} a_b \Omega^b \Omega_a = \frac{1}{s} a_b \omega^b \omega_a + \frac{1}{2} a_b \omega^b a_a - \frac{3}{2}\left(a_b \omega^b\right)^2 \omega_a - \frac{1}{2}\dot{a}_0 \omega_a - \dot{a}_b \omega^b \omega_a + O(s)$$

In these equations all the frame components on the right-hand side are evaluated at $\bar{x}$. In the above derivation, we also use $a_a(t) = a_a(u) - s\dot{a}_a(t) + O\left(s^2\right)$, which follows from (5.5.2), that is, $u = t - s + O\left(s^2\right)$. All the other terms, when transformed into FNC, have only leading terms which are trivial.

Substituting these results into the frame components of $\Phi_\alpha(x)$ in FNC, we obtain

$$\begin{aligned}\bar{\Phi}_0(t,s,\omega^a) &= \Phi_\alpha(x)\,\bar{e}^\alpha_0(x)\\ &= -\frac{1}{2}q\dot{a}_a\omega^a + \frac{1}{12}(1-6\xi)\,qR + \bar{\Phi}^{tail}_0 + O(s) \qquad (6.4.1)\\ \bar{\Phi}_a(t,s,\omega^a) &= \Phi_\alpha(x)\,\bar{e}^\alpha_a(x)\\ &= -\frac{q}{s^2}\omega_a - \frac{q}{2s}\left(a_a - a_b\omega^b\omega_a\right) + \frac{3}{4}qa_b\omega^b a_a - \frac{3}{8}q\left(a_b\omega^b\right)^2\omega_a + \frac{1}{8}q\dot{a}_0\omega_a + \frac{1}{3}q\dot{a}_a\\ &\quad - \frac{1}{3}qR_{a0b0}\omega^b + \frac{1}{6}qR_{b0c0}\omega^b\omega^c\omega_a + \frac{1}{12}q\left[R_{00} - R_{bc}\omega^b\omega^c - (1-6\xi)\,R\right]\omega_a\\ &\quad + \frac{1}{6}q\left(R_{ao} + R_{ab}\omega^b\right) + \bar{\Phi}^{tail}_a + O(s) \qquad (6.4.2)\end{aligned}$$

where all the frame components have their usual meanings and are evaluated at $\bar{x}$.

Now, we need to compute the averages of the frame components of $\Phi_\alpha(x)$ over the surface $S(t,s)$. These averages are equivalent to the mean value of the field at a fixed distance from the worldline at a fixed time, as measured in an instantaneously comoving frame of the particle. We need two angles θ^A $(A=1,2)$ to chart the two-surface S. A standard choice is polar angles given by $\omega^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, such that $\hat{x}^a = s\omega^a$ are the spatial coordinates of the point in FNC. Defining $\omega^a_A = \frac{\partial\omega^a}{\partial\theta^A}$, we obtain, from (5.2.15), the induced metric on this two-surface

$$ds^2 = s^2\left[\omega_{AB} - \frac{1}{3}s^2R_{AB} + O\left(s^3\right)\right]d\theta^A d\theta^B \qquad (6.4.3)$$

where $\omega_{AB} = \delta_{ab}\omega^a_A\omega^b_B$ (which is the metric of unit two-sphere) and $R_{AB} = R_{acbd}\omega^a_A\omega^c\omega^b_B\omega^d$. The surface area element is then given by

$$d\mathcal{A} = s^2\left[1 - \frac{1}{6}s^2R^c{}_{acb}\omega^a\omega^b + O\left(s^3\right)\right]d\omega \qquad (6.4.4)$$

where $d\omega = \sqrt{\det[\omega_{AB}]}d^2\theta$, which in our standard choice reduces to $d\omega = \sin\theta d\theta d\phi$. This is the solid angle element. Integrating the above equation gives the total surface area of $S(t,s)$, which is $\mathcal{A} = 4\pi s^2\left[1 - \frac{1}{18}s^2R^{ab}{}_{ab} + O\left(s^3\right)\right]$.

The averaged fields are given by

$$\left\langle\bar{\Phi}_0\right\rangle(t,s) = \frac{1}{\mathcal{A}}\oint_{S(t,s)}\bar{\Phi}_0(t,s,\omega^a)\,d\mathcal{A}, \quad \left\langle\bar{\Phi}_a\right\rangle(t,s) = \frac{1}{\mathcal{A}}\oint_{S(t,s)}\bar{\Phi}_a(t,s,\omega^a)\,d\mathcal{A} \qquad (6.4.5)$$

Before proceeding, we note the following integrals

$$\frac{1}{4\pi}\oint\omega^a d\omega = 0, \quad \frac{1}{4\pi}\oint\omega^a\omega^b d\omega = \frac{1}{3}\delta^{ab}, \quad \frac{1}{4\pi}\oint\omega^a\omega^b\omega^c d\omega = 0 \qquad (6.4.6)$$

which are easy to prove. Using the above integrals, it is very easy to obtain the average field

$$\langle\bar{\Phi}_0\rangle = \frac{1}{12}(1-6\xi)\,qR + \bar{\Phi}_0^{tail} + O(s) \tag{6.4.7}$$

$$\langle\bar{\Phi}_a\rangle = -\frac{q}{3s}a_a + \frac{1}{3}q\dot{a}_a + \frac{1}{6}qR_{ao} + \bar{\Phi}_a^{tail} + O(s) \tag{6.4.8}$$

which is still singular on the world line. Nevertheless, we take the limit $s \to 0$ of the above expressions, whence the tetrad $(\bar{e}_0^\alpha, \bar{e}_a^\alpha)$ coincides with $(u^{\bar{\alpha}}, e_a^{\bar{\alpha}})$. Going back to the general coordinates using the completeness relation (5.2.4), we obtain the average field given by

$$\langle\Phi_{\bar{\alpha}}\rangle = \lim_{s\to 0}\left(-\frac{q}{3s}\right)a_{\bar{\alpha}} - \frac{1}{12}(1-6\xi)\,qRu_{\bar{\alpha}} + q\left(g_{\bar{\alpha}\bar{\beta}} + u_{\bar{\alpha}}u_{\bar{\beta}}\right)\left(\frac{1}{3}\dot{a}^{\bar{\beta}} + \frac{1}{6}R^{\bar{\beta}}{}_{\bar{\gamma}}u^{\bar{\gamma}}\right) + \Phi_{\bar{\alpha}}^{tail} \tag{6.4.9}$$

where, from (6.3.2), we have

$$\Phi_{\bar{\alpha}}^{tail}(\bar{x}) = q\int_{-\infty}^{t^-}\nabla_{\bar{\alpha}}G_+(\bar{x},z)\,d\tau.$$

All the terms on the right hand side of the equation (6.4.9) are evaluated at the point $\bar{x} = z(t)$, which can now be considered to be an arbitrary point on the worldline γ.

6.5 Equations of motion

Since the retarded field $\Phi_\alpha(x)$ is singular on the worldline of the particle, it difficult to assess how the field acts on the particle. Hence, we model the particle to be hollow spherical shell of radius s_0 and compute the net force on this shell at propertime τ by finding the average of $q\Phi_\alpha(x)$ over the surface of the shell. Here, we assume that the field produced by the shell at $s = s_0$ is equal to the field produced by the particle. Taking the limit $s_0 \to 0$, using (6.4.9), we have, at an arbitrary point on the worldline $z^\mu(\tau)$,

$$q\langle\Phi_\mu\rangle = -(\delta m)\,a_\mu - \frac{1}{12}(1-6\xi)\,q^2Ru_\mu + q^2\left(g_{\mu\nu} + u_\mu u_\nu\right)\left(\frac{1}{3}\dot{a}^\nu + \frac{1}{6}R^\nu{}_\lambda u^\lambda\right) + q\Phi_\mu^{tail} \tag{6.5.1}$$

where

$$\delta m = \lim_{s_0\to 0}\left(\frac{q^2}{3s_0}\right) \tag{6.5.2}$$

is evidently a divergent quantity, and

$$q\Phi_\mu^{tail} = q^2\int_{-\infty}^{\tau^-}\nabla_\mu G_+\left(z(\tau), z(\tau')\right)d\tau' \tag{6.5.3}$$

is the tail part of the force.

Substituting (6.5.1) and (6.5.3) into (6.1.6) yields the equation of motion for scalar charge,

$$(m+\delta m)\,a^\mu = q^2\left(\delta^\mu{}_\nu + u^\mu u_\nu\right)\left[\frac{1}{3}\dot{a}^\nu + \frac{1}{6}R^\nu{}_\lambda u^\lambda + \int_{-\infty}^{\tau^-}\nabla^\nu G_+\left(z(\tau), z(\tau')\right)d\tau'\right] \tag{6.5.4}$$

where $m = m_0 - q\Phi(z)$, the dynamical mass, is also a formally divergent quantity. The combination $m_{obs} = m + \delta m$ is taken to be finite and to give true measure of inertia of the particle. Substituting (6.5.1) and (6.5.3) into the equation (6.1.7) gives the differential equation for m_{obs}

$$\frac{dm_{obs}}{d\tau} = -\frac{1}{12}(1-6\xi)\, q^2 R - q^2 u^\mu \int_{-\infty}^{\tau^-} \nabla_\mu G_+\left(z(\tau), z(\tau')\right) d\tau' \tag{6.5.5}$$

It is worth noting that the observed mass is *not* conserved. This is a very crucial property of dynamics of scalar charge in curved spacetime. This means that, because of the time-dependent metric, the particle can emit monopole radiation, the energy for which comes from the inertial mass of the particle. A similar fact has also been discussed in the section dealing with Rohrlich's paper where the radiation energy emitted by a particle in a general motion is supplied partly by mechanical energy and partly by bound part of electromagnetic energy.

It is to be noted that the expression $\delta m = \frac{q^2}{3s_0}$ for the spherical shell is *wrong* and its actual value is $\delta m = \frac{q^2}{2s_0}$. This discrepancy is believed to originate from the assumption that the fields of the shell and particle on the surface $s = s_0$ are equal. However, except for these divergent terms, the remainder of the field remains the same for the shell ans the particle, since their difference vanishes in the limit $s_0 \to 0$. Hence, the equation of motion is reliable even though the expression for δm is incorrect.

Another important point to note is that the equation of motion derived above is a third order differential equation in $z^\mu(\tau)$ and such an equation is known to admit unphysical solutions like *runaway* solutions and *pre-acceleration* solutions. Also, so far in the derivation we have not considered the external force. Both these issues can be solved in a single stroke. In the classical picture, a point particle is only an idealization of an extended object whose internal structure is irrelevant and hence, our equations provide only an approximate description of the actual motion of the particle. Within this approximation, we can replace the acceleration on the right hand side of the equation of motion by $\frac{F^\mu_{ext}}{m}$. This yields the equation of motion

$$m\frac{Du^\mu}{d\tau} = F^\mu_{ext} + q^2\left(\delta^\mu{}_\nu + u^\mu u_\nu\right)\left[\frac{1}{3m}\dot{F}^\nu_{ext} + \frac{1}{6}R^\nu{}_\lambda u^\lambda + \int_{-\infty}^{\tau^-} \nabla^\nu G_+\left(z(\tau), z(\tau')\right) d\tau'\right] \tag{6.5.6}$$

This equation is free from third order terms in $z^\mu(\tau)$ and hence, does not admit unphysical solutions.

6.6 Discussion

Comparing the equations (2.4.29) and (6.5.4), we can clearly see that, apart from *curvature* and *tail* terms, which are to be expected since we are working in curved spacetime, the two equations match except for a factor of 2. After a similar analysis as above, we can obtain the equation of motion of a point electric charge, which is quoted here without proof (detailed derivation can be found in Part VI, Chapter 18 of [9])

$$(m+\delta m)\, a^\mu = q^2\left(\delta^\mu{}_\nu + u^\mu u_\nu\right)\left[\frac{2}{3}\dot{a}^\nu + \frac{1}{3}R^\nu{}_\lambda u^\lambda\right] + 2q^2 u_\nu \int_{-\infty}^{\tau^-} \nabla^{[\mu} G^{\nu]}_{+\lambda'}\left(z(\tau), z(\tau')\right) u^{\lambda'} d\tau'$$

We can see that the non-curvature and non-tail part of the above equation exactly matches the Lorentz-Dirac equation. The factor of 2, we believe, is attributed to the degrees of freedom of the field. For a scalar field, the number of degrees of freedom is 1 and it is well known that the number of degrees of freedom is 2 which explains the factor of 2.

The appearance of the *tail* term suggests that the equation of motion is not instantaneous, that is the motion of the particle is not decided by the quantities evaluated at that particular instant. Rather, it depends on the entire history of the particle's motion.

Bibliography

[1] Feynman R, Leighton R, and Sands M, *The Feynman Lectures on Physics*, Vol 2 (Addison-Wesley Publishing Company, California, 1964),

[2] Eric Poisson, *An introduction to Lorentz-Dirac equation*, http://arXiv.org/abs/gr-qc/9912045v1,

[3] Fulton and Rohrlich, *Classical Radiation from a uniformly accelerating charge*, Annals of Physics: **9,** 499-517 (1960),

[4] F Rohrlich, *The Principle of Equivalence*, Annals of Physics: **22,** 169-191 (1963),

[5] F Rohrlich, *The definition of electromagnetic radiation*, Il Nuovo Cimento, 1 September 1961, Volume 21, Issue 5, pp 811-822,

[6] C Teitelboim, *Splitting of Maxwell Tensor: Radiation reaction without advanced fields*, Physical Review D, Vol 1, Number 6, 15 March 1970, pp 1572-1582,

[7] T Hirayama, *Classical Radiation formula in Rindler frame*, Progress of Theoretical Physics, Vol. 108, No. 4, October 2002, pp 679-688,

[8] DeWitt and Brehme, *Radiation damping in a gravitational field*, Annals of Physics: **9,** 220-259 (1960),

[9] Eric Poisson, *The motion of point particles in curved spacetime*, Living Rev. Relativity, 14, (2011), 7, http://www.livingreviews.org/lrr-2011-7