DYNAMICS OF A TIME-DELAYED NONLINEAR MODEL FOR BUSINESS CYCLES

A Project Report

submitted by

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(EE09B101)

in partial fulfilment of the requirements

for the award of the degree of

BACHELOR OF TECHNOLOGY

in

ELECTRICAL ENGINEERING

DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY, MADRAS.

MAY 2013

PROJECT CERTIFICATE

This is to certify that the project titled **Dynamics of a time-delayed non-linear model for business cycles**, submitted by **Sagar Sahu (EE09B101)**, to the Indian Institute of Technology, Madras, for the award of the degree of **Bachelor of Technology**, is a bona fide record of the project work done by him under my supervision. The contents of this report, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Date: May 21, 2013

ACKNOWLEDGEMENTS

Gratitude is the hardest emotion to express and often does not find adequate words to convey. A B. Tech. Project is not an effort of a single person but is a contribution of many hands and brains. Therefore, I would like to thank all those who have helped me directly or indirectly during my project.

With an ineffable sense of gratitude I take this opportunity to express my deep sense of indebtedness to Dr. Gaurav Raina for providing me the opportunity to work on this project. His constant support and guidance really helped me in completing this project successfully.

I am especially thankful to Vaze Sagar Rajiv who worked with me on this project.

I also thank my parents and friends who have been there as a constant source of support in all my endeavors.

ABSTRACT

KEYWORDS: Kaldor's model; economic cycles; time-delay; stability switches; Hopf bifurcation.

Feedback in real systems is rarely instantaneous. In this paper, our focus is on the dynamical characteristics of a time-delayed nonlinear model of the trade cycle.

Kaldor's model of the trade cycle is formulated as a feedback control system. We establish a necessary and sufficient condition for local stability. This, along with a study of the system's rate of convergence, allows us to understand the relationships between system parameters and local stability. Rather surprisingly, we find that the presence of delays in the system can produce multiple stability switches. Variation in time-delay is shown to induce a Hopf bifurcation leading to the emergence of limit cycles. In essence, the system can switch between a stable state and a Hopf induced limit cycle a finite number of times. Further, using the theory of normal forms and the center manifold analysis, we show that the Hopf bifurcation is supercritical. Based on our analysis, some guidelines for the control of business cycles are outlined.

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CHAPTER 1

INTRODUCTION

At a suitable level of abstraction, models of economic phenomena should have both *nonlinear elements* and *time-delays*. Business cycles form an integral part of economic dynamics. Early work on business cycle models recognized the need for nonlinear behavioral functional forms; for example, see (2), (5), (7), and (17), and time-delays; for example, see (8). The purpose of the present paper is to investigate the impact of both nonlinear elements and time-delays on a model of business cycle dynamics. As feedback is rarely instantaneous, delays make the model more realistic.

In the class of business cycle models, the one inspired by Kaldor (7) continues to generate interest. This early and elegant model considered nonlinear investment and savings functions which shift over time in response to capital accumulation (7). Numerous aspects of this model have been examined over the years. For analytical treatment of some dynamical characteristics of Kaldor's model, see (1), (9), (10), (14), and references therein. The fact that feedback is often time-delayed has not received adequate consideration. This innocuous looking, but realistic assumption can lead to unexpected and sophisticated dynamical behavior (3), (4). In this paper, we extend the simplifying assumption by Kaldor that there are no gestation lags in capital accumulation. This leads us to study a model which combines the essential arguments by Kaldor (7) and Kalecki (8).

The importance of feedback delays has been recognized in (11), (12). We build on the model given in (11), (12) by assuming non instantaneous dependence of the level of economic activity on the capital stock. Additionally, we also propose functional forms for investments and savings and show that they adhere to the guidelines suggested by Kaldor. By analyzing the stability and bifurcation properties of these functional forms, we have quantified the impact of various model parameters on the dynamics of the trade cycle.

We first frame the model as a time-delayed nonlinear control system. Using simulations, we justify the functional forms of the savings and investment functions. Using time domain analysis, we provide an explicit characterization of the necessary and sufficient condition for local stability. This condition does depend on the delay which highlights the importance of this parameter. We show that the system can undergo multiple stability switches. In essence, the system can lose and regain local stability as the delay in the system varies. This is established analytically and verified numerically. We also investigate the local rate of convergence. Finally, we show that loss of local stability occurs via a Hopf bifurcation which leads to the emergence of limit cycles. The theory of normal forms and center manifold analysis is used to establish the stability of limit cycles. Stability charts and bifurcation diagrams accompany our analysis. Although we restrict ourselves to a discussion on business cycles, the accompanying mathematical treatment is quite general. Systems governed by second order delay differential equations occur frequently in various areas, and our analysis could be easily adapted for such systems.

CHAPTER 2

MODELING THE TRADE CYCLE

In this section, we represent the trade cycle as a time-delayed, nonlinear feedback control system. Such a representation is shown in Figure 2.1. The investment (I) and savings (S) are represented by nonlinear functions of the present level of economic activity (x) and the capital (K). The level of economic activity (x) is the state variable of interest and is determined by the I and S functions. Thus, I and S are modeled as feedback loops in the system. There is also a dependence of capital stock on investment and the same is shown in Figure 2.1.

The parameter α captures the proportional dependence of rate of change of the present level of economic activity on the difference between investment and savings. The parameter β represents the proportional dependence of rate of change of capital stock on investment. To account for the noninstantaneous dependence of I and S on both x and K, we have incorporated a time-delay parameter (T) into the feedback, as well as the feedforward, paths of this system. It is important to note that variation of this parameter (T) causes the system to exhibit a wide range of sophisticated behavior.

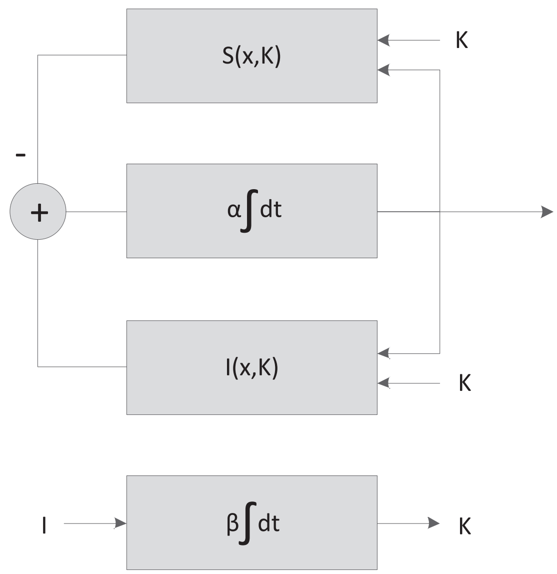

Figure 2.1: A control theoretic representation of a trade cycle. The accumulation of capital (K) determines the level of savings (S) and investment (I) (which leads to cycling (7)).

2.1 The Model

We work with the equations (2.1) and (2.2) for the system represented in Figure 2.1. which have been suggested in (11)

$$\frac{dx}{dt} = \alpha\Big[I\big(x(t-T), K(t-T)\big) - S\big(x(t-T), K(t-T)\big)\Big] \tag{2.1}$$

$$\frac{dK}{dt} = \beta I\big(x(t-T), K(t-T)\big) - \delta K\big(t-T\big)\ , \tag{2.2}$$

where I is the investment function, S represents the savings function, x denotes the level of economic activity and K represents the capital stock. Note that α, β, and δ are the system parameters as described earlier.

We propose functional forms for I and S, which adhere to the suggestions made by Kaldor (7)

$$I = \frac{a}{1+e^{-x(t-T)}} - \frac{a}{2} + \eta K(t-T) \tag{2.3}$$

$$S = \gamma \ln\left(\frac{bx(t-T)+1}{1-bx(t-T)}\right) . \tag{2.4}$$

The parameters in this model are

1. η describes the influence of the past value of capital stock on the present investment.
2. γ determines the propensity to save.
3. a and b represent the value of the slope of the I and S curves in the *normal* region as defined by Kaldor (7).
4. K captures the shift in the curves (7) due to accumulation of capital stock.

Differentiating (2.3), (2.4) with respect to x and using the condition $\frac{dI}{dx} > \frac{dS}{dx}$ (7), we obtain a condition on a and b as $\frac{a}{4} > 2b$. We numerically validate this model in MATLAB. For the purpose of simulation, the parameter values are chosen as $\alpha = 1$, $\gamma = 1$, $a = 4$, $b = \frac{1}{3}$, $kd = 0.02$ (kd captures the effect of η). The dependence on K in these curves is captured by shifting the I and S curves. The numerical results obtained using simulations are shown in Figure 2.2. It can be readily seen that for these parameter values, this model predicts a cycle, i.e. the economy goes from a state of high activity to a state of low activity and back.

We now proceed to analyze various dynamical aspects of this model.

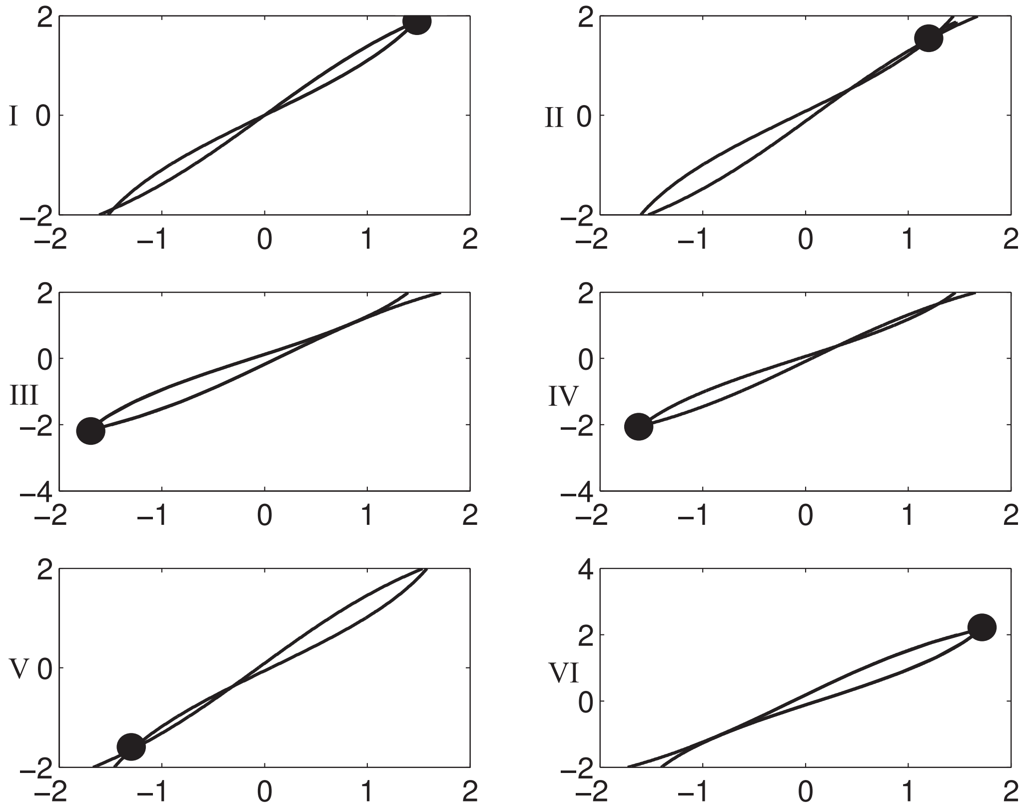

Figure 2.2: Numerical verification of model (2.3), (2.4). The dot indicates the current level of economic activity and the curves correspond to the savings and investment functions (7). The occurrence of cycles can be seen in the plots I to VI.

CHAPTER 3

STABILITY ANALYSIS

In this section, we establish the necessary and sufficient condition for stability of the system described by (2.1) and (2.2) using time domain analysis. For this, the linearization of (2.1) and (2.2) is carried out for the functional forms of I and S as given in (2.3) and (2.4).

$$\frac{dx}{dt} = \frac{a\alpha}{1-e^{-x(t-T)}} - \frac{a\alpha}{2} + \eta\alpha K(t-T) - \gamma\alpha \ln\left(\frac{bx(t-T)+1}{1-bx(t-T)}\right) \quad (3.1)$$

$$\frac{dK}{dt} = \frac{a\beta}{1-e^{-x(t-T)}} - \frac{a\beta}{2} + \eta\beta K(t-T) - \delta K(t)\ . \quad (3.2)$$

We now prove the existence and uniqueness of an equilibrium point E^* of the (2.1) and (2.2). We follow the style of analysis done in (6).

The conditions for existence of a unique equilibrium point are

1. there exists a constant $L > 0$ such that $|I(x)| \leq L$ for all $x \in R$;
2. $I(0) > 0$;

Then there exists a unique equilibrium point $E^* = (x^*, K^*)$ where x^* is the solution of

$$I(x) = \gamma \ln\left(\frac{bx^*+1}{1-bx^*}\right)\ ,$$

and K^* is determined by

$$K^* = \frac{\beta\gamma}{\delta} \ln\left(\frac{bx(t-T)+1}{1-bx(t-T)}\right)\ .$$

Proof. Let the equilibrium point be (x^*, K^*). At this equilibrium point, the values of $\frac{dx}{dt}$ and $\frac{dK}{dt}$ are zero.

$$\frac{dx}{dt} = \frac{dK}{dt} = 0 \ ,$$

that is

$$0 = \alpha I(x) - \gamma\alpha \ln\left(\frac{bx(t-T)+1}{1-bx(t-T)}\right) \tag{3.3}$$

$$0 = \beta I(x) - \delta K(t) \ . \tag{3.4}$$

Let us assume that $x > 0$ and $K > 0$ satisfy (3.3) and (3.4). Then

$$I(x) = \gamma \ln\left(\frac{bx(t-T)+1}{1-bx(t-T)}\right) \ ,$$

and

$$K = \frac{\beta\gamma}{\delta} \ln\left(\frac{bx(t-T)+1}{1-bx(t-T)}\right) \ .$$

which is a unique equilibrium point of (2.1) and (2.2).

To linearize about the equilibrium point, we write $x(t-T) = y(t-T) + x^*$ and $K(t) = k(t) + K^*$ where $y(t)$ and $k(t)$ are small perturbations about the equilibrium point. (3.1) and (3.2) linearize to

$$\frac{dy}{dt} = Ay(t-T) + \eta\alpha k(t-T) \tag{3.5}$$

$$\frac{dk}{dt} = By(t-T) + \eta\beta k(t-T) - \delta k(t) \ , \tag{3.6}$$

where

$$A = -\frac{a\alpha e^{-x^*}}{(1-e^{-x^*})^2} - \frac{2b\gamma\alpha}{1-b^2x^{*2}}$$

$$B = -\frac{a\beta e^{-x^*}}{(1-e^{-x^*})^2} .$$

Using (3.5) and (3.6), we get a second order delay differential equation which has the characteristic equation

$$\lambda^2 + a\lambda + b\lambda e^{-\lambda T} + c + de^{-\lambda T} = 0 . \tag{3.7}$$

Here the values of $a,\ b,\ c,\ d$ and T are constants, which can be calculated in terms of system parameters.

3.1 Local Stability Analysis

The critical points of the system are obtained by constraining the roots of the characteristic equation to be purely imaginary.

It is clear that there is no need to analyze the system for the cases where both a and b are simultaneously zero, as this can only lead to unstable solutions.

For the case where $b = 0$ (which corresponds to a delay in the feedback path), a purely imaginary solution of the characteristic equation is computed. Let this solution be iv, where v is given by

$$v^2 = \frac{2c - a^2 \pm \sqrt{a^4 - 4a^2c + 4d^2}}{2} .$$

At the critical point, the system satisfies

$$vT = \sin^{-1}\left(\frac{av}{d}\right).$$

Hence, the necessary and sufficient condition for stability is given by

$$T < \frac{1}{v}\sin^{-1}\left(\frac{av}{d}\right). \tag{3.8}$$

It can be seen that for a valid solution to exist for v, the term $a^4 - 4a^2c + 4d^2$ must be positive. This can be rewritten as

$$c < \frac{a^2}{4} + \left(\frac{d}{a}\right)^2. \tag{3.9}$$

If the condition given in (3.9) is satisfied, we can always find a critical point of the system and so a stability switch will exist. This is the case of *delay dependent* stability.

The parameter c represents the feedback term in the system without delay. If (3.9) is not satisfied, the system does not experience stability switches and is always stable. This is the case of *delay independent stability*. This behavior can be intuitively explained by focusing on the fact that when c is large enough, the instantaneous feedback term dominates the delayed feedback term and hence the effect of time-delay vanishes.

For the case where $a \neq 0,\ b \neq 0,\ c \neq 0$ and $d \neq 0$, a similar analysis, as the done previously for the case $b = 0$, is carried out which leads to the following necessary and sufficient condition for stability

$$vT < \tan^{-1}\left(\frac{bv}{d}\right) + \tan^{-1}\left(\frac{v^2 - c}{a}\right), \tag{3.10}$$

where v is given by

$$v^2 = \frac{b^2 + 2c - a^2 \pm \sqrt{(b^2 + 2c - a^2)^2 - 4(c^2 - d^2)}}{2}. \qquad (3.11)$$

The local stability analysis is carried out numerically for (3.7) and stability charts are prepared.

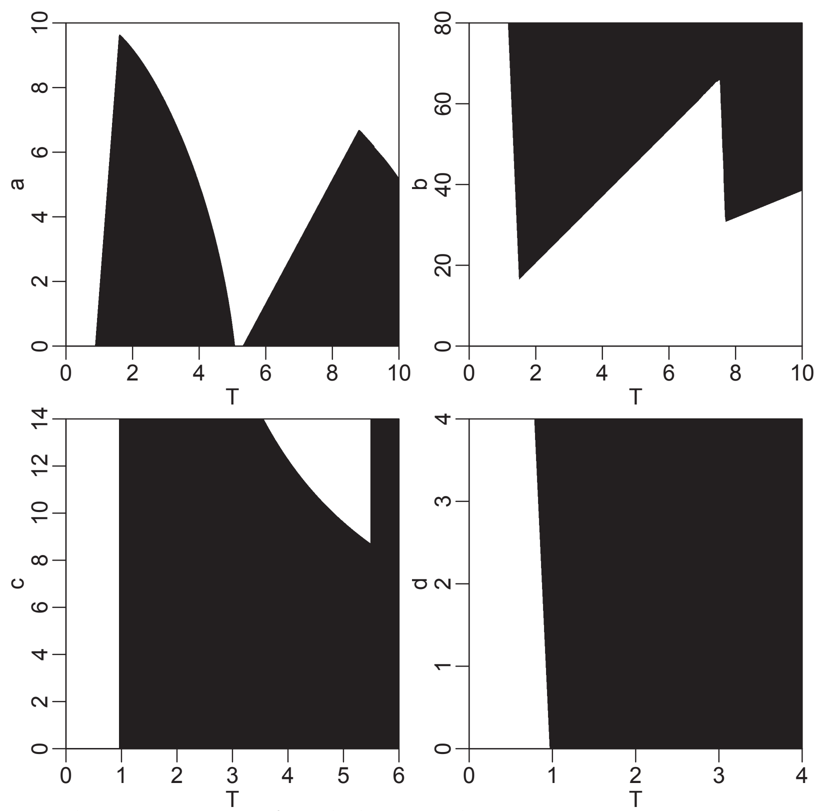

Figure 3.1: The variation of stability with parameters $a,\ b,\ c,\ d$ and delay T for the system (3.7) (white regions represent stable solutions whereas black regions represent unstable ones). When a is varied, $b = 10$, $c = 10$ and $d = 1$. When b is varied, $a = 1$, $c = 50$ and $d = 0.1$. When c is varied, $a = 0.5$, $b = 10$ and $d = 1$. When d is varied, $a = 1$, $b = 10$ and $c = 10$.

3.2 Stability Switches

In this section, we provide an explicit characterization of the conditions for (3.7) to undergo *multiple stability switches*. A plot demonstrating the presence of stability switches in (3.7) is shown in Figure 3.2.

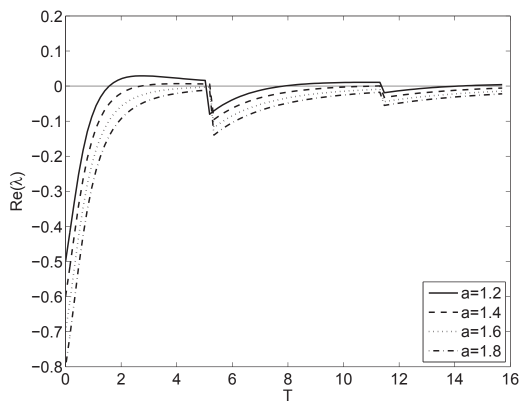

Figure 3.2: Stability switches in (3.7) with varying T when (3.9) is satisfied. Here, $b = 0$, $c = 1$, $d = 1$.

It is worth noting that for the choice of parameter values in Figure 3.2, the condition on c as specified in (3.9) is satisfied. If c does not satisfy (3.9), stability switches do not occur. This is demonstrated in Figure 3.3.

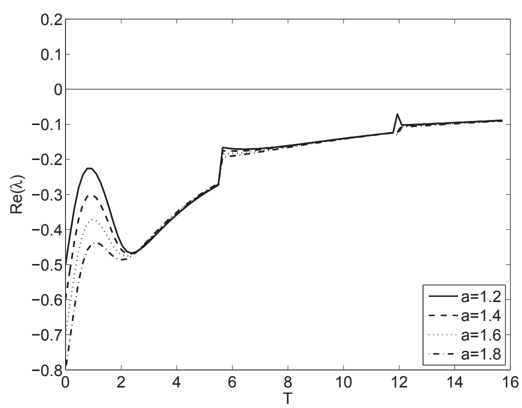

Figure 3.3: The absence of stability switches in (3.7) with varying T when (3.9) is not satisfied. Here, $b = 0$, $c = 5$, $d = 1$.

For an analytic characterization of stability switches, consider the value of v as given by (3.11) which can be rewritten as

$$v^2 = \frac{b^2 + 2c - a^2 \pm \sqrt{(b^2 - a^2)^2 + 4c(b^2 - a^2) + 4d^2}}{2}. \quad (3.12)$$

We establish conditions on $a, b, c,$ and d for one or more positive roots v to exist.

Table 3.1: Conditions on parameters of the system (3.7) that give rise to different behavior in terms of stability.

S. No.	Conditions on parameters	Number of positive roots of v
1	$c^2 < d^2$	1
2	$b^2 < a^2 - 2c$ $d^2 < c^2$	0
3	$a^2 - 2c < b^2 < a^2$ $(a^2 - b^2)(c - \frac{1}{4}(a^2 - b^2)) < d^2 < c^2$	2
4	$a^2 - 2c < b^2 < a^2$ $d^2 < (a^2 - b^2)(c - \frac{1}{4}(a^2 - b^2))$	0
5	$a^2 < b^2$ $d^2 < c^2$	2

Case 1) As can be seen from (3.11), if condition given in case 1 is satisfied, then the value of $\sqrt{(b^2 + 2c - a^2)^2 - 4(c^2 - d^2)}$ will be greater than $b^2 + 2c - a^2$. In this case, v^2 has one solution. Hence, v has just one positive root and the system will have only one bifurcation point where it will switch from a stable to an unstable state.

Case 2) From (3.11), we realize that if the conditions given in case 2 are satisfied, then v^2 does not have a solution. Hence, v has no solution and the system always remains stable i.e. the system does not undergo Hopf bifurcation. This is an instance of delay independent stability.

Case 3) If the conditions given in case 3 are satisfied, from (3.12) we can say that v^2 has two positive roots and thus there are two positive roots for v. Hence, the system undergoes two stability switches i.e. the system transitions from being stable to being unstable twice.

Case 4) From (3.11), we can see that if the conditions given in case 4 are satisfied, then v^2 does not have a solution. Hence v has no solution and the system always remains stable i.e. the system does not undergo Hopf bifurcation. This is another instance of delay independent stability.

Case 5) If the conditions given in case 5 are satisfied, then v^2 has two positive roots, as can be seen from (3.12). Hence v has two positive roots and the system undergoes stability switches twice.

CHAPTER 4

RATE OF CONVERGENCE

We vary the parameters c and d to analyze their effect on the rate of convergence of (3.7) and plot the same. The rate of convergence is estimated by calculating the eigenvalue of the system with the lowest magnitude of its real part. This eigenvalue represents the most slowly decaying exponential in the time evolution of the system and serves as a measure of its rate of convergence.

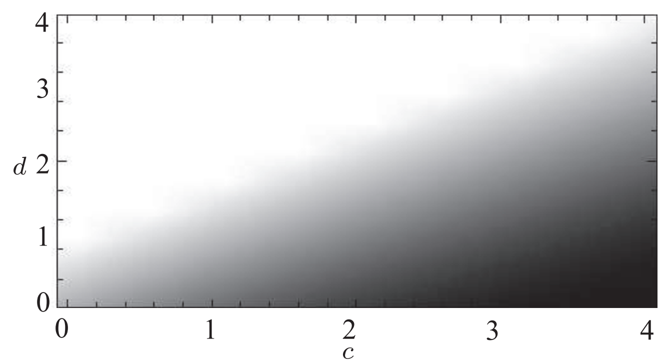

Figure 4.1: The variation of the rate of convergence of (3.7) with parameters c and d. Here $a = 1$, $b = 0$ and $T = 1.5$. Darker regions represent faster convergence.

Referring to (3.7), the system will be overdamped if the imaginary part of the eigenvalue is zero and the real part is negative. To investigate this, we substitute $\lambda = u$ in (3.7), which gives

$$\begin{aligned} u^2 + au + bue^{-uT} + c + de^{-uT} &= 0 \\ \frac{-u^2 + au + c}{bu + d} &= e^{-uT} \\ \frac{u}{b} + \frac{1}{b}\left(a - \frac{d}{b}\right) + \frac{c - \frac{d}{b}\left(a - \frac{d}{b}\right)}{bu + d} &= -e^{-uT}\,. \end{aligned} \tag{4.1}$$

We solve (4.1) graphically to find a solution for a particular choice of parameters.

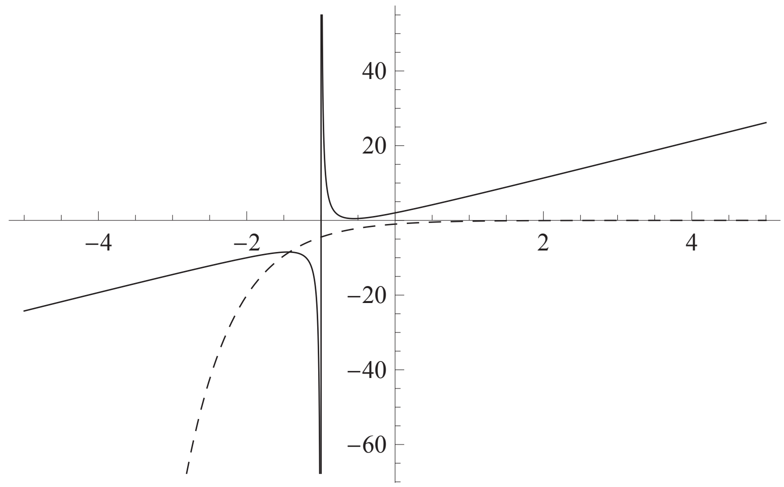

Figure 4.2: Solving (4.1) graphically for the overdamped system (3.7). Here $a = 1$, $b = 0.2$, $c = 1$ and $d = 1$.

By looking at Figure 4.2, we can see that the shape of the two graphs always leads to a solution. This implies that the system is always overdamped. This conclusion is highly counter intuitive and hence makes us question the possibility of having both real and complex roots at the same time, and rate of convergence being decided by the root whose real part has the least magnitude. This possibility has not been investigated further in this paper.

CHAPTER 5

BIFURCATION ANALYSIS

In nonlinear systems, transition from stability to instability is generally accompanied by a Hopf Bifurcation (3), (15). In this section, we prove that our model undergoes a Hopf bifurcation as the value of time-delay is increased, as long as the conditions for delay dependent stability as given in Table 1 are satisfied. In an economy, there are a number of economic agents acting at different time scales. These agents may have a stabilizing or destabilizing effect on the economy based on the delay in the response to their actions. Hence we choose T for bifurcation analysis.

5.1 Transversality Condition

To demonstrate that the loss of stability occurs via a Hopf bifurcation, we prove the transversality condition. To that end, we differentiate (3.7) with respect to T which leads to

$$\lambda'\big(2\lambda + a + be^{-\lambda T} - b\lambda e^{-\lambda T} - dTe^{-\lambda T}\big) - b\lambda^2 e^{-\lambda T} - d\lambda e^{-\lambda T} \quad = \quad 0 \, ,$$

where $\lambda' = \frac{d\lambda}{dT}$. Rewriting,

$$\lambda' \quad = \quad \frac{\lambda\big(b\lambda + d\big)}{\big(2\lambda + a\big)e^{\lambda T} + \big(b - b\lambda T - dT\big)} \, . \tag{5.1}$$

From (3.7),

$$e^{\lambda T} = \frac{b\lambda + d}{\lambda^2 + a\lambda + c} . \tag{5.2}$$

Substituting (5.2) in (5.1),

$$\lambda' = \frac{\lambda}{-\frac{(2\lambda+a)}{\lambda^2+a\lambda+c} + \frac{b}{b\lambda+d} - T} .$$

Considering each term in the denominator separately and substituting $\lambda = iv$, the first term can be written as

$$\frac{2iv + a}{-v^2 + iav + c} = \frac{av^2 + ac + i\big(2cv - 2v^3 - a^2v\big)}{A} ,$$

where $A = (c - v^2)^2 + a^2v^2$, is always positive. Similarly the second term can be written as

$$\frac{b}{ibv + d} = \frac{bd - i\big(b^2v\big)}{B} ,$$

where $B = b^2v^2 + d^2$, is always positive. To simplify (3.12), we multiply both numerator and denominator with AB. It is seen that since both A and B are positive, we need not consider the AB term appearing in the numerator. The denominator can be written as

$$\begin{aligned} Denominator &= \big(bdA - aBv^2 - acB - TAB\big) \\ &\quad -i\big(2Bcv - 2Bv^3 - a^2Bv + Ab^2v\big) . \end{aligned}$$

To simplify, we make the denominator a real number by multiplying both the numerator and the denominator with the conjugate of the denominator. In the modified equation obtained, it is not necessary to consider the denominator any longer, as it is a sum of squares of two real numbers and so is always positive. As we are interested in the real part of λ' and as the numerator of (5.1) is purely imaginary for $\lambda = iv$, we consider the term obtained by multiplying iv with the imaginary part of the conjugate of the denominator. This gives us the real part of the numerator as

$$\Re\big(Numerator\big) \;=\; -2Bv^2 + 2Bc - a^2B + Ab^2 \; .$$

Substituting the values of A and B

$$\Re\big(Numerator\big) \;=\; -b^2v^4 - 2d^2v^2 + 2cd^2 - a^2d^2 + b^2c^2 \; . \qquad (5.3)$$

If we equate (5.3) to zero and solve for v, we get a value of v which does not satisfy the (3.7). So we can say that the real part of $\frac{d\lambda}{dT}$ is always non zero and so there exists a $T_{critical}$ at which the system undergoes a Hopf bifurcation.

5.2 Stability of Limit Cycles

In this section, we address the question of stability of the emerging limit cycles. We do a Taylor expansion of (2.1) and (2.2) about the equilibrium point.

$$\frac{dy}{dT} \;=\; F_1 y(t-T) + F_2 y^2(t-T) + \eta\alpha k(t-T) \qquad (5.4)$$

$$\frac{dk}{dT} \;=\; G_1 y(t-T) + G_2 y^2(t-T) + \eta\beta k(t-T) - \delta k(t) \; , \qquad (5.5)$$

where

$$
\begin{aligned}
F_1 &= \frac{d}{dy}\left(\frac{1}{1-e^{-y(t-T)}}\right) \\
F_2 &= 0.5\frac{dF_1}{dy} \\
G_1 &= \frac{d}{dy}\left(\ln\left(\frac{by(t-T)+1}{1-by(t-T)}\right)\right) \\
G_2 &= 0.5\frac{dG_1}{dy} \ .
\end{aligned}
$$

We introduce an exogenous non-dimensional parameter $\kappa = \kappa_c + \mu$ to establish the stability of the limit cycle (4), where $\kappa_c = 1$ and the Hopf bifurcation takes place at $\mu = 0$.

$$
\frac{dy}{dT} = \kappa F_1 y(t-T) + \kappa F_2 y^2(t-T) + \kappa\eta\alpha k(t-T) \tag{5.6}
$$

$$
\frac{dk}{dT} = G_1 y(t-T) + G_2 y^2(t-T) + \eta\beta k(t-T) - \delta k(t) \ , \tag{5.7}
$$

We perform calculations that enable us to address questions about the form of the bifurcating solutions of (5.6) and (5.7) as it transits from stability to instability via a Hopf bifurcation. For our analysis, we follow the style given in (4). Consider the following autonomous delay differential equation.

$$
\frac{d}{dt}\boldsymbol{u}(t) = \mathcal{L}_\mu \boldsymbol{u}_t + \mathcal{F}(\boldsymbol{u}_t, \mu) \ , \tag{5.8}
$$

where $t > 0$, $\mu \in R$, where for $T > 0$

$$
\boldsymbol{u}_t(\theta) = \boldsymbol{u}(t+\theta) \qquad \boldsymbol{u} : [-T, 0] \to R^2, \qquad \theta \in [-T, 0].
$$

$\mathcal{L}_\mu$ is a one-parameter family of continuous (bounded) linear operators. The operator $\mathcal{F}(\boldsymbol{u}_t, \mu)$ contains the nonlinear terms. We assume that $\mathcal{F}$ is analytic and that $\mathcal{F}$ and $\mathcal{L}_\mu$ depend analytically on the bifurcation parameter μ for small $|\mu|$. Note that (5.6) is of the form (5.8), where $\boldsymbol{u} = [y, k]^T$. The objective now is to cast (5.8) into the form

$$\frac{d}{dt}\boldsymbol{u}_t \;=\; \mathcal{A}(\mu)\boldsymbol{u}_t + \mathcal{R}\boldsymbol{u}_t \;, \tag{5.9}$$

which has $\boldsymbol{u}_t$ rather than both $\boldsymbol{u}$ and $\boldsymbol{u}_t$. We transform the linear problem $d\boldsymbol{u}(t)/dt = \mathcal{L}_\mu \boldsymbol{u}_t$. By Riesz representation theorem, there exists an $n \times n$ matrix function $\eta(\cdot, \mu) : [-T, 0] \to R^{n^2}$, such that the components of η have bounded variation and for all $\boldsymbol{\phi} \in C[-T, 0]$

$$\mathcal{L}_\mu \boldsymbol{\phi} \;=\; \int_{-T}^{0} d\eta(\theta, \mu)\boldsymbol{\phi}(\theta) \;.$$

In particular,

$$\mathcal{L}_\mu \boldsymbol{\phi} \;=\; \int_{-T}^{0} d\eta(\theta, \mu)\boldsymbol{u}(t+\theta) \;. \tag{5.10}$$

We observe that

$$d\eta(\theta, \mu) \;=\; \begin{bmatrix} \kappa F_1 \delta(\theta + T) d\theta & \kappa\eta\alpha\delta(\theta + T) d\theta \\ G_1 \delta(\theta + T) d\theta & (\eta\beta\delta(\theta + T) - \delta(\theta)) d\theta \end{bmatrix}$$

satisfies (5.10). For $\boldsymbol{\phi} \in C[-T, 0]$, we define

$$\mathcal{A}(\mu)\boldsymbol{\phi}(\theta) \;=\; \begin{cases} \frac{d\boldsymbol{\phi}(\theta)}{d\theta}, & \theta \in [-T, 0) \\ \int_{-T}^{0} d\eta(s, \mu)\boldsymbol{\phi}(s) \equiv \mathcal{L}_\mu \boldsymbol{\phi}, & \theta = 0 \end{cases} \tag{5.11}$$

and

$$\mathcal{R}\boldsymbol{\phi}(\theta) = \begin{cases} 0, & \theta \in [-T, 0) \\ \mathcal{F}(\boldsymbol{\phi}, \mu), & \theta = 0 \end{cases} . \tag{5.12}$$

As $d\boldsymbol{u}_t/d\theta = d\boldsymbol{u}_t/dt$, (5.8) becomes (5.9) as desired.

The bifurcating periodic solutions $\boldsymbol{u}\big(t, \mu(\epsilon)\big)$ of (5.8) have amplitude $\mathcal{O}(\epsilon)$, period $\mathcal{P}(\epsilon)$ and non-zero Floquet exponent $\mathcal{B}(\epsilon)$, where the expressions for μ, $\mathcal{P}$ and $\mathcal{B}$ are given by

$$\begin{aligned} \mu &= \mu_2\epsilon^2 + \mu_4\epsilon^4 + \cdots \\ \mathcal{P} &= \frac{2\pi}{\omega_0}(\mathcal{T}_2\epsilon^2 + \mathcal{T}_4\epsilon^4 + \cdots) \\ \mathcal{B} &= \mathcal{B}_2\epsilon^2 + \mathcal{B}_4\epsilon^4 + \cdots . \end{aligned}$$

The sign of μ_2 determines the direction of bifurcation: $\mu_2 > 0$ implies a supercritical bifurcation and $\mu_2 < 0$ implies a subcritical bifurcation. The sign of $\mathcal{B}_2$ determines the stability of bifurcation: asymptotic orbital stability if $\mathcal{B}_2 < 0$ and instability if $\mathcal{B}_2 > 0$.

We only need to compute the expressions at $\mu = 0$, hence we set $\mu = 0$. Let $\boldsymbol{q}(\theta)$ be the eigenfunction for $\mathcal{A}(0)$ corresponding to $\lambda(0)$, namely

$$\mathcal{A}(0)\boldsymbol{q}(\theta) = i\omega_0\boldsymbol{q}(\theta) . \tag{5.13}$$

To find ω_0 and $\boldsymbol{q}(\theta)$, let $\boldsymbol{q}(\theta) = \boldsymbol{q}_0 e^{i\omega_0\theta}$, where $\boldsymbol{q}_0 = [1, q_{02}]^T$. Substituting this in (5.13) and using the expression for $\mathcal{A}$ as in (5.11) we get

$$\int_{-T}^{0} \begin{bmatrix} \kappa F_1 \delta(\theta+T)d\theta & \kappa\eta\alpha\delta(\theta+T)d\theta \\ G_1\delta(\theta+T)d\theta & (\eta\beta\delta(\theta+T)-\delta(\theta))d\theta \end{bmatrix} \begin{bmatrix} e^{iw_0\theta} \\ q_{02}e^{iw_0\theta} \end{bmatrix} = \begin{bmatrix} iw_0\theta \\ q_{02}iw_0\theta \end{bmatrix} .$$

Solving, we get

$$\begin{aligned} F_1 e^{-i\omega_0 T} + \eta\alpha q_{02} e^{-i\omega_0 T} &= iw_0 \\ G_1 e^{-i\omega_0 T} + \eta\beta q_{02} e^{-i\omega_0 T} - q_{02} &= iw_0 q_{02} \ . \end{aligned}$$

Solving for q_{02} we get,

$$\eta\alpha(i\omega_0+1)q_{02}^2 + F_1(1+i\omega_0 - i\omega_0\eta\beta)q_{02} - i\omega_0 G_1 = 0 \ .$$

From this quadratic equation, we find the value of q_{02} which is used to find the value of ω_0.

We define the adjoint operator $\mathcal{A}^*(0)$ as

$$\mathcal{A}^*(0)\boldsymbol{\alpha}(s) = \begin{cases} \frac{d\boldsymbol{\alpha}(s)}{ds}, & s \in [0,T) \\ \int_{-T}^{0} d\eta^T(t,0)\boldsymbol{\alpha}(-t), & s = 0. \end{cases}$$

Note that the domains of $\mathcal{A}$ and $\mathcal{A}^*$ are $C^1[-T,0]$ and $C^1[0,T]$ respectively. As

$$\mathcal{A}(0)\boldsymbol{q}(\theta) = \lambda(0)\boldsymbol{q}(\theta) \ ,$$

$\overline{\lambda}(0)$ is an eigenvalue for $\mathcal{A}^*$, and

$$\mathcal{A}^*(0)\boldsymbol{q}^* = -i\omega_0\boldsymbol{q}^*$$

for some non-zero vector $\boldsymbol{q}^*$. For $\boldsymbol{\phi} \in C[-T,0]$ and $\boldsymbol{\psi} \in C[0,T]$, we define an inner product

$$\langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle = \overline{\boldsymbol{\psi}}^T(0)\boldsymbol{\phi}(0) - \int_{\theta=-T}^{0} \int_{\tau=0}^{\theta} \overline{\boldsymbol{\psi}}^T(\tau-\theta)d\eta(\theta)\boldsymbol{\phi}(\tau)d\tau \,. \tag{5.14}$$

Then, $\langle \boldsymbol{\psi}, \mathcal{A}\boldsymbol{\phi} \rangle = \mathcal{A}^* \langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle$ for $\boldsymbol{\phi} \in \mathrm{Dom}(\mathcal{A})$ and $\boldsymbol{\psi} \in \mathrm{Dom}(\mathcal{A})$.

Let $\boldsymbol{q}^*(s) = De^{i\omega_0 s}$ be an eigenvector of $\mathcal{A}^*$ corresponding to eigenvalue $-i\omega_0$. We now find D such that the eigenvectors $\boldsymbol{q}$ and $\boldsymbol{q}^*$ satisfy conditions $\langle \boldsymbol{q}^*, \boldsymbol{q} \rangle = 1$ and $\langle \boldsymbol{q}^*, \overline{\boldsymbol{q}} \rangle = 0$. These two equations are solved for variables D_1 and D_2, where $\boldsymbol{D} = [D_1, D_2]^T$. Using the expression (5.14) for the inner product, we get

$$\begin{aligned} 1 &= \overline{D}_1 + \overline{D}_2 q_{02} + \overline{D}_1 \kappa_c T F_1 e^{-i\omega_0 T} + \overline{D}_1 \kappa_c T \eta \alpha q_{02} e^{-i\omega_0 T} + \overline{D}_2 \kappa_c T G_2 e^{-i\omega_0 T} \\ &\quad + \overline{D}_2 \kappa_c T \eta \beta q_{02} e^{-i\omega_0 T} \\ 0 &= i2\omega_0 \overline{D}_1 + i2\omega_0 \overline{D}_2 \overline{q_{02}} + \left(e^{i\omega_0 T} - e^{-i\omega_0 T}\right)\left(\overline{D_1} F_1 + \overline{D_1} \eta \alpha \overline{q_{02}} + \overline{D_2} G_1 + \overline{D_2} \eta \beta \overline{q_{02}}\right) \,. \end{aligned} \tag{5.15}$$

From (5.15), we solve for D_1 and D_2. For $\boldsymbol{u}_t$, a solution of (5.9) at $\mu = 0$, we define

$$\begin{aligned} z(t) &= \langle \boldsymbol{q}^*, \boldsymbol{u}_t \rangle \\ \boldsymbol{w}(t,0) &= \boldsymbol{u}_t(\theta) - 2\mathcal{RE}(z(t)\boldsymbol{q}(\theta)). \end{aligned}$$

Then, on the manifold, C_0, $\boldsymbol{w}(t,\theta) = \boldsymbol{w}(z(t), \overline{z}(t), \theta)$, where

$$\boldsymbol{w}(z, \overline{z}, \theta) = \boldsymbol{w}_{20}(\theta)\frac{z^2}{2} + \boldsymbol{w}_{11}(\theta) z\overline{z} + \boldsymbol{w}_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots \,. \tag{5.16}$$

In effect, z and $\overline{z}$ are local coordinates for C_0 in C in the directions of $\boldsymbol{q}^*$ and $\overline{\boldsymbol{q}}^*$, respectively. Note that $\boldsymbol{w}$ is real if $\boldsymbol{u}_t$ is real and we deal only with real solutions. The existence of the center manifold enables the reduction of (5.9) to an ordinary differential equation for a single complex variable on C_0. At $\mu = 0$, this is

$$\begin{aligned} z^{'}(t) &= \langle \boldsymbol{q}^*, \mathcal{A}\boldsymbol{u}_t + \mathcal{R}\boldsymbol{u}_t \rangle \\ &= i\omega_0 z(t) + \overline{\boldsymbol{q}}_*(0) \cdot \mathcal{F}\Big(\boldsymbol{w}(z, \overline{z}, \theta) + 2\mathcal{RE}\big(z\boldsymbol{q}(\theta)\big)\Big) \\ &= i\omega_0 z(t) + \overline{\boldsymbol{q}}_*(0) \cdot \mathcal{F}_{\prime}(z, \overline{z}), \end{aligned} \tag{5.17}$$

which is abbreviated to

$$z^{'}(t) = i\omega_0 z(t) + g(z, \overline{z}). \tag{5.18}$$

The next objective is to expand g in powers of z and $\overline{z}$. However, we also have to determine the coefficients $\boldsymbol{w}_{ij}(\theta)$ in (5.16). Once the $\boldsymbol{w}_{ij}$ have been determined, (5.17) would be explicit (as abbreviated in (5.17)). Expanding $g(z, \overline{z})$ in powers of z and $\overline{z}$, we have

$$\begin{aligned} g(z, \overline{z}) &= \overline{\boldsymbol{q}}^*(0) \cdot \mathcal{F}_0(z, \overline{z}) \\ &= g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} \cdots . \end{aligned}$$

Following (4), we write

$$\boldsymbol{w}^{'} = u_t^{'} - z^{'}\boldsymbol{q} - \overline{z}^{'}\overline{\boldsymbol{q}}$$

and using (5.9) and (5.18), we obtain

$$\boldsymbol{w}^{'} = \begin{cases} \mathcal{A}\boldsymbol{w} - 2\mathcal{RE}\left(\overline{\boldsymbol{q}}^*(0) \cdot \mathcal{F}_0 \boldsymbol{q}(\theta)\right), & \theta \in [-T, 0] \\ \mathcal{A}\boldsymbol{w} - 2\mathcal{RE}\left(\overline{\boldsymbol{q}}^*(0) \cdot \mathcal{F}_0 q(0)\right) + \mathcal{F}_0, & \theta = 0 \end{cases}$$

which is rewritten as

$$\boldsymbol{w}' = \mathcal{A}\boldsymbol{w} + \boldsymbol{h}(z, \overline{z}, \theta) \tag{5.19}$$

using (5.16), where

$$\boldsymbol{h}(z, \overline{z}, \theta) = \boldsymbol{h}_{20}(\theta)\frac{z^2}{2} + \boldsymbol{h}_{11}(\theta)z\overline{z} + \boldsymbol{h}_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots . \tag{5.20}$$

Now on C_0, near the origin

$$\boldsymbol{w}' = \boldsymbol{w}_z z' + \boldsymbol{w}_{\overline{z}}\overline{z}'.$$

Use (5.16) and (5.18) to replace $\boldsymbol{w}_z$, z' (and their conjugates by their power series expansion) and equating this with (5.19), we get

$$\begin{aligned} (2i\omega_0 - \mathcal{A})\boldsymbol{w}_{20}(\theta) &= \boldsymbol{h}_{20}(\theta) \\ -\mathcal{A}\boldsymbol{w}_{11}(\theta) &= \boldsymbol{h}_{11}(\theta) \\ -(2i\omega_0 + \mathcal{A})\boldsymbol{w}_{02}(\theta) &= \boldsymbol{h}_{02}(\theta). \end{aligned} \tag{5.21}$$

We start by observing

$$\begin{aligned} \boldsymbol{u}_t(\theta) &= \boldsymbol{w}(z, \overline{z}, \theta) + \boldsymbol{q}(\theta)z + \overline{\boldsymbol{q}}(\theta)\overline{z} \\ &= \boldsymbol{w}_{20}(\theta)\frac{z^2}{2} + \boldsymbol{w}_{11}(\theta)z\overline{z} + \boldsymbol{w}_{02}(\theta)\frac{\overline{z}^2}{2} + \boldsymbol{q}_0 e^{i\omega_0\theta}z + \overline{\boldsymbol{q}}_0 e^{-i\omega_0\theta}\overline{z} + \cdots \end{aligned}$$

from which we obtain $\boldsymbol{u}_t(0)$ and $\boldsymbol{u}_t(-T)$. We have actually looked ahead and we require only the coefficients of z^2, $z\overline{z}$, $\overline{z}^2$ and $z^2\overline{z}$. Hence we keep only the relevant terms in the expansions that follow. We see that we have only one nonlinear term, $y^2(t-T)$, in (5.6).

$$\begin{aligned} y^2(t-T) &= \left[\boldsymbol{u_t}(-T)\boldsymbol{u_t}^T(-T)\right]_{11} \\ &= e^{-2i\omega_0 T}z^2 + e^{2i\omega_0 T}\overline{z}^2 + 2z\overline{z} \\ &\quad + \left(w_{201}(-T)e^{i\omega_0 T} + 2w_{111}(-T)e^{-i\omega_0 T}\right) z^2\overline{z} + \cdots \end{aligned}$$

where $[w_{ij1}, w_{ij2}]^T = \boldsymbol{w}_{ij}$. Recall that,

$$g(z,\overline{z}) = \overline{\boldsymbol{q}}^*(0)\cdot\mathcal{F}_0(z,\overline{z}) \equiv \overline{D}_1\mathcal{F}_{01}(z,\overline{z}) + \overline{D}_2\mathcal{F}_{02}(z,\overline{z}) \tag{5.22}$$

where $[\mathcal{F}_{01}, \mathcal{F}_{02}]^T = \mathcal{F}_0$, and

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2}\cdots. \tag{5.23}$$

Comparing (5.22) and (5.23), we get

$$g_{20} = 2\overline{D}_1\kappa F_2 e^{-2i\omega_0 T} + 2\overline{D}_2 G_2 e^{-2i\omega_0 T} \tag{5.24}$$

$$g_{11} = 2\overline{D}_1\kappa F_2 + 2\overline{D}_2 G_2 \tag{5.25}$$

$$g_{02} = 2\overline{D}_1\kappa F_2 e^{2i\omega_0 T} + 2\overline{D}_2 G_2 e^{2i\omega_0 T} \tag{5.26}$$

$$g_{21} = \left(2\overline{D}_1\kappa F_2 + 2\overline{D}_2 G_2\right)\left(w_{201}(-T)e^{i\omega_{-T}T} + 2w_{111}(-T)e^{-i\omega_0 T}\right). \tag{5.27}$$

For the expression of g_{21}, we still need to evaluate $\boldsymbol{w}_{11}(0)$, $\boldsymbol{w}_{11}(-T)$, $\boldsymbol{w}_{20}(0)$ and $\boldsymbol{w}_{20}(-T)$. Now for $\theta \in [-T, 0)$

$$\begin{aligned} \boldsymbol{h}(z,\overline{z},\theta) &= -2\mathcal{RE}\left(\overline{\boldsymbol{q}}^*(0)\cdot\mathcal{F}_0\boldsymbol{q}(\theta)\right) \\ &= -2\mathcal{RE}\left(g(z,\overline{z})\boldsymbol{q}(\theta)\right) \\ &= -g\left(z,\overline{z}\right)\boldsymbol{q}(\theta) - \overline{g}(z,\overline{z})\overline{\boldsymbol{q}}(\theta) \\ &= \left(g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2}\right)\boldsymbol{q}(\theta) - \left(\overline{g}_{20}\frac{z^2}{2} + \overline{g}_{11}z\overline{z} + \overline{g}_{02}\frac{\overline{z}^2}{2}\right)\overline{\boldsymbol{q}}(\theta) \end{aligned}$$

which when compared to (5.20), yields

$$\begin{aligned}\boldsymbol{h}_{20}(\theta) &= -g_{20}\boldsymbol{q}(\theta) - g_{02}\overline{\boldsymbol{q}}(\theta)\\ \boldsymbol{h}_{11}(\theta) &= -g_{11}\boldsymbol{q}(\theta) - g_{11}\overline{\boldsymbol{q}}(\theta).\end{aligned}$$

From (5.11) and (5.21), we get

$$\boldsymbol{w}_{20}^{'}(\theta) = 2i\omega_0\boldsymbol{w}_{20}(\theta) + g_{20}\boldsymbol{q}(\theta) + \overline{g}_{02}\overline{\boldsymbol{q}}(\theta) \tag{5.28}$$

$$\boldsymbol{w}_{11}^{'}(\theta) = g_{11}\boldsymbol{q}(\theta) + \overline{g}_{11}\overline{\boldsymbol{q}}(\theta)\ . \tag{5.29}$$

Solving (5.28) and (5.29), we obtain

$$\boldsymbol{w}_{20}(\theta) = -\frac{g_{20}}{i\omega_0}\boldsymbol{q}_0 e^{i\omega_0\theta} - \frac{\overline{g}_{02}}{3i\omega_0}\overline{\boldsymbol{q}}_0 e^{-i\omega_0\theta} + \boldsymbol{e}e^{2i\omega_0\theta} \tag{5.30}$$

$$\boldsymbol{w}_{11}(\theta) = \frac{g_{11}}{i\omega_0}\boldsymbol{q}_0 e^{i\omega_0\theta} - \frac{\overline{g}_{11}}{i\omega_0}\overline{\boldsymbol{q}}_0 e^{-i\omega_0\theta} + \boldsymbol{f} \tag{5.31}$$

for some $\boldsymbol{e} = [e_1, e_2]^T$ and $\boldsymbol{f} = [f_1, f_2]^T$, which we determine. For $\boldsymbol{h}(z, \overline{z}, 0) = -2\mathcal{RE}\left(\overline{\boldsymbol{q}}^*(0) \cdot \mathcal{F}_0 q(0)\right) + \mathcal{F}_0$

$$\begin{aligned}\boldsymbol{h}_{20}(0) &= -\,g_{20}q(0) - g_{02}\overline{\boldsymbol{q}}(0) + \begin{bmatrix} 2\kappa F_2 e^{-2i\omega_0 T} \\ 2G_2 e^{-2i\omega_0 T} \end{bmatrix}\\ \boldsymbol{h}_{11}(0) &= -\,g_{11}q(0) - g_{11}\overline{\boldsymbol{q}}(0) + \begin{bmatrix} 2\kappa F_2 \\ 2G_2 \end{bmatrix}.\end{aligned}$$

Again, from (5.11) and (5.21), we obtain

$$g_{20}\boldsymbol{q}(0) + g_{02}\overline{\boldsymbol{q}}(0) = \begin{bmatrix} 2\kappa F_2 e^{-2i\omega_0 T} \\ 2G_2 e^{-2i\omega_0 T} \end{bmatrix} + \begin{bmatrix} \kappa F_1 w_{201}(-T) + \kappa\eta\alpha w_{202}(-T) - 2i\omega_0 w_{201}(0) \\ \eta\beta w_{202}(-T) + G_1 w_{201}(-T) - 2i\omega_0 w_{202}(0) - w_{202}(0) \end{bmatrix} \tag{5.32}$$

$$g_{11}\boldsymbol{q}(0) + g_{11}\overline{\boldsymbol{q}}(0) = \begin{bmatrix} 2\kappa F_2 \\ 2G_2 \end{bmatrix} + \begin{bmatrix} \kappa F_1 w_{111}(-T) + \kappa\eta\alpha w_{112}(-T) \\ G_1 w_{111}(-T) + \eta\beta w_{112}(-T) - w_{112}(0) \end{bmatrix}. \tag{5.33}$$

Substituting the expression for $w_{ijk}(x), x \in [-T, 0]$ from (5.30) and (5.31) into (5.32) and (5.33), and finally solving for e_1, e_2, f_1 and f_2, we obtain

$$\boldsymbol{e} \equiv \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \frac{n_2 A_1 - m_2 A_2}{m_1 n_2 - m_2 n_1} \\ \frac{m_1 A_2 - n_1 A_1}{m_1 n_2 - m_2 n_1} \end{bmatrix} \tag{5.34}$$

and

$$\boldsymbol{f} \equiv \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{(\eta\beta - 1)B_1 - \kappa\eta\alpha B_2}{\kappa F_1(\eta\beta - 1) - \kappa\eta\alpha G_1} \\ \frac{\kappa F_1 B_2 - G_1 B_1}{\kappa F_1(\eta\beta - 1) - \kappa\eta\alpha G_1} \end{bmatrix}, \tag{5.35}$$

where

$$
\begin{aligned}
M_1 &= -2i\omega_0 + \kappa F_1 e^{-2i\omega_0 T} \\
M_2 &= \kappa\eta\alpha e^{-2i\omega_0 T} \\
N_1 &= G_1 e^{2i\omega_0 T} \\
N_2 &= \eta\beta e^{-2i\omega_0 T} - 2i\omega_0 - 1 \\
A_1 &= g_{20} + g_{02} - 2\kappa F_2 e^{-2i\omega_0 T} + \kappa F_1 \left(\frac{g_{20}}{i\omega_0} e^{-i\omega_0 T} + \frac{\overline{g}_{02}}{3i\omega_0} e^{i\omega_0 T} \right) \\
&\quad - 2\left(g_{20} + \frac{\overline{g}_{02}}{3} \right) + \kappa\eta\alpha \left(\frac{\overline{g}_{02}}{3i\omega_0} \overline{q_{02}} e^{i\omega_0 T} - \frac{g_{20}}{i\omega_0} q_{02} e^{-i\omega_0 T} \right) \\
A_2 &= g_{20} q_{02} + g_{02} \overline{q}_{02} - 2G_2 e^{-2i\omega_0 T} + G_1 \left(\frac{g_{20}}{i\omega_0} e^{-i\omega_0 T} + \frac{\overline{g}_{02}}{3i\omega_0} e^{i\omega_0 T} \right) \\
&\quad + \eta\beta \left(\frac{\overline{g}_{02}}{3i\omega_0} \overline{q_{02}} e^{i\omega_0 T} - \frac{g_{20}}{i\omega_0} q_{02} e^{-i\omega_0 T} \right) - (2i\omega_0 + 1) \left(\frac{g_{20} q_{02}}{i\omega_0} + \frac{\overline{g}_{02} \overline{q}_{02}}{3i\omega_0} \right) \\
B_1 &= 2g_{11} - 2\kappa F_2 - \kappa F_1 \left(\frac{g_{11}}{i\omega_0} e^{-i\omega_0 T} - \frac{\overline{g}_{11}}{i\omega_0} e^{i\omega_0 T} \right) \\
&- \kappa\eta\alpha \left(-\frac{\overline{g}_{11}}{i\omega_0} \overline{q_{02}} e^{i\omega_0 T} + \frac{g_{11}}{i\omega_0} q_{02} e^{-i\omega_0 T} \right) \\
B_2 &= g_{11}(q_{02} + \overline{q_{02}}) - 2G_2 - G_1 \left(\frac{g_{11}}{i\omega_0} e^{-i\omega_0 T} - \frac{\overline{g}_{11}}{i\omega_0} e^{i\omega_0 T} \right) \\
&- \eta\beta \left(-\frac{\overline{g}_{11}}{i\omega_0} \overline{q_{02}} e^{i\omega_0 T} + \frac{g_{11}}{i\omega_0} q_{02} e^{-i\omega_0 T} \right) .
\end{aligned}
$$

Using the values of $\boldsymbol{e}$ and $\boldsymbol{f}$ in (5.30) and (5.31), followed by substituting $\theta = 0, -T$, we obtain the values for $\boldsymbol{w}_{11}(0)$, $\boldsymbol{w}_{11}(-T)$, $\boldsymbol{w}_{20}(0)$ and $\boldsymbol{w}_{20}(-T)$. Using these we evaluate g_{21}. Hence, we have the expressions for g_{20}, g_{11}, g_{02} and g_{21}.

All the quantities required for the stability analysis of the Hopf bifurcation have been calculated. We can now comment on the nature of the

Hopf bifurcation by finding out the values of μ_2 and $\mathcal{B}_2$ (4)

$$\begin{aligned} \mu_2 &= \frac{-\mathcal{RE}\big(c_1(0)\big)}{\alpha'(0)} \\ \mathcal{B}_2 &= 2\mathcal{RE}\big(c_1(0)\big) \end{aligned} \tag{5.36}$$

where, as in (4),

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}\,. \tag{5.37}$$

Now, μ_2 is calculated for various values of μ in the neighborhood of the bifurcation point i.e. $\mu = 0$. The values of μ_2 are plotted in Figure 5.1.

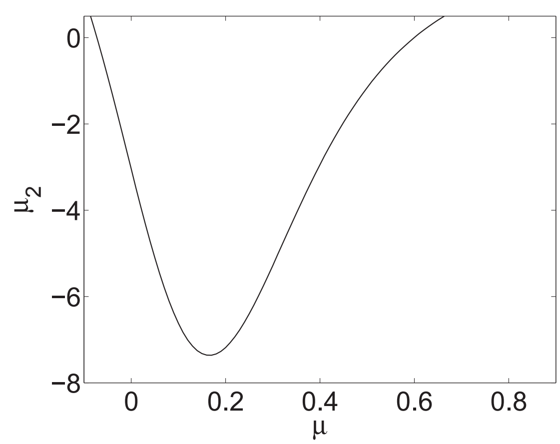

Figure 5.1: Values of μ_2 in the neighborhood of the bifurcation point i.e. $\mu = 0$.

As, can be clearly seen in Figure 5.1, $\mu_2 < 0$ at the bifurcation point, $\mu = 0$. Hence, the system (3.1) and (3.2) transitions from stability to instability through a *supercritical* Hopf bifurcation and the emerging limit cycles are stable.

Also, the expressions for period $\mathcal{P}(\epsilon)$ and Floquet exponent $\mathcal{B}(\epsilon)$ are given by

$$\begin{aligned}
\mathcal{P}(\epsilon) &= \frac{2\pi}{\omega_0}\left(1+\epsilon^2\mathcal{T}_2+\mathcal{O}\left(\epsilon^4\right)\right) \\
\mathcal{T}_2 &= -\frac{\mathcal{IM}\left(c_1(0)\right)+\mu_2\omega'(0)}{\omega_0} \\
\mathcal{B}(\epsilon) &= \mathcal{B}_2\epsilon^2+\mathcal{O}\left(\epsilon^4\right) \\
\epsilon &= \sqrt{\frac{\mu}{\mu_2}}\,.
\end{aligned} \tag{5.38}$$

5.3 Bifurcation Diagrams

The bifurcation diagrams for (3.1) and (3.2) are plotted using MATLAB. They are shown in Figure 5.2.

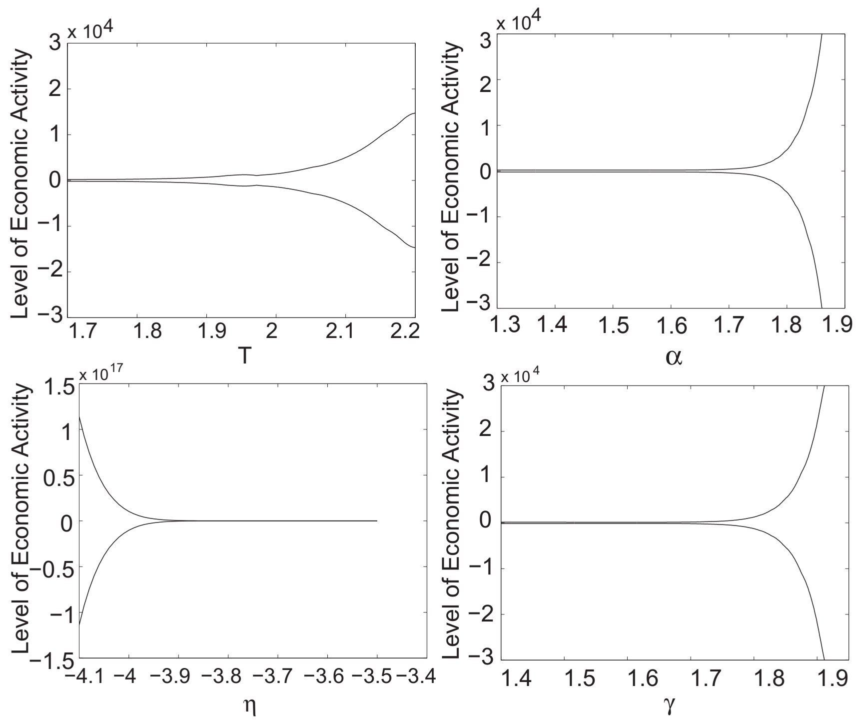

Figure 5.2: Hopf bifurcation in (3.1), (3.2) for varying parameters $T,\ \alpha,\ \eta$ and γ. In the bifurcation diagram, all parameters, except the one being varied, are set to 1.

CHAPTER 6

OUTLOOK

The presence of periodic cycles in the economy provides motivation to study some nonlinear models of economic dynamics. In this paper, we analyzed a certain nonlinear Kaldor-Kalecki business cycle model.

We proposed functional forms of the savings and investment functions in a Kaldor-Kalecki framework, which is modeled as a nonlinear control system. We then outlined necessary and sufficient condition for local stability and analyzed rate of convergence of the linearized system. An extremely interesting feature of the model was its ability to undergo multiple stability switches: from a stable equilibrium to a Hopf induced limit cycle. We showed that loss of stability occurs via a Hopf bifurcation which leads to the emergence of limit cycles. We proved that the bifurcation is supercritical and hence the limit cycles are stable.

As can be seen from the bifurcation diagrams in Figure 5.2, the system undergoes a transition from stability to a stable limit cycle. It can be seen that the radius of the limit cycle is largest for bifurcation due to variation of η. As η represents the effect of past capital stock on the present investment, a larger magnitude of η implies a greater dependence of the present investment on the past value of capital stock. This can be seen as a stronger influence of the delayed capital stock, which tends to destabilize the economy.

Intuitively it may seem desirable to maximize the influence of the current investment on the level of economic activity (captured by α). Contrary to this expectation, the bifurcation with increasing α demonstrates that unrestrained maximization of this parameter has a destabilizing effect on the economy, which may manifest itself as rapid fluctuations in economic indicators such as GDP, level of employment etc.

Avenues for further research

Mathematically, it would be natural to characterize the limit cycles. Of particular interest would be the relationship between system parameters and the amplitude of the resulting cycles. From a modeling perspective, it would be natural to extend the model and analysis to cater for inputs to the system, and to consider external disturbances. This would be a step towards developing a framework which might be able to offer policy guidelines.

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