On the Lie algebras of BPS states

A Project Report

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HERSHDEEP SINGH
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THESIS CERTIFICATE

This is to certify that the thesis titled **On the Lie algebras of BPS states**, submitted by **Hershdeep Singh**, to the Indian Institute of Technology, Madras, for the award of the degree of **Master of Technology**, is a bona fide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Prof. Suresh Govindarajan
Research Guide
Professor
Department of Physics
IIT Madras, 600 036

Place: Chennai
Date: 19th May 2014

Prof. Nandita DasGupta
Research Guide
Professor
Department of Electrical Engineering
IIT Madras, 600 036

Place: Chennai
Date: 19th May 2014

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ABSTRACT

KEYWORDS: Lie algebras; representation theory; Weyl denominator formula; modular forms; Jacobi forms; Siegel modular forms; additive lift; eta products

We begin with a review of the classficiation of finite semisimple Lie algebras and affine Kac-Moody Lie algebras. We work out two applications of the Weyl denominator formula and find modular forms naturally appearing in the denominator formulae for affine Kac-Moody algebras. In order to study Borcherds algebras, we take a detour into modular forms and, in particular, Jacobi forms and Siegel modular forms. Finally, we establish one case of two conjectures that concern the additive lift of a weight 1 and index $1/2$ Jacobi form $\Psi_{1,1/2}$ that produces the Siegel modular form Δ_1. We also tabulate the characters of all multiplicative eta products corresponing to conjugacy classes of M_{12} under modular transformations.

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CHAPTER 1

Finite Lie Algebras

1.1 What is a Lie Algebra?

A Lie algebra L is just vector space over a field F with an operation, formally called the *Lie Bracket*, $[\cdot,\cdot] : L \times L \to L$ satisfying the following properties:

(L1) $(x, y) \mapsto [x, y]$ is linear for all $x, y \in L$

(L2) $[x, x] = 0$ for all $x \in L$

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [y, x]] = 0$ for all $x, y, z \in L$

Notice that (L1) and (L2) together imply antisymmetricity of the Lie bracket $[x, y] = -[y, x]$. Axiom (L3) is called **Jacobi Identity**.

1.2 Some Mundane Definitions

If H, K are subspaces of L, then $[HK]$ is defined as the subspace spanned by all products of the form $[hk]$ with $h \in H, k \in K$.

A subspace $H \subset L$ is a **subalgebra** if it is closed under the lie bracket i.e. $[HH] \subset H$

An **ideal** I of L is a special kind of subalgebra with the added property that it is closed under multiplication with L, that is $[IL] \in I$

Suppose we have two lie algebras L_1, L_2 over k. A map $\theta : L_1 \to L_2$ is a **homomorphism** if θ is linear and if it respects the Lie bracket,

$$\theta([xy]) = [\theta(x), \theta(y)] \quad \text{for all } x, y \in L_1$$

A bijective homomorphism is called **isomorphism** of Lie algebras.

An **associative algebra** A is a vector space equipped with an associative bilinear map $(\cdot,\cdot) : A \times A \to A$ that takes $(x, y) \to xy$ for $x, y \in A$. Given any associative algebra, we can form a Lie algebra by just defining the lie bracket to be the commutator $[x, y] = xy - yx$. Its easy to see that the Jacobi identity (L3) is satisfied by the virtue of associativity.

1.3 Representations

If V is a vector space over k, then $GL(V)$ is an associative algebra. As mentioned above, it can be made into a Lie algebra by setting $[x, y] = xy - yx$. This is called the **general linear Lie algebra** $\mathfrak{gl}(V)$

Any subalgebra of the general linear Lie algebra is called a **linear Lie algebra**.

A **representation** is a just a homomorphism $\rho : L \to \mathfrak{gl}(V)$. Incidentally, this also makes V into an L**-module** by defining

$$x \cdot v = \rho(x)v \quad \text{for all } x \in L, v \in V \tag{1.1}$$

A L-module is to a vector space what a vector space is to a field. That is, it defines a way in which elements of L multiply with elements of V to give a product in V

1.4 The Adjoint Representation

The most important example of a representation for a lie algebra is the **Adjoint representation** $\text{ad} : L \to \mathfrak{gl}(V)$ which send $x \to \text{ad}\, x$ where $\text{ad}\, x(y) = [x, y]$. In this case, the vector space V is chosen to be L itself and L becomes as an L-module.

1.5 Cartan Subalgebra

A Cartan subalgebra of a Lie algebra is a maximal nilpotent subalgebra. More formally, define the **normaliser** $N(H)$ of a subalgebra H as

$$N(H) = \{x \in L : [h, x] \in H \; \forall h \in H\}$$

H is called **Cartan subalgebra** if it is nilpotent and $N(H) = H$.

The next theorem helps us to find Cartan subalgebras explicitly:

Theorem 1. *Let x be a regular element of L. Then the null component $L_{0,x}$ is a Cartan subalgebra of L.*

The converse of the above theorem is also true:

Theorem 2. *If H is a Cartan subalgebra, then there exists an $x \in L$ such that $L_{0,x} = H$*

1.6 Killing Form

The Killing form (x, y) of x and y is defined as

$$(x, y) = \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) \tag{1.2}$$

Note that this makes sense as the trace of a matrix is independent of the basis.

In general, a bilinear form $\beta(x, y)$ is called **non-degenerate** if its **radical** S is 0, where $S = \{x \in L : \beta(x, y) = 0\ \forall y \in L\}$. Since the Killing form is associative, its radical is in fact an ideal of L. This tells us that

Theorem 3. *Let L be a Lie algebra. It is semisimple if and if only if the Killing form is non-degenerate.*

1.7 The Root System

For a semisimple algebra, the Cartan subalgebra is abelian. Since H is abelian, $\mathrm{ad}_L\, H$ is a commuting set of endomorphisms of L. This tells us that $\mathrm{ad}_L\, H$ is simultaneously diagonalizable and thus L permits a decomposition as a direct sum $L = \oplus L_\alpha$ where $L_\alpha = \{x \in L : [hx] = \alpha(h)x\ \forall h \in H\}$, where $\alpha \in H^*$ are called the **roots** and $\dim L_\alpha$ is the multiplicity of the root. The vector $x \in L$ for which $[h, x] = \alpha(h)x$ is called the **root vector**.

The dimension of Cartan subalgebras of L is called the **rank** of L.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a fixed basis of roots so that every element of H_*^0 can be written as $\rho = \sum_i c_i\alpha_i$. Define an ordering on H_0^* as:

(i) $\rho > 0$ if the first non-zero $c_i > 0$

(ii) $\rho < 0$ if the first non-zero $c_i < 0$

(iii) $\rho > \sigma$ iff $\rho - \sigma > 0$

Given a set of roots, we can write it as a disjoint union of positive roots and negative roots. A **simple root** is one which can not be written as a sum of two or more positive roots.

1.8 Cartan Matrix

For an n rank algebra, the Cartan matrix is the $n \times n$ matrix

$$A_{ij} = \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \tag{1.3}$$

where $\alpha_1, \ldots, \alpha_n$ are simple roots.

For example, the Cartan Matrix for $\mathfrak{sl}(2)$ is

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{1.4}$$

1.9 Weyl Group

The **Weyl group** is the group generated by reflections about the roots. Given an element in $\alpha \in L$, the weyl group element w_α acts on element $\beta \in L$ as

$$w_\alpha(\beta) = \alpha - \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\beta$$

It turns out that the Weyl group is in fact generated by just the reflections about the simple roots. So the Cartan matrix provides all the information necessary to compute the Weyl group reflections.

$$\begin{aligned} w_{\alpha_i}(\alpha_j) &= \alpha_j - 2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \\ &= \alpha_j - A_{ji}\alpha_i \end{aligned}$$

1.10 Dynkin Diagrams

The determination of all possible Cartan matrices is possible using **Dynkin Diagrams**. A Dynkin diagram of a root system with n simple roots is simply a graph with n nodes. The ith node represents the ith simple root and the ith and jth nodes are connected by $A_{ij} \cdot A_{ji}$ lines.

If two nodes are connected by more than one lines, then we mark the larger root by an arrow pointing towards it. The larger root among node i and j can be determined by finding the smaller one of A_{ij} and A_{ji}.

It turns out that the class of admissible Dynkin diagrams that correspond to root systems is rather restricted. This allows to classify all semisimple lie algebras, as shown in figure 1.1.

1.11 Classification

Semisimple Lie algebras have a out to have a very elegant classification.

Once we have the adjoint representation and the Cartan subalgebra, the action of Lie algebra L on H via the adjoint representation gives the root system. The scalar product on the root space gives us a connection between between H and its dual space H^*. We then establish a one to one correspondence between semisimple lie algebras and root systems. This reduces the problem to one of classification of root systems. The information about simple roots is completely contained in the Cartan Matrix, which can be classified using Dynkin Diagrams, as shown in figure 1.1.

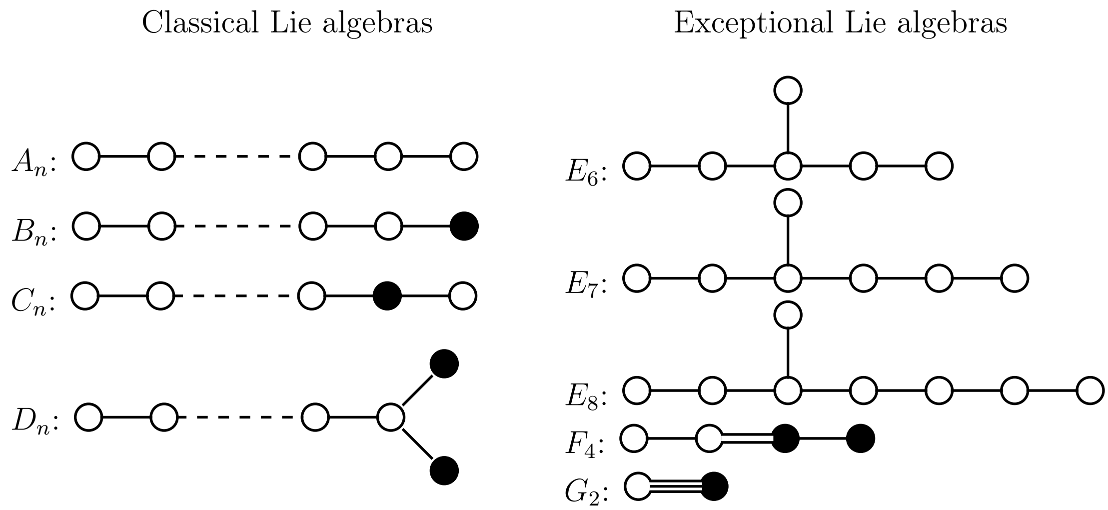


Figure 1.1: The complete classification of all Semisimple Finite Lie algebras. Apart from the four families of classical Lie algebras defined as subalgebras of $\mathfrak{gl}_n$, we find five exceptional Lie algebras.

CHAPTER 2

Beyond Finite Lie Algebras

By relaxing the conditions on the admissible Cartan matrices, and consequently on the types of simple roots, we get generalizations of the finite Lie algebras studied in the previous chapter. Starting with finite Lie algebras, which contain only positive normed simple roots, we get other (often infinite-dimensional) Lie algebras by adding suitable simple roots.

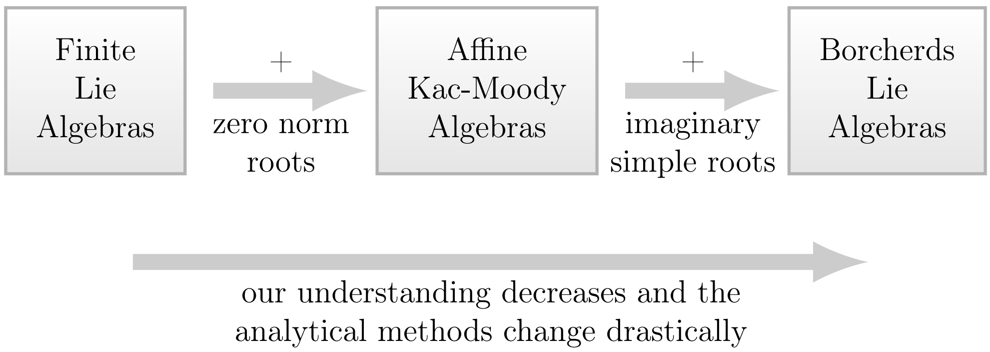


Figure 2.1: Summary of the types of Lie algebras

2.1 Generalized Cartan Matrix

We will move on to more general Lie algebras, associated with a **generalized Cartan matrix** (abbreviated as GCM). An $n \times n$ matrix A is a GCM if it satisfies

$$a_{ij} = 2 \quad \text{for } i = 1, \ldots, n \tag{2.1}$$

$$a_{ij} \in \mathbb{Z} \text{ and } a_{ij} \leq 0 \quad \text{if } i \neq j \tag{2.2}$$

$$a_{ij} = 0 \implies a_{ji} = 0 \tag{2.3}$$

As can be easily seen, the GCM reduces to a Cartan matrix if the off-diagonal entries are non-positive.

2.2 Trichotomy of GCMs

If A is an **indecomposable** GCM, then it is exactly one of the three types

(a) Finite
(b) Affine
(c) Indefinite

A GCM has a finite type if

(i) $\det A \neq 0$
(ii) there exists $u > 0$ with $Au > 0$
(iii) $Au \geq 0$ implies $u > 0$ or $u = 0$

A GCM has affine type if

(i) corank $A = 1$
(ii) there exists $u > 0$ with $Au = 0$
(iii) $Au \geq 0$ implies $Au = 0$

A GCM has indefinite type if

(i) there exists $u > 0$ with $Au < 0$
(ii) $Au \geq 0$ and $u \geq 0$ imply $u = 0$.

It is possible to construct Lie algbras corresponding to a given GCM. Such algebras are of the finite, affine and indefinite type respectively.

2.3 Affine Kac-Moody Lie Algbras

As shown in figure 2.1, Lie algebras of the affine type are obtained by adding zero normed roots to the existing list of finite Lie algebras. The representation theory of Affine Lie algebras is well understood. The complete list of affine Lie algebras can infact be constructed from the finite Lie algebras list in a simple fashion.

2.4 Borcherds Lie algebras

A further generalization to our story of Lie algebras occurs when we allow *imaginary simple roots*. Such Lie algebras are called Borcherds Kac-Moody (or Generalized Kac-Moody) algebras. Other than that, a Borcherds algebra is defined in a very similar way to a Kac-Moody algebra.

CHAPTER 3

Weyl Denominator Formula

One of the main objects of our study is the **Weyl denominator formula**, which is a special case of the **Weyl character formula** for the trivial one dimensional representation. In the following sections, we shall illustrate this interesting formula with two cases: one of finite semisimple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and the other of the affine Lie algebra $\hat{A}_1$. We shall see how the latter case naturally leads us to a study of modular forms, an important aspect in our later endeavours.

3.1 Finite Lie Algebra: $\mathfrak{su}(3)$

As a very simple application of the Weyl denominator formula, lets take the case of the semi-simple lie algebra $\mathfrak{sl}_2(\mathbb{C})$ or, in the Cartan classification, A_2. The Cartan matrix and the Dynkin diagram for A_2 is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The Weyl denominator formula for the finite semi simple lie algebras is

$$e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{w \in W} \epsilon(w) e^{w(\rho)} \tag{3.1}$$

where

ρ is half the sum of the positive roots,
Φ^+ is the set of all positive roots,
W is the Weyl group,
$\epsilon(w)$ is the order of the Weyl group element w.

Let the simple roots be $\{\alpha_1, \alpha_2\}$ and the set of the positive roots will then be $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_3 = \alpha_1 + \alpha_2$.

ρ, half the sum of all positive roots, is simply α_3.

The Weyl group is generated by the action of the simple roots only. The action of the generators is thus

$$\begin{aligned} w_{\alpha_1}: \quad & \alpha_1 \to -\alpha_1 \\ & \alpha_2 \to \alpha_1 + \alpha_2 \\ & \rho = \alpha_1 + \alpha_2 \to \alpha_2 \\ w_{\alpha_2}: \quad & \alpha_2 \to -\alpha_2 \\ & \alpha_1 \to \alpha_1 + \alpha_2 \\ & \rho = \alpha_1 + \alpha_2 \to \alpha_1 \end{aligned}$$

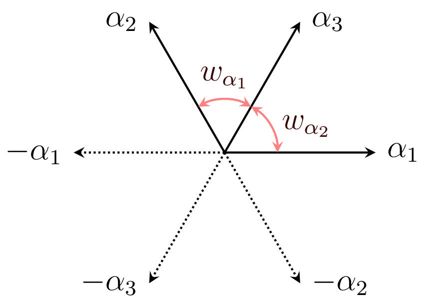


Figure 3.1: Roots of the finite lie algebra A_2 or $\mathfrak{su}(3)$

The complete Weyl group is composed of six elements

$$W = \{1,\, w_{\alpha_1},\, w_{\alpha_2},\, w_{\alpha_2}w_{\alpha_1},\, w_{\alpha_1}w_{\alpha_2},\, w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}\}$$

The LHS of equation (3.1) can be written down

$$e^{\alpha_3}(1 - e^{-\alpha_1})(1 - e^{-\alpha_2})(1 - e^{-\alpha_3})$$

Writing $e^{-\alpha_1} = x$ and $e^{-\alpha_2} = y$, we get the LHS as

$$x^{-1}y^{-1}(1 - x)(1 - y)(1 - xy)$$

whereas the RHS is

$$e^{\alpha_3} - e^{\alpha_2} - e^{\alpha_1} + e^{-\alpha_2} + e^{-\alpha_1} - e^{-\alpha_1 - \alpha_2}$$

which is just

$$x^{-1}y^{-1} - x^{-1} - y^{-1} + x + y - xy.$$

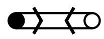
We can now see that Weyl denominator formula for SU(3) is just the algebraic identity

$$(1-x)(1-y)(1-xy) = 1 - x - y + x^2y + xy^2 - x^2y^2.$$

3.2 Affine Lie Algebra: $\hat{A}_1$

The Cartan matrix and the Dynkin diagram for the affine Kac-Moody lie algebra $\hat{A}_1$ is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$



The roots are

$$\begin{aligned} \text{I: } &\{n\alpha_0 + (n-1)\alpha_1 \mid n \in \mathbb{Z}\} \\ \text{II: } &\{(n-1)\alpha_0 + n\alpha_1 \mid n \in \mathbb{Z}\} \\ \text{III: } &\{n\alpha_0 + n\alpha_1 \mid n \in \mathbb{Z}, n \neq 0\} \end{aligned}$$

The Weyl group W here turns out to be $\mathbb{Z}_2 \ltimes \mathbb{Z}$. Lets prove this. The Weyl group is generated by the reflections about simple roots, w_{α_0} and w_{α_1}. Using the reflection formula $S_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$ and the Cartan matrix, we can work out how the Weyl group acts on all the roots.

$$\begin{aligned} w_{\alpha_0}(\alpha_0) &= -\alpha_0 \\ w_{\alpha_0}(\alpha_1) &= \alpha_1 - 2\frac{\langle \alpha_1, \alpha_0 \rangle}{\langle \alpha_0, \alpha_0 \rangle}\alpha_0 \\ &= \alpha_1 + 2\alpha_0 \\ w_{\alpha_1}(\alpha_0) &= \alpha_0 - 2\frac{\langle \alpha_0, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\alpha_1 \\ &= \alpha_0 + 2\alpha_1 \\ w_{\alpha_1}(\alpha_1) &= -\alpha_1 \end{aligned}$$

Using these results to get

$$\begin{aligned}
w_{\alpha_0}(n\alpha_0 + (n-1)\alpha_1) &= -n\alpha_0 + (n-1)(2\alpha_0 + \alpha_1) \\
&= (n-2)\alpha_0 + (n-1)\alpha_1 \\
w_{\alpha_1}(n\alpha_0 + (n-1)\alpha_1) &= n(\alpha_0 + 2\alpha_1) + (n-1)(\alpha_1) \\
&= n\alpha_0 + (n+1)\alpha_1 \\
w_{\alpha_0}(n(\alpha_0 + \alpha_1)) &= w_{\alpha_1}(n(\alpha_0 + \alpha_1)) \\
&= n(\alpha_0 + \alpha_1) \\
w_{\alpha_1}(n(\alpha_0 + \alpha_1)) &= n(\alpha_0 + \alpha_1)
\end{aligned}$$

The action of the Weyl group on the roots is shown in Figure 3.2. For convenience, write $x = w_{\alpha_0}$ and $y = w_{\alpha_0}w_{\alpha_1}$ with the relations $x^2 = 1$, $(xy)^2 = 1$. The group $H = \langle x \rangle$ is isomorphic to $\mathbb{Z}_2$ and the group $N = \langle y \rangle$ is isomorphic to $\mathbb{Z}$. The complete Weyl group W is generated by compositions of x and y, that is $\langle x, y \rangle$. To prove that W is, in fact, the *semidirect product* $H \ltimes N$, we need to show that N is a normal subgroup, $H \cap N = \{1\}$ and $W = HN$. $(xy)^2 = 1 \implies xy = y^{-1}x^{-1} = y^{-1}x$. This lets you 'pull' all x to the left in an term containing x, y in arbitrary order, which shows that any element can be written as $x^m y^n$, $m \in \{0, 1\}, n \in \mathbb{Z}$. Therefore, $W = HN$. Similarly, in $wy^m w^{-1}$, where $w \in W$, all x can be pulled to the left. The power of x has to be even due to the presence of both w and w^{-1}. This shows that $wy^m w^{-1} = y^k$ for some $k \in \mathbb{Z}$. Hence N is a normal subgroup and $W = \mathbb{Z}_2 \ltimes \mathbb{Z}$

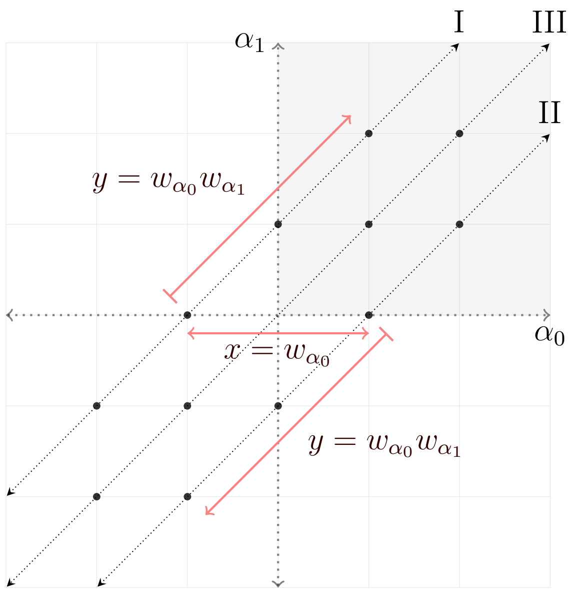


Figure 3.2: Roots for Affine Lie Algebra $\hat{A}_1$. The red lines show the action of Weyl group elements on the roots. The imaginary roots (Set III) are invariant under the action of the Weyl group. The elementary reflections (w_{α_0} and w_{α_1}) interchange Set I and Set II, and an even number of elementary reflections translate the roots along the lines.

Positive roots are the ones in the positive quadrant

$$L_+ = \{(n-1)\alpha_0 + n\alpha_1 \mid n \geq 0\} \cup \{n\alpha_0 + (n-1)\alpha_1 \mid n \geq 0\} \cup \{n\delta \mid n > 0\}$$

The Weyl vector ρ is defined as half the sum of all positive roots.

Weyl denominator formula is

$$\sum_{w \in W} (\det w) e^{w(\rho)-\rho} = \prod_{\alpha \in L_+} (1 - e^{-\alpha})^{\operatorname{mult} \alpha} \tag{3.2}$$

We can calculate $w(\rho) - \rho$ in three steps, one for each set of roots (I, II and III). For each of the sets, notice that $w(\rho)$ and ρ are both infinite sums. To get around this, we look at $w(\rho) - \rho$. The roots in $w(\rho)$ that are not present in ρ belong to

$$\Phi_w = \{w(\alpha) \mid w(\alpha) < 0, \alpha > 0\}$$

and the roots in ρ that do not constitute $w(\rho)$ belong to

$$\Phi'_w = \{w(\alpha) \mid w(\alpha) > 0, \alpha < 0\}.$$

But $\Phi'_w = -\Phi_w$, and so the infinite sum reduces to a finite sum,

$$\sum_{w \in W} (\det w) e^{\sum_{\alpha \in \Phi_w} w(\alpha)}.$$

1. Set I $\{(n-1)\alpha_0 + n\alpha_1 \mid n \geq 1\}$

   (a) $w = y^m$, $m < 0$ (as $m > 0$ just pushes the roots further towards the positive direction and does not contribute to the sum)

   $$y^m((n-1)\alpha_0 + n\alpha_1) = (n + 2m - 1)\alpha_0 + (n + 2m)\alpha_1$$

   The roots which become negative on applying y^m are those with

   $$\begin{aligned} n + 2m &\leq 0 \\ n &\leq -2m \\ \implies 1 \leq n &\leq -2m \end{aligned}$$

We can therefore do the inner summation:

$$
\begin{aligned}
\sum_{n=1}^{-2m}(y^m((n-1)\alpha_0+n\alpha_1)) &= \sum_{n=1}^{-2m}((n+2m-1)\alpha_0+(n+2m)\alpha_1)) \\
&= -\sum_{n'=1}^{-2m} n'\alpha_0 - \sum_{n'=1}^{-2m-1} n'\alpha_1 \\
&= -\left(\frac{(-2m+1)(-2m)}{2}\alpha_0 + \frac{(-2m)(-2m-1)}{2}\alpha_0\right) \\
&= -m[(2m-1)\alpha_0+(2m+1)\alpha_1]
\end{aligned}
$$

(b) $w = xy^m$, $m < 0$

$$
xy^m((n-1)\alpha_0+n\alpha_1) = (n+2m+1)\alpha_0+(n+2m)\alpha_1
$$

The part that contributes to the sum is

$$
\begin{aligned}
n+2m+1 &\leq 0 \\
n &\leq -2m-1 \\
\implies 1 \leq n &\leq -2m-1
\end{aligned}
$$

The sum becomes

$$
\begin{aligned}
\sum_{n=1}^{-2m-1}(xy^m((n-1)\alpha_0+n\alpha_1)) &= \sum_{n=1}^{-2m-1}((n+2m+1)\alpha_0+(n+2m)\alpha_1)) \\
&= -(2m+1)[(m+1)\alpha_0+m\alpha_1]
\end{aligned}
$$

2. Set II $\{n\alpha_0+(n-1)\alpha_1 \mid n \geq 1\}$

(a) $w = y^m$, $m > 0$

$$
y^m(n\alpha_0+(n-1)\alpha_1) = (n-2m)\alpha_0+(n-2m-1)\alpha_1
$$

The limits on the sum are

$$
1 \leq n \leq 2m,
$$

and, therefore, the sum is

$$
\begin{aligned}
\sum_{n=1}^{2m}(y^m(n\alpha_0+(n-1)\alpha_1)) &= \sum_{n=1}^{2m}((n-2m)\alpha_0+(n-2m-1)\alpha_1)) \\
&= -m[(2m-1)\alpha_0+(2m+1)\alpha_1]
\end{aligned}
$$

(b) $w = xy^m$, $m \geq 0$

$$xy^m(n\alpha_0 + (n-1)\alpha_1) = (n-2m-2)\alpha_0 + (n-2m-1)\alpha_1$$

The limits on the sum are

$$1 \leq n \leq 2m+1,$$

and, therefore, the sum is

$$\begin{aligned}\sum_{n=1}^{2m+1}(y^m(n\alpha_0 + (n-1)\alpha_1)) &= \sum_{n=1}^{2m+1}((n-2m)\alpha_0 + (n-2m-1)\alpha_1)) \\ &= -(2m+1)[(m+1)\alpha_0 + m\alpha_1]\end{aligned}$$

Using the above results and clubbing them together, we get the LHS of the Weyl denominator formula [Equation (3.2)]

$$\begin{aligned}\sum_{w\in W}(\det w)e^{w(\rho)-\rho} &= 1 + \sum_{m<0} e^{m[(2m-1)\alpha_0+(2m+1)\alpha_1]} - \sum_{m<0} e^{(2m+1)[(m+1)\alpha_0+m\alpha_1]} \\ &\quad + \sum_{m>0} e^{m[(2m-1)\alpha_0+(2m+1)\alpha_1]} - \sum_{m\geq 0} e^{(2m+1)[(m+1)\alpha_0+m\alpha_1]} \\ &= \sum_{m\in\mathbb{Z}} e^{m[(2m-1)\alpha_0+(2m+1)\alpha_1]} - \sum_{m\in\mathbb{Z}} e^{(2m+1)[(m+1)\alpha_0+m\alpha_1]}\end{aligned}$$

The RHS of the Weyl denominator formula is easy to write down. It is just the product of terms involving positive roots,

$$\prod_{\alpha\in L_+}(1-e^{-\alpha})^{\text{mult}\,\alpha} = \prod_{m\geq 1}(1-e^{-((n-1)\alpha_0+n\alpha_1)})(1-e^{-n(\alpha_0+\alpha_1)})(1-e^{-(n\alpha_0+(n-1)\alpha_1)})$$

Finally, we get the Weyl denominator formula for the Affine Kac-Moody algebra $\hat{A}_1$ to be

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} e^{m[(2m-1)\alpha_0+(2m+1)\alpha_1]} - \sum_{m\in\mathbb{Z}} e^{(2m+1)[(m+1)\alpha_0+m\alpha_1]} \\ &\quad = \prod_{m\geq 1}\left(1-e^{-((n-1)\alpha_0+n\alpha_1)}\right)\left(1-e^{-n(\alpha_0+\alpha_1)}\right)\left(1-e^{-(n\alpha_0+(n-1)\alpha_1)}\right) \quad (3.3)\end{aligned}$$

3.2.1 Jacobi Triple Product Identity

Setting $e^{-\alpha_0} = r$ and $e^{-\alpha_1} = qr^{-1}$, we get the RHS

$$\prod_{m\geq 1}(1-q^n)(1-q^n r^{-1})(1-q^{n-1}r)$$

and the LHS as

$$\sum_{n\in\mathbb{Z}} q^{n(2n+1)}r^{-2n} - \sum_{n\in\mathbb{Z}} q^{n(2n+1)}r^{2n+1}$$

In the first summation, put $k = 2n$ so that the sum runs over all even integers and in the second summation, we set $k = 2n+1$ so that it runs over all odd integers.

$$\sum_{n\in 2\mathbb{Z}} q^{\frac{k(k+1)}{2}}r^{-k} - \sum_{n\in 2\mathbb{Z}+1} q^{\frac{(k-1)k}{2}}r^{k}$$

Replacing k by $-k$ in the first sum and grouping both the sums together, we get RHS

$$\sum_{n\in\mathbb{Z}} (-1)^k q^{\frac{k(k-1)}{2}}r^{k}$$

The denominator identity becomes the Jacobi triple product identity for the Jacobi Theta Function

$$\begin{aligned}\prod_{n\geq 1}(1-q^n)(1-q^n r^{-1})(1-q^{n-1}r) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{k(k-1)}{2}}r^{k} \\ &= q^{-1/8}r^{1/2}\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(k-1/2)^2}{2}}r^{(k-1/2)} \end{aligned} \tag{3.4}$$

We use the following definition of the ϑ function[1],

$$\vartheta_1(\tau, z) = \iota \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(n-1/2)^2}{2}}r^{(n-1/2)} \tag{3.5}$$

which is related to the usual ϑ function as

$$-\vartheta_1(\tau, z) = \vartheta_{11}(\tau, z) = e^{\pi\iota(\frac{\tau}{4}+z+\frac{1}{2})}\vartheta\left(\tau, z+\frac{1}{2}\tau+\frac{1}{2}\right) \tag{3.6}$$

with $q = e^{2\iota\pi\tau}$ and $r = e^{2\iota\pi z}$. Finally,

$$\begin{aligned}-\iota\vartheta_1(\tau, z) &= q^{1/8}r^{-1/2}\prod_{n\geq 1}(1-q^n)(1-q^n r^{-1})(1-q^{n-1}r) \\ &= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(k-1/2)^2}{2}}r^{(k-1/2)}\end{aligned}$$

This is a remarkable result. It suggests that the denominator formulas for more complicated lie algebras might be expressible as identities of modular forms.

[1] This is also (-ϑ_{11}). See this Wikipedia article: https://en.wikipedia.org/wiki/Jacobi_theta_functions_(notational_variations)

Infact, for the more complex cases, it be possible to recover the lie algebra associated with such special functions by considering their product identities. We will therefore take a detour in the next chapter to study modular forms in some detail.

3.3 Borcherds Denominator Formula

Without going into the technicalities, we merely state the denominator formula for Borcherds Lie algebras,

$$e(\rho)\prod_{\alpha\in\phi^+}(1-e(-\alpha))^{m_\alpha}=\sum_{w\in W}\epsilon(w)w\left(e(\rho)\sum_{\Psi}(-1)^{|\Psi|}e(-\sum\Psi)\right) \tag{3.7}$$

where

$\mu \rightarrow e(\mu)$ is an isomorphism between the additive group of weights and the corresponding multiplicative group,

Φ^+ is the set of *positive roots*,

W is the Weyl group,

$m_\alpha = \dim L_\alpha$,

Ψ runs over all finite subset of mutually orthogonal imaginary fundamental roots, and

ρ is any element of $Q \otimes \mathbb{R}$ such that

$$\langle\rho, \alpha_i\rangle = \frac{1}{2}\langle\alpha_i, \alpha_i\rangle$$

Notice the factor $e^{(-\sum\Psi)}$ on the right hand side of equation (3.7). It appears due to the inclusion of the imaginary simple roots.

CHAPTER 4

Modular Forms

4.1 Modular Group and Congruence Subgroups

The **modular group** is the group of all 2×2 matrices with integer entries and unit determinant.

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

The *upper half space* is defined as

$$\mathcal{H} = \{\tau : \Im(\tau) > 0\}$$

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Its easy to see that the modular group acts on the upper half space by the transformation

$$z \to \gamma(\tau) = \frac{az + b}{cz + d}$$

Definition 1. *A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k, where k is an integer, if*

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \quad \textit{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \textit{ and } \tau \in \mathcal{H} \tag{4.1}$$

and f is holomorphic on $\mathcal{H}$ and at ∞.

If we replace $\mathrm{SL}_2(\mathbb{Z})$ by a subgroup Γ, then we get a more generalized notion of a modular form. We say that such a function f is a modular form on the subgroup Γ. A *congruence subgroup* is a subgroup obtained by imposing some congruence conditions on the elements of $\mathrm{SL}_2(\mathbb{Z})$.

The *principle congruence subgroup of level N* is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

A few other important congruence subgroups are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\} \tag{4.2}$$

Any subgroup $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$ for which there exists an $N \in \mathbb{Z}^+$ such that $\Gamma(N) \in \Gamma$ is a *congruence subgroup of level* N.

4.2 Generalizations

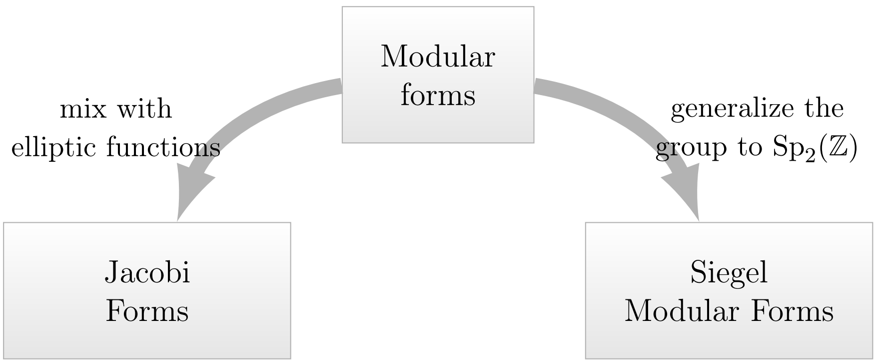


Figure 4.1: Generalizations of modular forms

4.2.1 Jacobi Forms

For a subgroup $\Gamma_1 \in \Gamma$, we define

$$(\phi|_{k,m}M)(\tau, z) := (c\tau + d)^{-k} e^{2\pi\iota m(\frac{-cz^2}{c\tau+d})} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$$(\phi|_{k,m}X)(\tau, z) := e^{2\pi\iota m(\lambda^2\tau+2\lambda z)}\phi(\tau, z + \lambda\tau + \mu) \qquad X = [\lambda\ \mu] \in \mathbb{Z}^2$$

Definition 2. *A* ***Jacobi form of weight*** k ***and index*** m *on a subgroup* $\Gamma \subset \Gamma_1$ *is a holomorphic function* $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ *satisfying*

$$(\phi|_{k,m}M) = \phi \qquad M \in \Gamma$$
$$(\phi|_{k,m}X) = \phi \qquad X \in \mathbb{Z}^2$$

and having a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \le 4nm}} c(n, r) e^{2\pi\iota(n\tau + rz)}.$$

The following properties can be checked

$$(\phi|_{k,m}M)|_{k,m}M' = \phi|_{k,m}(MM')$$
$$(\phi|_{k,m}X)|_{k,m}X' = \phi|_{k,m}(X+X')$$
$$(\phi|_{k,m}M)|_{k,m}XM = (\phi|_{k,m}X)|_{k,m}M$$

where $X, X' \in \mathbb{Z}^2$ and $M \in \Gamma_1$. This shows that the full Jacobi group is a semidirect product $\Gamma_1^J := \Gamma_1 \ltimes \mathbb{Z}^2$

4.2.2 Siegel Modular Forms

The **Siegel upper half space** is defined as the set $\mathcal{H}_n$ of complex symmetric $n \times n$ matrices Z with positive-definite imaginary part. The group:

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \{M \in M_{2n}(\mathbb{R}) \mid MJ_{2n}M^T = J_{2n}\}, \quad J_{2n} = \begin{pmatrix} 0 & -I_n \\ -I_n & 0 \end{pmatrix},$$
$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A, B, C, D \in M_n(\mathbb{R}),\ AB^T = BA^T,\ CD^T = DC^T,\ AD^T - BC^T = I_n \right\}$$

acts on $\mathcal{H}_n$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}. \tag{4.3}$$

Note that $\mathrm{Sp}_2(\mathbb{Z})$ is just the usual modular group Γ_1.

A **Siegel modular form** of degree n and weight k with respect to the full **Siegel modular group** $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$ is a holomorphic function $F : \mathcal{H}_n \to \mathbb{C}$ satisfying

$$F(M \cdot Z) = \det{(CZ+D)}^k F(Z)$$

for all $z \in \mathcal{H}_n$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. If $n > 1$, such a function has a Fourier expansion of the form

$$F(Z) = \sum_{T \geq 0} A(T) e^{2\pi\iota \, \mathrm{Tr}(TZ)}$$

where T runs over positive semidefinite semi-integral $n \times n$ matrices, that is, $2t_{ij}$, $t_{ii} \in \mathbb{Z}$ for $i = 1, \dots, n$.

If $n = 2$, we write Z as

$$Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$$

with the positive definiteness of the imaginary part implying that $\operatorname{Im}(\det Z) > 0 \implies \operatorname{Im}(z)^2 < \operatorname{Im}(\tau)\operatorname{Im}(\tau')$. Similarly,

$$T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}$$

with $n, r, m \in \mathbb{Z}$ and $r^2 \leq 4nm$ as T is a positive semidefinite matrix (it can be taken to be symmetric without loss of generality since the antisymmetric part of T vanishes in $T \cdot Z$)

Writing $F(Z)$ as $F(\tau, a, \tau')$ and $A(T)$ as $A(n, r, m)$, we get the following Fourier expansion

$$F(\tau, z, \tau') = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,\, m,\, 4nm - r^2 \geq 0}} A(n, r, m) e^{2\pi\iota(n\tau + rz + m\tau')}$$

CHAPTER 5

Eta Products

5.1 Dedekind Eta Function

The Dedekind eta function $\eta : \mathcal{H} \to \mathbb{C}$ is a modular form of weight $\frac{1}{2}$.

$$\eta(\tau) = e^{\frac{\pi \iota \tau}{12}} \prod_{n=1}^{\infty} (1 - q^n), \tag{5.1}$$

$$= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{5.2}$$

where $q = e^{2\pi\iota\tau}$. It transforms with the character

$$\epsilon(a, b, c, d) = \begin{cases} \left(\frac{d}{c}\right) e^{\frac{\iota\pi}{12}[bd(1-c^2)+c(a+d)]} e^{\frac{\iota\pi(1-c)}{4}}, & \text{for } c \text{ odd} \\ \left(\frac{c}{d}\right) e^{\frac{\iota\pi d}{4}} e^{\frac{\iota\pi}{12}[bd(1-d^2)+d(b-c)]}, & \text{for } d \text{ odd} \end{cases}$$

The Legendre symbol, for $a \in \mathbb{Z}$ and p prime, is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \equiv x^2 \pmod{p} \text{ for some } x \in \mathbb{Z} \setminus \{0\} \\ -1 & \text{if } a \not\equiv x^2 \pmod{p} \text{ for any } x \in \mathbb{Z} \\ 1 & \text{if } a \equiv 0 \pmod{p} \end{cases}$$

The Jacobi symbol generalizes the Legendre symbol for odd composite p. If p can be prime factorized as $p = p_1^{d_1} p_2^{d_2} \dots p_k^{d_k}$, then the Jacobi symbol is defined as

$$\left(\frac{a}{p}\right) = \left(\frac{a}{p_1}\right)^{d_1} \left(\frac{a}{p_2}\right)^{d_2} \cdots \left(\frac{a}{p_k}\right)^{d_k}$$

5.2 Multiplicative Eta Products

As mentioned in [GK10], we associate to the cycle shape $\rho = 1^{a_1} 2^{a_2} \cdots N^{a_N}$, the following product of eta functions

$$\rho = 1^{a_1} 2^{a_2} \cdots N^{a_N} \mapsto g_\rho(\tau) = \eta(\tau)^{a_1} \eta(2\tau)^{a_2} \cdots \eta(N\tau)^{a_N}. \tag{5.3}$$

A **balanced** cycle shape is one for which there is an integer N for which the cycle shape $\prod_{i=1}^{N} m_i^{a_i}$ remains invariant under $m_i \mapsto N/m_i$.

A series $f(\tau) = \sum_{n=0}^{t} a_n q^n$, with $q = e^{2\pi\iota\tau}$ and τ in the upper half plane $\mathcal{H}$, is multiplicative if $a_{mn} = a_m a_n$ for all coprime pairs $(m, n) \in \mathbb{Z}^2$.

It was shown in [DKM85] that all the multiplicative eta products for M_{24} have balanced cycle shapes. In this chapter, we follow [DKM85] to derive the character for eta products of M_{12} and enumerate all the multiplicative eta products.

5.3 Modular Transformation of Eta Products

Lets take a moment to figure out how an eta product transforms under modular transformations. This is important as the character for an eta product transformation appears in the Hecke operator, as given in [CG11].

Let $m_1, \ldots, m_t$ be nonzero integers, $e_i = \operatorname{sgn}(m_i), 2k = \sum_{i=1}^{t} e_i$. We make the following assumptions:

(i) $\sum m_i \equiv 0 \pmod{12}$

(ii) $\sum N/m_i \equiv 0 \pmod{12}$

(iii) $\prod n_i \equiv N^k \operatorname{mod}^\times$ square integers

We have, if $f(x)$ is a product of eta functions,

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k \chi^k(d) f(z).$$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, with $c \geq 0$

$$\begin{aligned} f(\frac{a\tau + b}{c\tau + d}) &= \prod \eta \left(n_i \frac{a\tau + b}{c\tau + d} \right)^{e_i} \\ &= \prod \eta \left(\frac{a(n_i\tau) + bn_i}{(c/n_i)(n_i\tau) + d} \right)^{e_i} \\ &= \prod \left[\epsilon(a, bn_i, c/n_i, d) e^{-\iota\pi/4} (c\tau + d)^{1/2} \eta(n_i\tau) \right]^{e_i} \\ &= \left[\prod \epsilon(a, bn_i, c/n_i, d)^{e_i} \right] e^{-\iota\pi k/2} (c\tau + d)^k f(\tau) \end{aligned}$$

Note that since c/n_i appears in the above computation, we must have $c \equiv 0 \mod n_i$ for all n_i, $i = 1, \ldots, t$. As $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we need N to divisible by LCM $(n_1, \ldots, n_t)$.

Let

$$E = e^{-\iota\pi k/2} \prod_i \epsilon(a, bn_i, c/n_i, d)^{e_i}.$$

For d odd,

$$E = e^{-\iota\pi k/2} e^{\iota\pi kd/2} e^{\frac{\iota\pi}{12}\sigma} \prod_i \left(\frac{c/n_i}{d}\right)$$

where

$$\begin{aligned}\sigma &= \sum_i \left(a\frac{c}{n_i}(1-d^2) + d(bn_i - \frac{c}{n_i})\right) \\ &= a(1-d^2)\sum_i \frac{c}{n_i} + d\sum_i bn_i - d\sum_i \frac{c}{n_i}\end{aligned}$$

Thus,

$$\begin{aligned}E &= (-1)^{k\frac{d-1}{2}+\sigma'} \prod_i \left(\frac{c/n_i}{d}\right) \\ &= (-1)^{k\frac{d-1}{2}+\sigma'} \left(\frac{N}{d}\right)^k\end{aligned}$$

Simplifying σ'

We have $\sum n_i = 12l$ and $\sum 1/n_i = 12m/N$ for positive integers m, n. Also $c = Nc'$ for some integer c'. Putting this,

$$\begin{aligned}\sigma &= c(a(1-d^2) - d)\sum \frac{1}{n_i} + bd\sum n_i \\ &= Nc'(a(1-d^2)-d)\frac{12l}{N} + bd(12m) \\ &= 12\big(lc'(a(1-d^2)-d) + bdm\big)\end{aligned}$$

For N even, we have a, d odd,

$$\begin{aligned}\sigma &= 12\big(lc'(a(1-d^2)-d) + bdm\big) \\ \frac{\sigma}{12} &\equiv (lc' + mb) \mod 2\end{aligned}$$

Therefore,

$$E = (-1)^{k\frac{d-1}{2} + l\frac{c}{N} + mb} \left(\frac{N}{d}\right)^k$$

For k odd, we use $(-1)^{\frac{d-1}{2}} = \left(\frac{-1}{d}\right)$ to get

$$E = \begin{cases} (-1)^{l\frac{c}{N}+mb}\left(\frac{-N}{d}\right) & k \text{ odd} \\ (-1)^{l\frac{c}{N}+mb} & k \text{ even} \end{cases} \tag{5.4}$$

If N is odd, however, then d might be odd or even. If it is odd, then the above calculation goes through. Else if d is even, then c must be odd. So,

$$E = e^{-\iota\pi k/2} e^{\frac{\iota\pi}{4}\sum_i (1-\frac{c}{n_i})e_i} e^{\iota\pi\tau/12} \prod_i \left(\frac{d}{c/n_i}\right)$$

where

$$\begin{aligned} \tau &= \sum_i e_i[[bn_i d(1-(c/n_i)^2)] + c(a+d)/n_i] \\ &= (bd\sum n_i e_i) + (bdc^2 + c(a+d))\sum_i \frac{e_i}{n_i} \\ &= 12bd + (bdc^2 + c(a+d))\frac{12}{N} \\ \frac{\tau}{12} &= bd + bdc\frac{c}{N} + \frac{c}{N}(a+d) \\ &\equiv a\frac{c}{N} \mod 2 \qquad (\text{since } d \text{ is even, } c \text{ is odd}) \end{aligned}$$

The Jacobi symbol product can be written as

$$\prod_i \left(\frac{d}{c/n_i}\right) = \left(\frac{d}{N}\right)^k$$

Finally,

$$\begin{aligned} E &= e^{-\frac{\iota\pi}{4}c\frac{12}{N}} e^{\iota\pi\frac{ac}{N}} \left(\frac{d}{N}\right)^k \\ &= -e^{\iota\pi ac/N}\left(\frac{d}{N}\right)^k \end{aligned}$$

Combining the above results,

$$\boxed{\chi = \begin{cases} e^{\frac{c}{N}+b}\left(\frac{-N}{d}\right)^k & N \text{ even, or } N \text{ odd and } d \text{ odd} \\ -e^{\iota\pi ac/N}\left(\frac{d}{N}\right)^k & N \text{ odd and } d \text{ even} \end{cases}} \tag{5.5}$$

Using the above result for the character, we used a `sage` script to enumerate all balanced cycle shapes of M_{12} and checked the first few coefficients to determine which of them correspond to multiplicative eta products. The result is shown in table 5.1.

Cycle Shape	Character χ	N
	$k=1$	
$1^1 11^1$	$e^{\iota\pi(\frac{c}{11}+b)}\left(\frac{-11}{d}\right)$ for d odd $-e^{\iota\pi ac/11}\left(\frac{d}{11}\right)$ for d even	11
$10^1 2^1$	$e^{\iota\pi(\frac{c}{20}+b)}\left(\frac{-20}{d}\right)$	20
$9^1 3^1$	$e^{\iota\pi(\frac{c}{27}+b)}\left(\frac{-27}{d}\right)$ for d odd $-e^{\iota\pi ac/27}\left(\frac{d}{27}\right)$ for d even	27
$8^1 4^1$	$e^{\iota\pi(\frac{c}{32}+b)}\left(\frac{-32}{d}\right)$	32
$5^1 7^1$	$e^{\iota\pi(\frac{c}{35}+b)}\left(\frac{-35}{d}\right)$ for d odd $-e^{\iota\pi ac/35}\left(\frac{d}{35}\right)$ for d even	35
6^2	$e^{\iota\pi(\frac{c}{36}+b)}\left(\frac{-36}{d}\right)$	36
	$k=2$	
$1^1 2^1 3^1 6^1$	$e^{\iota\pi(\frac{c}{6}+b)}$	6
$1^2 5^2$	$e^{\iota\pi(\frac{c}{5}+b)}$ for d odd $-e^{\iota\pi ac/5}$ for d even	5
$2^2 4^2$	$e^{\iota\pi(\frac{c}{8}+b)}$	8
3^4	$e^{\iota\pi(\frac{c}{9}+b)}$ for d odd $-e^{\iota\pi ac/9}$ for d even	9
	$k=3$	
$1^3 3^3$	$e^{\iota\pi(\frac{c}{3}+b)}\left(\frac{-3}{d}\right)$ for d odd $-e^{\iota\pi ac/3}\left(\frac{d}{3}\right)$ for d even	3
2^6	$e^{\iota\pi(\frac{c}{4}+b)}\left(\frac{-4}{d}\right)$	4
	$k=4$	
$1^4 2^4$	$e^{\iota\pi(\frac{c}{2}+b)}$	2

Table 5.1: Character table for multiplicative eta products with cycle shapes in M_{12}. This is analogous to the character table in [DKM85] which enumerates cycle shapes of M_{24} instead.

CHAPTER 6

Additive Lift

6.1 Conjectures

In the following sections, our main goal will be to gather evidence for the following two conjectures of [GK10], for the case $N = 6$.

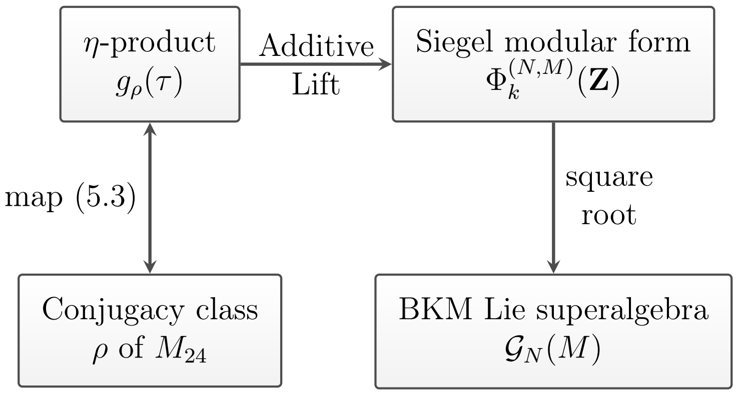


Figure 6.1: Summary of the conjectures

Conjecture 1. *Let the cycle shape be $\rho = 1\cdot 2\cdot 3\cdot 6$ (conjugacy class of M_{12}) and the corresponding eta product $g_\rho = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$. Let*

$$\Delta_1 = \mathrm{Lift}\left(\frac{\theta(\tau,z)}{\eta(\tau)^3}\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)\right)$$

The conjecture states that the sum and product representations of the Δ_1 are precisely the sum and product sides of the Weyl-Kac-Borcherds (WKB) denominator formula for the BKM Lie superalgebra with the generalized Cartan matrix

$$A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} \tag{6.1}$$

Conjecture 2. *Consider the square of the additive seed in the previous case. The conjecture is that the squaring operation and the additive lift 'commute'. More* *precisely, if*

$$\Phi_2 = \text{Lift}\left(\frac{\theta(\tau,z)^2}{\eta(\tau)^6}\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2\right),$$

then we must have

$$\Phi_2(Z) = [\Delta_1(Z)]^2 .$$

6.2 N=6

Lets take the cycle shape $1 \cdot 2 \cdot 3 \cdot 6$, which gives $g_\rho(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(4\tau)$.

The *additive seed* is a Jacobi form of weight $k = 1$ and index $t = 1/2$,

$$\begin{aligned}\Psi_{1,1/2}(\tau,z) &= \frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}g_\rho(\tau)\\ &= \frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)\end{aligned}$$

As seen in the table 5.1, this is a Jacobi form with respect to the subgroup $\Gamma_0(6)$. [CG11] defines the group

$$\Gamma_1(Nq,q) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{Nq},\ b \equiv 0 \pmod{q},\ a \equiv d \equiv 1 \pmod{Nq}\right\} \tag{6.2}$$

Its easily seen that $\Gamma_1(Nq,q) \subset \Gamma_0(N)$. Also, note that $\Gamma_1(N \cdot 1, 1) = \Gamma_0(N)$.

For $\Gamma_1(Nq,q)$, we use the Hecke operator

$$T^{(N)}(m) = \sum_{\substack{ad=m\\ (a,Nq)=1\\ b \bmod d}} \Gamma_1(Nq,q)\sigma_a \cdot \begin{pmatrix} a & qb\\ 0 & d\end{pmatrix} \tag{6.3}$$

where $a > 0$ and $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0\\ 0 & a\end{pmatrix}$ mod Nq. This element induces a Hecke operator on the Jacobi form $\tilde{\phi}(Z) = \phi(\tau, z)e^{2\pi\iota\omega}$

$$\tilde{\phi}|_k T_-^{(N)}(m)(Z) = m^{k-1} \sum_{\substack{ad=m \\ (a,Nq)=1 \\ b \bmod d}} d^{-k}\chi(\sigma_a)\phi(\frac{a\tau + bq}{d}, az)e^{2\pi\iota m t\omega}. \tag{6.4}$$

To fix q so that the given Hecke operator is applicable on a modular form of $\Gamma_0(6)$, observe that for $N = 6, q = 1$, the supposition that $\mathrm{Ker}(\chi) \supset \Gamma_1(Nq, q)$ does not hold true. But for $N = 6$ and $q = 2$, it does hold true since

$$\mathrm{Ker}(\chi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 6b \mod 12) \right\}$$

The additive lift, given by

$$F_\phi(Z) = \mathrm{Lift}_\mu(\phi)(Z) = \sum_{\substack{m \equiv \mu \bmod q \\ m>0}} \tilde{\phi}|_k T_-^{(N)}(m)(Z), \tag{6.5}$$

is a modular form for $\Gamma_{qt}(N)^+$ with a character $\chi_{t,\mu}$.

We define,

$$\begin{aligned}
\Delta_1(\mathbf{Z}) &= \mathrm{Lift}_1(\Psi_{1,1/2})(\mathbf{Z}) && (6.6) \\
&= \sum_{\substack{m \equiv 1 \bmod q \\ m>0}} m^{k-1} s^{mt} \sum_{\substack{ad=m \\ (a,Nq)=1 \\ b \bmod d}} d^{-k}\chi(\sigma_a)\Psi_{1,1/2}\left(\frac{a\tau + bq}{d}, az\right) \\
&= \sum_{\substack{m \equiv 1 \bmod 2 \\ m>0}} s^{\frac{m}{2}} \sum_{\substack{ad=m \\ (a,6q)=1 \\ b \bmod d}} d^{-1}\Psi_{1,1/2}\left(\frac{a\tau + bq}{d}, az\right). && (6.7)
\end{aligned}$$

We implemented the lift given in equation (6.7) in `Sage` (The source code listing is provided in section A.1). The resulting expansion is:

$$\begin{aligned}
\Psi_{1,1/2} =& q_h\left(r_h - \frac{1}{r_h}\right) \\
&+ q_h^3\left(-r_h^3 + 2\,r_h - \frac{2}{r_h} + \frac{1}{r_h^3}\right) \\
&+ q_h^5\left(-2\,r_h^3 + 4\,r_h - \frac{4}{r_h} + \frac{2}{r_h^3}\right) \\
&+ q_h^7\left(r_h^5 - 4\,r_h^3 + 7\,r_h - \frac{7}{r_h} + \frac{4}{r_h^3} - \frac{1}{r_h^5}\right) + \cdots && (6.8)
\end{aligned}$$

where we use $q_h = q^{1/2}, r_h = r^{1/2}, s_h = s^{1/2}$. The additive lift of equation (6.7) may be written as a series in s_h as

$$\Delta_1 = (1 + s_h^3 \, \mathrm{T}_3 + s_h^5 \, \mathrm{T}_5 + s_h^5 \, \mathrm{T}_5 + \cdots)\Psi_{1,1/2}, \tag{6.9}$$

where we have abbreviated the Hecke operator $\Psi|_k \, \mathrm{T}_-^N(m)$ as simply $\mathrm{T}_m \, \Psi$. The first few terms of the expansion are

$$\begin{aligned}\Delta_1 = s_h(-\mathbf{q_h^3 r_h^3} + q_h r_h + \frac{q_h^3}{r_h^3} - \frac{\mathbf{q_h}}{\mathbf{r_h}} + \cdots) \\ + s_h^3(2\, q_h r_h - \frac{2\, q_h}{r_h} + \frac{q_h}{r_h^3} - \mathbf{q_h r_h^3} + \cdots) + \cdots\end{aligned} \tag{6.10}$$

Let $\delta_1, \delta_2, \delta_3$ be the three simple roots. As given in appendix D.1 of [GK09], we can write

$$e^{-\pi\iota(\delta_1,\mathbf{Z})} = qr, \quad e^{-\pi\iota(\delta_2,\mathbf{Z})} = r_{-1}, \quad e^{-\pi\iota(\delta_3,\mathbf{Z})} = sr.$$

For the root $\alpha[n, l, m] = n\delta_1 + (m + n - l)\delta_2 + m\delta_3$, we have

$$q^n r^l s^m = e^{-\iota\pi(\alpha[n,l,m],\mathbf{Z})}.$$

In this notation, the real simple roots are $\{\alpha[1, 1, 0], \alpha[0, -1, 0], \alpha[0, 1, 1]\}$. Since we have the sum side of the denominator formula, each term in expansion of Δ_1 must be generated by Weyl reflections of the simple roots. For example, the roots corresponding to the real simple roots themselves are

$$q^{1/2}r^{1/2}s^{1/2}(qr, r^{-1}, sr) = (q^{3/2}r^{3/2}s^{1/2}, q^{1/2}r^{-/12}s^{1/2}, q^{1/2}r^{3/2}s^{3/2}),$$

all of which can be seen in the expansion (6.10). (The real simple roots are shown in boldface).

For conjecture (2), we evaluated coefficients of $\Phi_2(\mathbf{Z})$ using the additive lift for integer indices

$$\Phi_2(\mathbf{Z}) = \sum_{(n,l,m)>0} \sum_{d|(n,l,m)} \chi(d) d^{k-1} a\left(\frac{nm}{d^2}, \frac{l}{d}\right) q^n r^l s^m, \tag{6.11}$$

where $(n, l, m) > 0$ implies $n, m \in \mathbb{Z}_+$, $l \in \mathbb{Z}$ and $(4nm - l^2) > 0$. The first few terms of the expansion are

$$
\begin{aligned}
\Phi_2(\mathbf{Z}) = s &\left(q(r + \frac{1}{r} - 2) + q^2(-2\,r^2 + 6\,r + \frac{6}{r} - \frac{2}{r^2} - 8) + \cdots \right) \\
&+ s^2 \Bigg(q(-2\,r^2 + 6\,r + \frac{6}{r} - \frac{2}{r^2} - 8) \\
&\quad + q^2(6\,r^3 - 26\,r^2 + 54\,r + \frac{54}{r} - \frac{26}{r^2} + \frac{6}{r^3} - 68) + \cdots \Bigg) + \cdots
\end{aligned}
$$

The resulting expansion was found to be the same as the one obtained by squaring $\Delta_1(\mathbf{Z})$ directly, which confirms the conjecture

$$
[\Delta_1(\mathbf{Z})]^2 = \Phi_2(\mathbf{Z}).
$$

We have thus shown that the square-root of $\Phi_2(\mathbf{Z})$ is equal to $\Delta_1(\mathbf{Z})$ that appears in the Weyl Denominator Formula for a BKM Lie super algebra.

CHAPTER 7

Borcherds Lift

Having confirmed the sum side (in chapter 6), we now move to the other part of the denominator formula, the product side. The product series for the Siegel modular form $\Delta_1(\mathbf{Z})$ can be generated using a **Borcherds Lift**. Here we introduce the Borcherds Lift and describe its application in our case.

Recall that the number of non-equivalent cusps of the Hecke congruence subgroup $\Gamma_0(N)$ is equal to $\sum_{e|N,e>0}\varphi((e,\frac{N}{e}))$ where φ is the Euler's function and (a,b) is the greatest common divisor of a and b. Let $\mathcal{P}$ denote the set of all cusps

$$\mathcal{P}=\left\{\tfrac{f}{e} : e|N,\, e\geq 1,\, f \mod (e,\tfrac{N}{e}),\, (e,f)=1\right\}.$$

We associate to each cusp, the matrix $M_{f/e}$ via

$$\tfrac{f}{e}\mapsto M_{f/e}=\begin{pmatrix} f & * \\ e & * \end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}),\quad M_{f/e}\langle\infty\rangle = f/e.$$

Given a Jacobi form $\phi\in J_{0,t}^{nh}(\Gamma_0(N))$, we can now write its Fourier expansion at the cusp f/e using $M_{f/e}$

$$(\phi|_{0,t}M_{f/e})(\tau,z)=\sum_{n\in\mathbb{Z}/h_e}\sum_{l\in\mathbb{Z}}c_{f/e}(n,l)q^n r^l.$$

The following is a restatement of the theorem 3.1 from [CG11], which describes how to write the Borcherds product for a given modular form.

Theorem 4. *Let $\phi\in J_{0,t}^{nh}(\Gamma_0(N))$. Assume that for all cusps of $\Gamma_0(N)$ we have $\frac{h_e}{N_e}c_{f/e}(n,l)\in\mathbb{Z}$ if $4nmt-m^2\leq 0$. Then the product*

$$B_\phi(\mathbf{Z})=q^A r^B s^C\prod_{f/e\in\mathcal{P}}\prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}}\left(1-(q^n r^l s^{tm})^{N_e}\right)^{\frac{h_e}{N_e}c_{f/e}(n,l)},$$

where

$$(n,l,m)>0 \textit{ means } \begin{cases} \textit{if } m>0, & \textit{then } n\in\mathbb{Z},\, l\in\mathbb{Z}\\ \textit{if } m=0 \textit{ and } n>0, & \textit{then } l\in\mathbb{Z}\\ \textit{if } m=n=0, & \textit{then } l<0\end{cases}$$

h_e is the height of the cusp, $N_e = N/e$ and

$$A = \frac{1}{24} \sum_{\substack{f/e \in \mathcal{P} \\ l \in \mathbb{Z}}} h_e c_{f/e}(0,l) \ , \quad B = \frac{1}{2} \sum_{\substack{f/e \in \mathcal{P} \\ l \in \mathbb{Z}, l>0}} l h_e c_{f/e}(0,l) \ , \quad C = \frac{1}{4} \sum_{\substack{f/e \in \mathcal{P} \\ l \in \mathbb{Z}}} l^2 h_e c_{f/e}(0,l)$$

defines a meromorphic modular form of weight

$$k = \frac{1}{2} \sum_{f/e \in \mathcal{P}} \frac{h_e}{N_e} c_{f/e}(0,0)$$

with respect to $\Gamma_t(N)^+$ with a character χ.

For $\Gamma_0(6)$, let

$$\psi(\tau,z) := \frac{1}{6}\phi_{0,1}(\tau,z) - \left(\frac{1}{6}E_2^{(2)}(\tau) + \frac{1}{2}E_2^{(3)}(\tau) - \frac{5}{2}E_2^{(6)}(\tau)\right)\phi_{-2,1}(\tau,z) \in J_{0,t}^{nh}(\Gamma_0(6)) \ , \tag{7.1}$$

where

$$\phi_{0,1}(\tau,z) = 8\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right] \ ,$$
$$\phi_{-1,2}(\tau,z) = \eta(\tau)^{-6}\vartheta_1(\tau,z)^2 \ ,$$

are two basic Jacobi forms that generate the ring of Jacobi forms modulo multiplication by modular forms and

$$E_2^{(N)}(\tau) = \frac{12i}{\pi(N-1)}\,\partial_\tau\left[\ln\eta(\tau) - \ln\eta(N\tau)\right] = 1 + \frac{24}{N-1}\sum_{\substack{n_1,n_2\geq 1 \\ n_1\neq 0 \bmod N}} n_1 e^{2\pi i n_1 n_2 \tau} \ ,$$

are the weight two Eisenstein series at level N and $\vartheta_i(\tau,z)$ in the above equations are the Jacobi theta functions. The modular form $\psi(\tau,z)$ generates the Borcherds product for $\Phi_2(\mathbf{Z})$ that we constructed through the additive lift.

In order to use the theorem, we list the various cusps along with their heights as well as relevant Fourier coefficients in table 7.1. This provides the data that we need to apply the above theorem for our situation.

The Fourier expansions of $\psi(\tau,z)$ at all the cusps given in table 7.1 are obtained

Cusp	∞	$\frac{0}{1}$	$\frac{1}{3}$	$\frac{1}{2}$
h	1	6	2	3
N_e	1	6	2	3
$c(0,0)$	-2	2	2	2
$c(0,\pm 1)$	2	0	0	0

Table 7.1: Relevant data for the Jacobi form of $\Gamma_0(6)$

from the expansions of the following Jacobi forms as follows:

$$\begin{aligned}
\iota\infty :6F_6^{0,1}(\tau,z) &:= \frac{1}{6}\phi_{0,1}(\tau,z) - \left(\frac{1}{6}E_2^{(2)}(\tau) + \frac{1}{2}E_2^{(3)}(\tau) - \frac{5}{2}E_2^{(6)}(\tau)\right)\phi_{-2,1}(\tau,z) \\
&= \left(\tfrac{2}{r} - 2 + 2r\right) + q\left(-\tfrac{2}{r^2} + \tfrac{6}{r} - 8 + 6r - 2r^2\right) + \cdots \\
0 :6F_6^{1,0} &:= \frac{1}{6}\phi_{0,1}(\tau,z) - \frac{1}{12}\left(E_2^*(\tau) + E_2^*(\tfrac{\tau}{2}) + E_2^*(\tfrac{\tau}{3}) - E_2^*(\tfrac{\tau}{6})\right)\phi_{-2,1}(\tau,z) \\
&= 2 + \left(-\tfrac{2}{r} + 4 - 2r\right)q^{1/6} + \cdots \\
\frac{1}{2} :6F_6^{2,1} &:= \frac{1}{6}\phi_{0,1}(\tau,z) - \frac{1}{12}\left(E_2^*(\tau) + 4E_2^*(2\tau) + E_2^*(\tfrac{\tau+1}{3}) - 4E_2^*(\tfrac{2\tau-1}{3})\right)\phi_{-2,1}(\tau,z) \\
&= 2 + e^{4\pi i/3}\left(\tfrac{2}{r} - 4 + 2r\right)q^{1/3} + \cdots \\
\frac{1}{3} :6F_6^{3,1} &:= \frac{1}{6}\phi_{0,1}(\tau,z) - \frac{1}{12}\left(E_2^*(\tau) + E_2^*(\tfrac{\tau-1}{2}) + 9E_2^*(3\tau) - 9E_2^*(\tfrac{3\tau-1}{3})\right)\phi_{-2,1}(\tau,z) \\
&= 2 + \left(-\tfrac{2}{r} + 4 - 2r\right)\sqrt{q} + \left(\tfrac{2}{r^2} - \tfrac{4}{r} + 4 - 4r + 2r^2\right)q + \cdots
\end{aligned}$$

where $E_2^*(\tau) = 1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^m = \frac{12i}{\pi}\,\partial_\tau\left[\ln\eta(\tau)\right]$ is the holomorphic part of the weight two non-holomorphic modular form of $SL(2,\mathbb{Z})$.

Using the Fourier expansion, we see that $A = B = C = 1$ and $k = 2$ consistent with the Siegel modular form $\Phi_2(\mathbf{Z})$ that we have seen in the previous chapter. The Borcherds product formula provides us with explicit formula for the multiplicity of the positive roots of BKM Lie superalgebra. Since $\nabla_1(\mathbf{Z})$ is given by the square-root of $\Phi_2(\mathbf{Z})$, we need to take one-half of the exponent (or equivalently the Jacobi form that is the seed for the Borcherds product). For instance the root associated with r^{-1} has its multiplicity given by $\frac{1}{2}c_\infty(0,-1) = 1$, the roots qr and sr have multiplicities given by $\frac{1}{2}c_\infty(0,1) = 1$. This is consistent with the simple real roots appearing with multiplicity one. We have not proved that all multiplicities are even but that should follow from the integrality properties of the Eisenstein series.

APPENDIX A

Code Listings

A.1 **Sage** Script - Expansion for $\Phi_{2,1}$

```
# CYCLE SHAPE 1^2.2^2.3^2.6^2

# Parameters
num_truncate = 15 #Truncate the expansions num_truncate
n_max = 10 #Truncate the summation for Lift at n_max

q = var('q')
r = var('r')
s = var('s')

# To remove the fractional exponents
qh = var('q_h')
rh = var('r_h')
sh = var('s_h')

#######################
#  Generate the seed  #
#######################

# Generate the expansion for \theta_1
def theta1_qexpansion(num_truncate=10):
    i = var('i') #Dummy index
    theta1_term = (-1)^i*(r^(i+1/2))*q^(((i+1/2)^2)/2)
    theta1_series = sum(theta1_term, i, -num_truncate, num_truncate)
    return theta1_series

# The seed is a modular form of the group \Gamma_0(N)
N=6

# Need to divide the eta functions, PowerSeries works better than
# SymbolicExpression as it is faster and also keeps track of the order
# of the series.
q1 = qexp_eta(QQ[['q']], num_truncate)
q1_prec = q1.prec()
qt = qexp_eta(QQ[['q']], num_truncate).truncate(num_truncate)

# Power Series Ring over the field of Rational Numbers QQ in the
# variable q
S = PowerSeriesRing(QQ, q)

```

```
q2 = qt.subs(q=q^2); q2 = S(q2).O(2*q1_prec)
q3 = qt.subs(q=q^3); q3 = S(q3).O(3*q1_prec)
q6 = qt.subs(q=q^6); q6 = S(q6).O(6*q1_prec)

qpp = q2*q3*q6/q1^2

qp = qpp.truncate(num_truncate)
qp = q^(3/8)*qp

eta_product_exp = qp.expand()
eta_product_prec = qpp.prec()
###

# Generate the theta series only uptil the order of the eta product
# expansion.  The higher terms are useless, and it seems to be
# difficult to truncate a symbolic expression later.
theta1_num_truncate = ceil((sqrt(3*eta_product_prec)-1)/2)
theta1_exp = theta1_qexpansion(num_truncate=theta1_num_truncate)
theta1_exp = theta1_exp.simplify_exp().expand()

# This is valid only upto order
# Min(Order(theta1_exp^2),Order(eta_product_exp))^2.  Need to find a
# way to truncate it, to get rid of the higher order junk
phi_two_one = (theta1_exp*eta_product_exp)^2

phi_two_one = phi_two_one.simplify_exp()
phi_two_one = phi_two_one.expand()

# Information about \phi_{2,1}
weight = 2
index = 1

###################
#  Additive Lift  #
###################

# The additive lift in this case is straightforward and the character
# is trivial.
# Reference: [S. Govindarajan] [2009] [arXiv: 0907.1410] Equation 3.22
# Phi_2(Z) = \sum_{(n,l,m)>0} \sum_{d|(n,l,m) \\ d=1,5 mod 6} d^(k-1)
# a(mn/d^2,l/d) q^n r^l s^m
# where (n,l,m)>0 means n,m > 0 and 4nm-l^2>0

def phi_two_one_coeff(m,n):
    if m==0:
        x=flatten(phi_two_one.coeffs(q))
        for i, exp in enumerate(x[1::2]):
            if exp==0:
                return x[2*i].coeff(r^n)
    if n==0:
        x=flatten(phi_two_one.coeffs(r))
        for i, exp in enumerate(x[1::2]):
            if exp==0:
```

```python
                return x[2*i].coeff(q^m)

    return phi_two_one.coeff(q^m).coeff(r^n)

def additive_lift(n_max = 10):

    Phi=0
    for n in range(1,n_max+1,1):
        for m in range(1,n+1,1):

            gcd_mn = gcd(n,m)

            lmax = floor(2*sqrt(n*m))
            l_list = range(-lmax,lmax+1)

            for l in l_list:

                g = gcd(gcd_mn,l)
                d_list = [d for d in divisors(g) \
                        if d in union(range(1,g+1,6),range(5,g+1,6))]

                for d in d_list:

                    fc = phi_two_one_coeff((n*m)/(d^2), l/d)

                    if not fc==0:
                        term_coeff = d^(weight-1)*fc
                        if (n==m):
                            term_vars = q^(n)*r^(l)*s^(m)
                        else:
                            term_vars = \
                                    q^(n)*r^(l)*s^(m) + q^(m)*r^(l)*s^(n)
                        Phi += term_coeff*term_vars
    return Phi

Phi_2 = additive_lift(n_max)

# vim: ft=python
```

A.2 Sage Script - Character Table

The following Sage code enumerates all balanced cycle shapes of M_{12} and then checks the first few coeffecients to determine if they are multiplicative. This was used to generate the character table 5.1.

```
# Author: Hersh Singh [hershdeep@gmail.com]
# Date: April 18, 2014

N=12

plists = Partitions(N).list()

# Takes a Partition object and checks if it is balanced.
def CheckBalancedCycle(plist):
    M = plist[0]*plist[-1]

    if not M%lcm(plist) == 0:
        return False, 0

    plist_new = [M/i for i in plist]
    plist_new.sort()
    plist_new.reverse()

    if plist_new == plist:
        return True, M
    else:
        return False, 0

balanced_cycles=[]
def GetAllBalancedCycles(k=0):
    # If k=0, return the complete table
    # Else, return the table only for that value of k

    number_balancedcycles = 0
    if not k==0:
        for plist in [plist for plist in plists if len(plist)==2*k]:
            is_balanced, M = CheckBalancedCycle(plist)
            if is_balanced:
                number_balancedcycles = number_balancedcycles + 1
                balanced_cycles.append(plist)
                print plist.to_exp_dict()
    else:
        for k in range(1,N+1):
            print "k =",k
            for plist in [plist for plist in plists if len(plist)==k]:
                is_balanced, M = CheckBalancedCycle(plist)
                if is_balanced:
                    number_balancedcycles = number_balancedcycles + 1
                    balanced_cycles.append(plist)
                    print plist.to_exp_dict()
    print "\nTotal number of balanced cyles = ", number_balancedcycles
```

```
N=12

plists = Partitions(N).list()

num_truncate = 500 # Max order of stuff

# The eta product expansion is valid only till order
# num_truncate*min(plist)
n_max = num_truncate

# Generate a list of all pairs of coprimes such that their product is
# less than n_max
coprimes = [[x,y] for x in range(1,n_max) for y in range(1,x) \
        if gcd(x,y)==1 and x*y<n_max]

# Eta expansion
q = var('q')
q1 = qexp_eta(ZZ[['q']], num_truncate).truncate(num_truncate)
eta_exp = q1*q^(1/24)

# Get the coeff
def etacoeff(i):
    return etaproduct.coeff(q^(i))

# Number of possible multiplicative functions
num_multiplicative = 0

# Loop over all balanced cycles
for plist in balanced_cycles:

    # Generate the eta product expansion, valid only till order
    # num_truncate*min(plist)
    etaproduct = 1
    for num in plist:
        etaproduct = etaproduct * eta_exp.subs(q=q^num)
    etaproduct = etaproduct.simplify_exp().expand()

    # Check the first few coefficients
    is_multplicative=1
    for [x,y] in coprimes:
        if not etacoeff(x)*etacoeff(y) == etacoeff(x*y):
            is_multplicative=0
            break;

    if is_multplicative==1:
        print "Could be..", plist
        num_multiplicative += 1
    else:
        print "\tNope!", plist

print "No of possible multiplicative functions =", num_multiplicative

# vim: ft=python
```

REFERENCES

[Cah84] Robert N Cahn. *Semi-simple Lie algebras and their representations.* Benjamin/cummings Menlo Park, California, 1984.

[Car05] Roger William Carter. *Lie algebras of finite and affine type*, volume 96. Cambridge University Press, 2005.

[CG11] F. Clery and Valery Gritsenko. Siegel modular forms of genus 2 with the simplest divisor. *Proceedings of the London Mathematical Society*, 102(6):1024–1052, 2011.

[DKM85] D Dummit, H Kisilevsky, and J McKay. Multiplicative products of η functions. *Contemp. Math*, 45:89–98, 1985.

[DVV97] Robbert Dijkgraaf, Erik P. Verlinde, and Herman L. Verlinde. Counting Dyons in N=4 String Theory. *Nucl. Phys.*, B484:543–561, 1997.

[EZ85] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55. Birkhauser Boston, 1985.

[FN03] Alex J. Feingold and Hermann Nicolai. Subalgebras of Hyperbolic Kac-Moody Algebras. 2003.

[GK09] Suresh Govindarajan and K Gopala Krishna. Generalized Kac-Moody algebras from CHL dyons. *Journal of High Energy Physics*, 2009(04):032, 2009.

[GK10] Suresh Govindarajan and K Gopala Krishna. BKM Lie superalgebras from dyon spectra in $\mathbb{Z}_N$ CHL orbifolds for composite N. *Journal of High Energy Physics*, 2010(5):1–40, 2010.

[GN96] V. A. Gritsenko and V. V. Nikulin. Igusa modular forms and “the simplest” Lorentzian Kac-Moody algebras. *Mat. Sb.*, 187(11):27–66, 1996.

[GN97] Valeri A. Gritsenko and Viacheslav V. Nikulin. Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras. *Amer. J. Math.*, 119(1):181–224, 1997.

[GN98a] Valeri A. Gritsenko and Viacheslav V. Nikulin. Automorphic forms and Lorentzian Kac-Moody algebras. I. *Internat. J. Math.*, 9(2):153–199, 1998.

[GN98b] Valeri A. Gritsenko and Viacheslav V. Nikulin. Automorphic forms and Lorentzian Kac-Moody algebras. II. *Internat. J. Math.*, 9(2):201–275, 1998.

[GN02] V. A. Gritsenko and V. V. Nikulin. On the classification of Lorentzian Kac-Moody algebras. *Uspekhi Mat. Nauk*, 57(5(347)):79–138, 2002.

[Gov10] Suresh Govindarajan. Brewing moonshine for Mathieu. *arXiv preprint arXiv:1012.5732*, 2010.

[Gov11] Suresh Govindarajan. BKM Lie superalgebras from counting twisted CHL dyons. *Journal of High Energy Physics*, 2011(5):1–29, 2011.

[Hum72] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer, 1972.

[Kac94] Victor G Kac. *Infinite-dimensional Lie algebras*, volume 44. Cambridge university press, 1994.

[OEI11] The on-line encyclopedia of integer sequences, 2011. published electronically at http://oeis.org.

[Sen08] Ashoke Sen. Black Hole Entropy Function, Attractors and Precision Counting of Microstates. *Gen. Rel. Grav.*, 40:2249–2431, 2008.