Analysis of a Certain MAC Scheduling Algorithm

A Thesis
submitted by

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under the guidance of
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for award of the degree of

Bachelor of Technology

May 2013

DEPARTMENT OF ELECTRICAL ENGINEERING,

INDIAN INSTITUTE OF TECHNOLOGY.

THESIS CERTIFICATE

This is to certify that this thesis submitted by **Basil Mohammed**, to the Department of Electrical Engineering, Indian Institute of Technology, Madras, for the award of the degree of **Bachelor of Technology**, is a bonafide record of the research work done by him under my supervision.

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1 Introduction

If two nodes in a wireless netwrok transmit at the same time, their signals will interfere and the reciever might not be able to seperate the two signals. It is called collision. It is assumed in this paper that , in the event of collsion none of the messages are delivered.The function of MAC layer or Medium Access Control layer is to regulate the access to a shared communication channel so that efficient data transfer can take place with minimal collisions, high bandwidth efficency and so on. The challenge is to construct a MAC protocol with the given constraints of limited power, limited information exchange and limited error frequencies. One of the most common used approach to MAC is TDMA or Time Division Multiple Access. Time is divided into equal units called frames and frames are further divided into equal units called slots. Users agree upon using specific slots without any conflicts and transmit their messages in those slots in each frame. The schedulding part of the TDMA is usually centralized, i.e the slots are alloted with certain involvement from an external server or something of that sort. In this paper I analyse a simple but completely decentralized schedulding algorithms which will work in the framework of TDMA. Clearly, no deterministic algorithm is going to work here as the users are assumed to be indistinguishable. So, the users are going to do trial and error by attempting to send message during carefully chosen random slots and hope for the best.

Unless otherwise mentioned I assume that the time is divided into frames of size N slots, where each slot is a period extending for t seconds. Slots are numbered 1 to N and there are N users U_1through U_N who are trying to share the medium. I also assume each user(node) U_i will attempt to send a message in exaclty one slot in each frame. If collision occurs, their message is not send and get an immediete negetive feedback acknowledging the collision. If the no other users try to send a message in the same slot as a user, then its the message is transmitted successfully, they immedietly get a positive feedback. This the only feed back the users get. For example the users cannot listen to the network and find out which all slots are being used in the current frame. Topology of the network will be ignored and all the users will be assumed to be synchronized in time. This is actually not completely necessary since we only need synchronization upto slots. That is slot which U_1 Slot 1 could be Slot 4 for U_3 as long as starting and ending points of the slot are synchronized. One important efficiency indicating parameter in these kind of problems is the expected settling time (denoted by T_N which is defined to be the average time (measured in number of frames) before all the users are assigned a slot averaged over the random bits the users use for randomization.

2 Description of the Algorithm

A simple algorithm which the users can excecute independently can be descirbe as follows:

Algorithm X :

(1)Attempt to transmit in one of the slots in the current frame by choosing 1among N slots uniformly at random

(2)In case of a successful transmition go to step (3), else go to step (1) (after the current frame ends)

(3)Continue to transmit in the same slot in the future frames.That is he has been assigned that particular slot for future transmissions

Although in the algorithm described, the unsettled slots (i.e the slots without a single success till now) are attempted by unsettled users (i.e , the users who haven't been assigned a slot) unifromly, the users also attempt to transmit messages in already settled slots. This might make the algorithm sub optimal. On obvious remedy would be to make users learn which all slots are already settled with the help of the minimum feedback they recieve and not try to attempt those slots in the future. An algorithm of this sort is described in [3].But any such method will lead to unwanted localization in case of false detection. Although this would make the settling time smaller in some cases, the author conjuctures that Algorithm X is optimal, when what we are trying to minimize is the expected time.

3 Exact expected settling time

Denote by $T_{N,m}$, the expected additional settling time when m users are already settled (i.e assigned a slot) and the remaining $N-m$ users are still trying. Also denote by $q_{N,m,i}$ the probability that an additional i users will settle in the next frame when m users are already settled. Then we havethe basic recurrence relation

$T_N = T_{N,0,} = q_{N,0,0}(T_{N,0}+1) + q_{N,0,1}(T_{N,1}+1).. + q_{N,0,N-1}(T_{N,N-1}+1) + q_{N,0,N}$

which on simplifcation gives, $T_{N,0} = \frac{1}{1-q_{N,N,0}} \sum_{i=1}^{N-1} q_{N,0,i} T_{N,i}$ and more generally we have, $T_{N,m} = \frac{1}{1-q_{N,m,0}} \sum_{i=1}^{N-m-1} q_{N,m,i} T_{N,m-i}$ for $0 \leq m \leq N-2$. We also have the initial condition

$$T_{N,N-1} = \sum_{i=1}^{\infty} \frac{i}{N} \left(\frac{N-1}{N}\right)^{i-1} = N$$

Hence we can get the exact expected settling time once we calculate $q_{N,m,i}$.We have

$$q_{N,m,i} = \binom{N-m}{i}^2 P_{N,m,i}$$

where $P_{N,m,i}$ is the probability that exactly some specific set of i new users get settled in the first i unused slots when m users are already settled.Now $P_{N,m,i} = \frac{i!}{N^i} \frac{(N-i)^{N-m-i}}{N^{N-m-i}} q_{N-i,m,0}$. Now we can calculate $1 - q_{N-i,m,0}$ can be calculated by using inclusion exclusion formula. It is the probability that alteast on among the $N-r$ users will settle.

$$1 - q_{N-i,m.0} = \sum_{j=1}^{N-i-m} (-1)^{j-1} \binom{N-i-m}{j}^2 j! \frac{(N-i-j)^{N-m-i-j}}{(N-i)^{N-m-i}}$$

hence, we have

$$q_{N,m,i} = \binom{N-m}{i}^2 i! \frac{(N-i)^{N-m-i}}{N^{N-m}} [1 - \sum_{j=1}^{N-i-m} (-1)^{j-1} \binom{N-i-m}{j}^2 j! \frac{(N-i-j)^{N-m-i-j}}{(N-i)^{N-m-i}}]$$

4 A Complexity Upper Bound

Although it is possible to compute exactly the expected settling time, it doesn't help us much in figuring out its complexity. Here I will provide a simple upper bound.

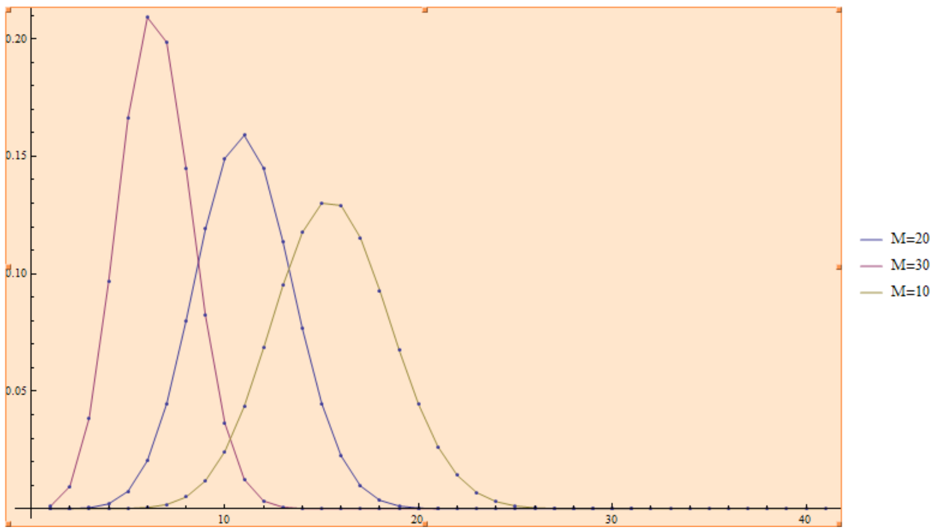

Figure 3.1: Graph of $q_{N,m,i}$vs i when $N = 50$

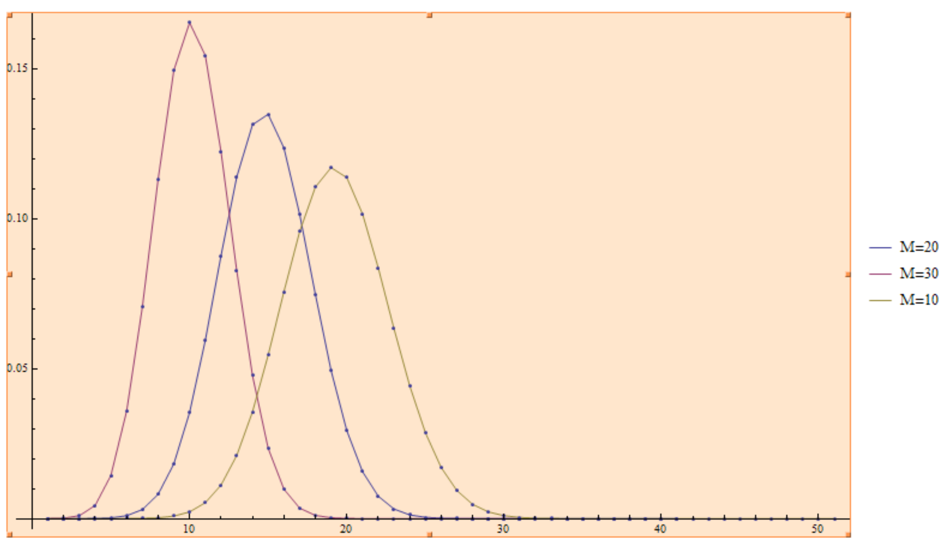

Figure 3.2: Graph of $q_{N,m,i}$vs i when $N = 60$

Lemma. *Let T_n and X_n asn varies from 0 to N be positive random variables whereX_n can takes only integer values between 0 and n such that $T_0 = 0$ and $T_n = 1 + T_{n-X_n}$.Also suppose $E[X_n]$ is non decreasing in n. Then for $n \geq 1$*

$$E[T_n] \leq \sum_{j=1}^{n} \frac{1}{E[X_j]}$$

Proof. We will prove it by induction on n . For $n = 1$, since $E[X_1] = q_1$, where q_1, is the probality that $X_1 = 1$ and $T_1$is a random variable with geometric distribution with mean $\frac{1}{q_1}$ we the inequality , $E[T] = \frac{1}{E[T_1]}$. Let $p_i = P[X_m = i]$

Assume its for $n \leq m - 1$. Le

$$E[T_m] = p_m + p_{m-1}(T_1 + 1) + p_{m-2}(T_2 + 1) + .. + p_0(T_m + 1) \quad (4.1)$$

$$(1 - p_0)E[T_m] = 1 + \sum_{i=1}^{m-1} p_{m-i}T_i \quad (4.2)$$

$$(1 - p_0)E[T_m] \leq 1 + \sum_{i=1}^{m-1} p_{m-i} \sum_{j=1}^{i} \frac{1}{E[X_j]} \quad (4.3)$$

$$(1 - p_0)E[T_m] \leq 1 + \sum_{i=1}^{m-1} p_{m-i}(\sum_{j=1}^{m} \frac{1}{E[X_j]} - \sum_{j=i+1}^{m} \frac{1}{E[X_j]}) \quad (4.4)$$

$$(1 - p_0)E[T_m] \leq 1 + (1 - p_0 - p_m) \sum_{j=1}^{m} \frac{1}{E[X_j]} - \sum_{i=1}^{m} \frac{p_{m-i}(m - i)}{E[X_m]} \quad (4.5)$$

$$(1 - p_0)E[T_m] \leq 1 + (1 - p_0 - p_m) \sum_{j=1}^{m} \frac{1}{E[X_j]} - \frac{E[X_m] - mp_m}{E[X_m]} \quad (4.6)$$

$$(1 - p_0)E[T_m] \leq (1 - p_0 - p_m) \sum_{j=1}^{m} \frac{1}{E[X_j]} + \frac{mp_m}{E[X_m]} \quad (4.7)$$

$$(1 - p_0)E[T_m] \leq (1 - p_0) \sum_{j=1}^{m} \frac{1}{E[X_j]} - \frac{mp}{E[X_m]} + \frac{mp_m}{E[X_m]} \quad (4.8)$$

$$(1 - p_0)E[T_m] \leq (1 - p_0) \sum_{j=1}^{m} \frac{1}{E[X_j]} \quad (4.9)$$

$$E[T_m] \leq \sum_{j=1}^{m} \frac{1}{E[X_j]} \quad (4.10)$$

I have used the induction hypothesis in (4.1) and used the monotonicity of expectation in (4.6) and (4.8) □

It can be also noted that it is impossible to get a lower bound solely in terms of expectation. If X_N takes the value n with probability α and 0 with probability $1-\alpha$, then $E[T_N]$ is independent of $E[X_n]$ if $n < N$. The equality holds when all $E[X_n]$ are the same. The author also noted that a more general version of this lemma is proved in [2]

Here is an attempt to get a lower bound.Define a markov chain with states S_n as n varies from 0 to N such that X_n is the expected length of state transition when you are at state S_n. The Markov chain should be also such that the transition probabilities $p_{ij}$are only non zero when $i \geq j.E[T_N]$ is just the expected time before one reach S_0.Suppose one reaches the state S_0 through the following transitions : $S_n = S_{i_1} \to S_{i_2} \to ..S_{i_m} = S_0$. Then for all i from 1 to N Define $t_i = \frac{1}{i_r - i_{r+1}}$ where r is such that $i_r \geq i$ and $i_{r+1} < i$ .Then clearly $E[T_N] \geq \sum_{1=1}^{N} t_i$ (The only time missing will be the times at which the state remain unchanged). Now let us calculate the $E[t_i]$ by conditioning on those S_{i_r} (call the corresponding events e_k's). Then by Jenson's inequality we have $E[t_i|e_k] = E[\frac{1}{X_k}|X_k > k-i] \geq \frac{1}{E[X_k|X_k > k-i]}$. Then if we could figure out an upper bound for $E[X_k|X_k > k-i]$, then we have ourselves a lower bound given by

$$E[T_n] \geq \sum_{j=1}^{n} \frac{1}{\underset{k \geq i \geq j}{max} \{E[X_i|X_i \geq i-j]\}}$$

Fact. *Suppose there are k balls and n bins out of which $n-k$ bins are occupied (i.e there is already a ball in it). Also suppose that each of the k balls try to select one among the n bins uniformly at random, Then the expected number of balls getting their own bins (ie, there is no other ball is there in the bin they attempted) is given by*

$$e_{k,n} = \frac{k^2}{n}(1-\frac{1}{n})^{k-1} \approx \frac{k^2}{n}\exp(\frac{-k+1}{n})$$

Proof. If r balls are thrown into S bins, then the expected number of balls givern by $f_{r,s} = r(1-\frac{1}{s})^{r-1}$. To see that, let X_i , $0 \leq i \leq r$ denote the random variable which is zero when i^{th} ball doesn't get a bin of its own and 1 otherwise. Then we are looking at the expectation of $\sum_{j=1}^{r} X_j$ which is equal to $\sum_{j=1}^{r} P[X_j = 1] = rP[X_1 = 1] = r(1 - \frac{1}{s})^{r-1}$

Now if Y denote the random variable counting the number of balls getting their own bin, then the required expectation. Let $E_m$be the event that m balls attempt already occupied bins

$$E[Y] = \sum_{m=0}^{k-1} E[Y|E_m]P[E_m] \tag{4.11}$$

$$= \sum_{m=0}^{k-1} f_{k-m,k} \binom{k}{m} (\frac{n-k}{n})^m (\frac{k}{n})^{k-m} \tag{4.12}$$

$$= \sum_{m=0}^{k-1} (k-m)(\frac{k-1}{k})^{k-m-1} \binom{k}{m} \frac{(n-k)^m k^{k-m}}{n^k} \tag{4.13}$$

$$= \frac{k}{n^k} \sum_{m=0}^{k-1} (k-m)(k-1)^{k-m-1} \binom{k}{m} (n-k)^m \tag{4.14}$$

$$= \frac{k}{(k-1)n^k} \sum_{m=0}^{k} m \binom{k}{m} (k-m)(k-1)^{k-m}(n-k)^m \tag{4.15}$$

$$= \frac{k}{(k-1)n^k} \times k(k-1)(n-1)^{k-1} = \frac{k^2}{n}(1-\frac{1}{n})^{k-1} \tag{4.16}$$

□

Now the following directly follows from the lemma and the fact above (It is straightforward to verify that $E[X_n]$ is non decreasing in n in this case. See Appendix)

Theorem. *Let T_n denote the expected time before everyone gets their own slots, if each of the n users attempt n slots by choosing 1 of them uniformly at random, until a successful transmission occurs, after which he sticks with the same slot , then we have*

$$T_n \leq \sum_{k=1}^{n} \frac{n}{k^2}(1-\frac{1}{n})^{1-k}$$

Since $(1-\frac{1}{n})^{1-k} \leq (1-\frac{1}{n})^{1-n} \leq \frac{1}{e}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, we have $T_n \leq \frac{\pi^2}{6e}n$.Hence T_n is at most $O(n)$ .As seen from the Figure 4.2, the error increases as n increases. This is also clear from the proof in which the errors add up

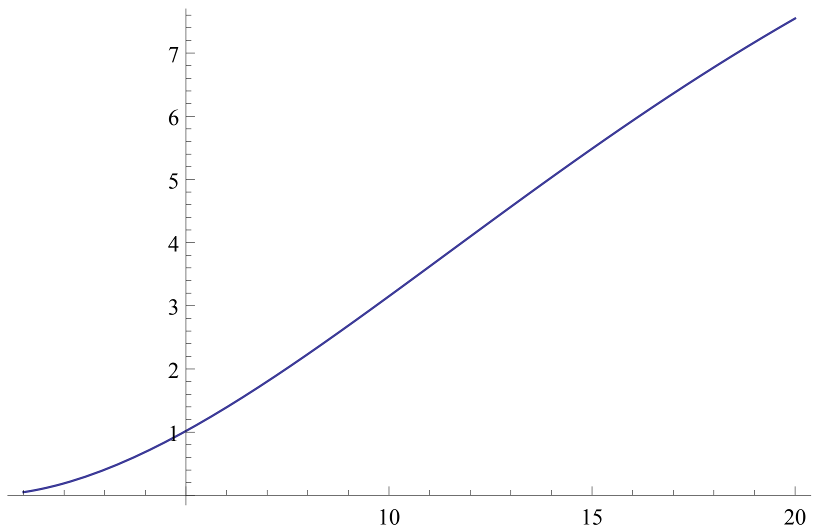

Figure 4.1: Graph of $E[X_k]$ vs k when $N = 20$

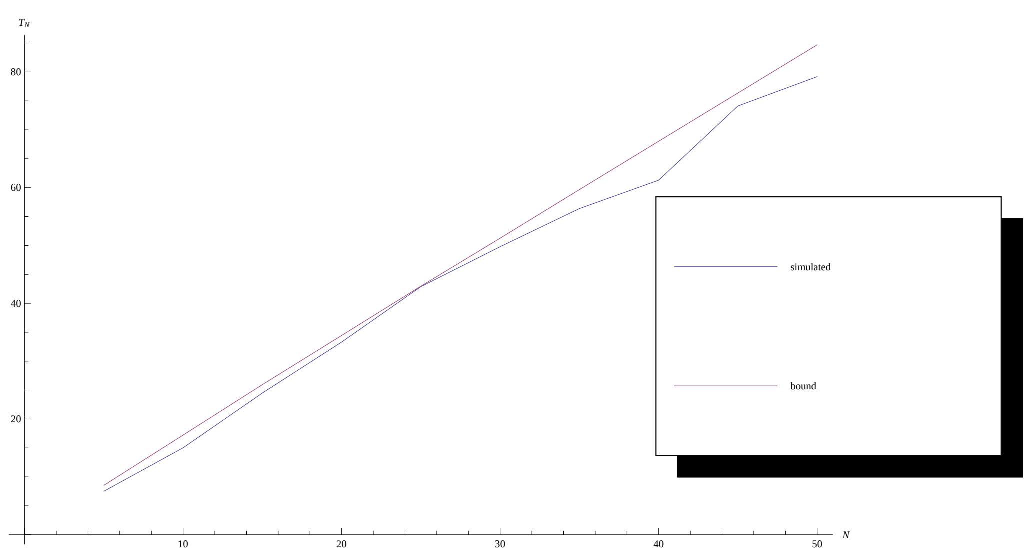

Figure 4.2: Graph ofT_N vs N

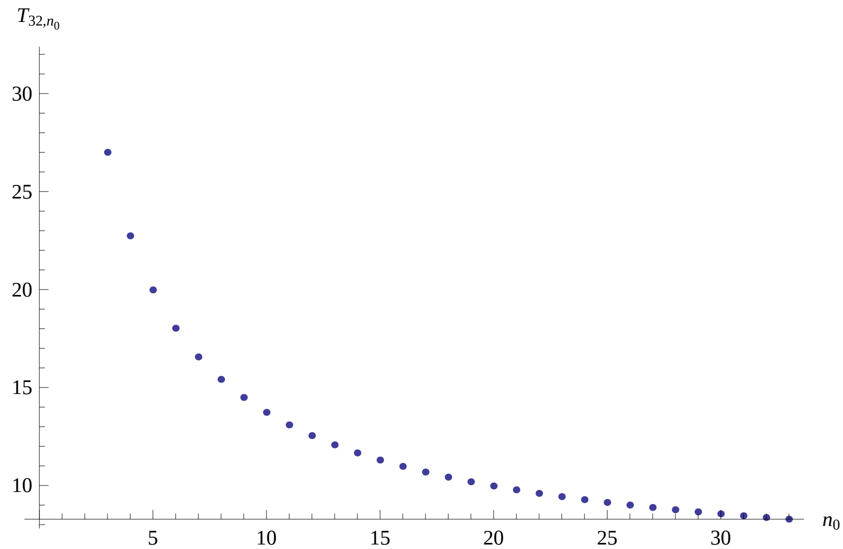

Figure 5.1: Graph of T_{32,n_0}vs n_0

inductively. It should be noted that all arguments made above holds with minor modifications when we relax the condition that the number of users is the same as the number of slots. (That is, if we only need an ordering). In the appendix I have stated some of the corresponding results without proof.

5 APPENDIX

Fact. *Suppose there are k balls and $n + n_0$ bins out of which $n - k$ bins are occupied (i.e there is already a ball in it). Also suppose that each of the k balls try to select one among the $n + n_0$ bins uniformly at random, Then the expected number of balls getting their own bins (ie, there is no other ball is there in the bin they attempted) is given by*

$$e_{k,n,n_0} = \frac{k(k+n_0)}{n+n_0}(1 - \frac{1}{n+n_0})^{k-1}$$

and hence

Theorem. *Let T_{n,n_0} denote the expected time before everyone gets their own slots, if each of the n users attempt $n + n_0$ slots by choosing 1 of them uniformly* *at random, until a successful transmission occurs, after which he sticks with the same slot , then we have*

$$T_{n,n_0} \le \sum_{k=1}^{n} \frac{n+n_0}{k(k+n_0)}(1-\frac{1}{n})^{1-k} = B_{n,n_0}$$

As expected it is decreasing in n_0 at a very fast rate. Hence if the duration of communication is small enough, it is better to have just an ordering instead of one hundred percent efficient slot assignment.Since the bounds are pretty close to the actual values when n is small, $B_{n,0} - B_{n,n_0}$will give us a good measure of how faster the settling time is in the situation with extra slots relative to the original algorithm. We have now $B_{n,0} - B_{n,n_0} \ge \sum_{k=1}^{n}(\frac{n}{k^2} - \frac{n+n_0}{k(k+n_0)})(1-\frac{1}{n+n_0})^{1-k}$. But since $(1-\frac{1}{n+n_0})^{1-k} \ge 1$, we have $B_{n,n_0} - B_{n,0} \ge \sum_{k=1}^{n}(\frac{n}{k^2} - \frac{n+n_0}{k(k+n_0)})$. It is easy two see that $\frac{n}{k^2} - \frac{n+n_0}{k(k+n_0)} = \frac{n_0(n-k)}{k^2(k+n_0)}$ is a decreasing function function in n. Hence we have $\sum_{k=1}^{n} \frac{n_0(n-k)}{k^2(k+n_0)} \ge \frac{n_0 n}{1+n_0} + \int_1^n \frac{n_0(n-k)}{k^2(k+n_0)}\, dk$. This integral is evaluated to be $(n-1)+\left(\frac{n}{n_0}+1\right)(\log(n+n_0) - \log(n_0(n_0+1)))$ whose derivate with respect to $n_0$is $\frac{n(\log(n_0)+\log(n_0+1)-\log(n+n_0))}{n_0^2} - \left(\frac{n}{n_0}+1\right)\left(\frac{1}{n_0+1} - \frac{1}{n+n_0} + \frac{1}{n_0}\right)$. Hence for small n_0 the decrement is around $\log(n_0)$.

Fact. $f(k) = \frac{k(k+n_0)}{n+n_0}(1-\frac{1}{n+n_0})^{k-1}$ *is non decreasing for* k *in* $[1,n]$ *for all non negetive* n_0

Proof. $\frac{f(k+1)}{f(k)} = (1-\frac{1}{n+n_0})(1+\frac{1}{k})(1+\frac{1}{k+n_0})$ which is greater than or equal to $(1-\frac{1}{n+n_0})(1+\frac{1}{n})(1+\frac{1}{n+n_0}) = (1-\frac{1}{(n+n_0)^2})(1+\frac{1}{n})$ which is at least $1-\frac{1}{n^3}+\frac{1}{n}-\frac{1}{n^2} \ge 1$ □

5.1 A lower bound

Fact. *Suppose there are* k *balls and* n *bins out of which* $n-k$ *bins are occupied (i.e there is already a ball in it). Also suppose that each of the* k *balls try to select one among the* n *bins uniformly at random, Then we have the followng bound for the expected number of balls getting their own bins (ie, there is no other ball is there in the bin they attempted) conditioned on the fact that at least* r *balls gets their own bin given by*

$$e_{k,n,r} \le \frac{(k-r)^2}{n}(1-\frac{1}{n})^{k-r-1} + r$$

Proof. The required exectation will be atmost the expectation obtained by conditioning on some fixed r balls getting their own bin. (Essentially we are the splitting calculation of expectation (of a positive random variable) by conditioning over non disjoint events).Hence $e_{k,n,r} \leq e_{k-r,n} + r$. □

Hence with same notations as above, we have$E[X_k|X_k > k-i] \leq \frac{(i-1)^2}{n}(1-\frac{1}{n})^{i-2} + k - i + \leq n - i + \frac{(i-1)^2}{n}(1-\frac{1}{n})^{i-2}$. And hence we will have the bound $T_N \geq \sum_{i=1}^{N}(n-i+\frac{(i-1)^2}{n}(1-\frac{1}{n})^{i-2})^{-1}$. As expected this bound turns out to be extremely weak. Computations show that the sum is bounded above by 2. The weakness is due to the weakness of the bound on the expectation.

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