On the eigenvalue and entropy distribution of random pure state reduced density matrices

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THESIS CERTIFICATE

This is to certify that the report entitled "On the eigenvalue and entropy distribution of random pure state reduced density matrices" submitted by Barath A (EE09B067) to the Indian Institute of Technology, Madras in partial fulfilment of the requirements for the award of Master of Technology is a bonafide record of work carried out by him under our supervision. The contents of this thesis, in full or parts have not been submitted to any other Institute or University for the award of any Degree or Diploma.

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Abstract

In this project, I have studied the eigenvalue distributions that arise from using different generating probability distributions for the pure state. The entropy of entanglement of these distributions are also studied and the interesting localisation behaviour in the case of Cauchy distributed pure states is detailed.

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1 Introduction

1.1 A review of Dirac's Bra-Ket notation

The Bra-Ket notation as introduced by Paul A.M Dirac has found widespread adoption in the modern physics community to describe quantum mechanics (where states are basically vectors in a linear vector space).

In this notation, an $N \times 1$ column vector belonging to the N-dimensional linear vector space $\mathcal{F}^N$ over field $\mathcal{F}$ (usually the field $\mathbb{C}$ of complex numbers) is represented by $|\psi\rangle$ and its dual (the conjugate transpose) is represented by $\langle\psi|$.

The operations defined on the vectors follow the usual axioms of a linear vector space, namely,

- Closure over addition i.e if $|\psi\rangle \in \mathcal{F}^N$ and $|\phi\rangle \in \mathcal{F}^N$, then, $|\psi\rangle + |\phi\rangle \equiv |\psi + \phi\rangle \in \mathcal{F}^N$.
- Commutativity and associativity over addition i.e $|\psi\rangle + |\phi\rangle = |\phi\rangle + |\psi\rangle$ and $|\phi\rangle + (|\psi\rangle + |\chi\rangle) = (|\phi\rangle + |\psi\rangle) + |\chi\rangle$.
- Existence of additive identity(null vector) and inverse i.e $\exists\ |\Phi\rangle$, and given any vector $|\psi\rangle$, $\exists\ |-\psi\rangle$ such that $|\psi\rangle + |-\psi\rangle = |\Phi\rangle$.
- Distributivity of scalar multiplication over vectors and scalars i.e given a, $b \in \mathcal{F}$,
  - $a|\psi\rangle + a|\phi\rangle = a(|\psi\rangle + |\phi\rangle)$
  - $(a+b)|\psi\rangle = a|\psi\rangle + b|\psi\rangle$

The inner product between two vectors $|\phi\rangle$ and $|\psi\rangle$ is denoted by

$$\langle\phi|\psi\rangle = \sum_i \phi_i^* \psi_i \tag{1}$$

and the outer product respectively is given by

$$|\phi\rangle\langle\psi| = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \begin{pmatrix} \psi_1^* & \dots & \psi_N^* \end{pmatrix} \tag{2}$$

where the ψ_i and ϕ_i are the components of the respective vectors.

Some of the important properties of the inner product and outer product follow for convenience as well as to illustrate the elegance the notation brings to handling these objects.

Let a, $b \in \mathbb{C}$, then:

- $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$ (skew-symmetry)
- $\langle\phi|a\psi + b\chi\rangle = a\langle\phi|\psi\rangle + b\langle\phi|\chi\rangle$ (linear w.r.t second argument)
- $\langle a\phi + b\chi|\psi\rangle = a^*\langle\phi|\psi\rangle + b^*\langle\chi|\phi\rangle$ (anti-linear w.r.t first argument)
- $\langle\psi|\psi\rangle \geq 0$ with equality holding iff $|\psi\rangle$ is the null vector.
- $|\psi\rangle\langle\phi| = |\phi\rangle\langle\psi|^\dagger$ (skew-symmetry)
- $|a\phi + b\chi\rangle\langle\psi| = a|\phi\rangle\langle\psi| + b|\chi\rangle\langle\psi|$ (linear w.r.t first argument)
- $|\phi\rangle\langle a\psi + b\chi| = a^*|\phi\rangle\langle\psi| + b^*|\phi\rangle\langle\chi|$ (anti-linear w.r.t second argument)
- $(|\psi\rangle\langle\phi|) \times (|\phi\rangle\langle\psi|) = |\psi\rangle\langle\phi|\phi\rangle\langle\psi| = \langle\phi|\phi\rangle|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|$ with the last equality holding when $|\phi\rangle$ is a normalised state vector.
- Given a basis $\{|v_i\rangle\}_{i=1}^N$, we have the following relation: $\sum\limits_i |v_i\rangle\langle v_i| = I_N$ where I_N is the identity operator i.e identity matrix of order N.

1.2 bipartite pure states

Consider two systems A and B with corresponding hilbert spaces $\mathcal{H_A}$ and $\mathcal{H_B}$, then states in the composite system AB can be represented using states that belong to the tensor product structure $\mathcal{H_A} \otimes \mathcal{H_B}$.

A pure state $|\psi\rangle_{AB}$ of system AB is termed bipartite, if that state is represented as

$$|\psi\rangle_{AB} = \sum_{i,\alpha} x_{i\alpha} |i_A\rangle \otimes |\alpha_B\rangle \tag{3}$$

where $|i_A\rangle$ and $|\alpha_B\rangle$ are some orthonormal bases of $\mathcal{H_A}$ and $\mathcal{H_B}$ respectively. The coefficients $x_{i\alpha}$ are in general drawn from the set of complex numbers such that $\sum\limits_i \sum\limits_\alpha |x_{i\alpha}|^2 = 1$ (normalisation condition).

The question of determining whether a given state $|\psi\rangle_{AB}$ is entangled is in general difficult. However, if that state $|\psi\rangle_{AB}$ is a bipartite pure state, we can introduce a well defined function known as the von neumann entropy to assess the strength of the entanglement. Before we do so, let us introduce the required machinery of Schmidt decomposition and Density matrices.

1.3 Schmidt decomposition

Suppose $|\psi\rangle$ is a pure state of a composite system AB, then there exist orthonormal states $\left|\lambda_i^A\right\rangle$ of system A and orthonormal states $\left|\lambda_i^B\right\rangle$ of system B such that,

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} \left|\lambda_i^A\right\rangle \left|\lambda_i^B\right\rangle \tag{4}$$

satisfying $\sum_i \lambda_i = 1$ where $\sqrt{\lambda_i}$ are non-negative real numbers.[1]

Proof: Let $|j\rangle$ and $|k\rangle$ be any fixed orthonormal bases for systems A and B, respectively. Then $|\psi\rangle$ can be written as,

$$|\psi\rangle = \sum_{j,k} x_{jk} |j\rangle |k\rangle \tag{5}$$

for some matrix X of complex numbers x_{jk}. By singular value decomposition, $X = UDV$ where D is a diagonal matrix with non-negative elements, and, U and V are unitary matrices. Therefore,

$$|\psi\rangle = \sum_{i,j,k} u_{ji} d_{ii} v_{ik} |j\rangle |k\rangle \tag{6}$$

Setting $|\lambda_i^A\rangle \equiv \sum_j u_{ji} |j\rangle$, $|\lambda_i^B\rangle \equiv \sum_k v_{ik} |k\rangle$, and $\sqrt{\lambda_i} = d_{ii}$, we see that this gives,

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |\lambda_i^A\rangle |\lambda_i^B\rangle \tag{7}$$

To conclude the proof, we notice that the unitary nature of U and V, and the orthonormality of the bases $|j\rangle$ and $|k\rangle$ leads to the orthonormality of the bases $|\lambda_i^A\rangle$ and $|\lambda_i^B\rangle$ respectively.

1.4 Density Matrix and Reduced Density Matrix

1.4.1 Definition and Properties of Density Matrix

The Density operator formalism is an equivalent approach to the state vector formulation of Quantum Mechanics. It is convenient to describe our system via the density operator or density matrix approach in certain scenarios, for example, in the description of individual subsystems of a composite quantum system.

Suppose a quantum system is in one of a number of normalised states $|\psi_i\rangle$, with respective probabilities p_i, namely, it exists as an ensemble of pure states: $\{p_i, |\psi_i\rangle\}$ where $\sum_i p_i = 1$. Then, the density matrix ρ of the system is defined by the following equation:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| \tag{8}$$

In particular, if the state of the system is found to be a normalised pure state vector $|\psi\rangle$, then the density matrix associated with this state is simply the outer product $|\psi\rangle \langle\psi|$.

It is seen easily from the above definition of ρ that,

[1] notice that in equation (2), the product $|i_A\rangle |i_B\rangle$ represents, in shorthand, the tensor product of the two states

- $\rho^\dagger = \rho$ and $\langle\chi|\rho|\chi\rangle \geq 0$ i.e ρ is hermitian as well as positive semi-definite matrix.
- $\mathrm{Tr}(\rho) = 1$
- $\mathrm{Tr}(\rho^2) \leq 1$ with equality iff ρ is the DM of a pure state. In addition, we have for a pure state DM $\rho^2 = \rho$.

1.4.2 Reduced Density Matrix (rdm)

Now, we will look at the main application of the density matrix formalism, namely in the description of subsystems of a composite quantum system. This is achieved by computing the Reduced Density Matrix for the subsystem in question.

Suppose ρ^{AB} is the Density Matrix of a composite system AB, then the Reduced Density Matrix for subsystem A is given by

$$\rho^A = \mathrm{Tr}_\mathrm{B}(\rho^{AB}) \tag{9}$$

where Tr_B is the partial trace over system B. The partial trace is defined by

$$\mathrm{Tr}_\mathrm{B}(|\mathrm{a}_1\rangle\langle\mathrm{a}_2|\otimes|b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2|\mathrm{Tr}(|\mathrm{b}_1\rangle\langle\mathrm{b}_2|) = |a_1\rangle\langle a_2|\langle b_2|b_1\rangle \tag{10}$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B.

Reformulating the same in a more general fashion, consider system B: Let $|\alpha_B\rangle$ denote a complete basis which lies in the Hilbert Space $\mathcal{H}_\mathcal{B}^{(\mathrm{M})}$, then,

$$\rho_A = \mathrm{Tr}_\mathrm{B}(\rho^\mathrm{AB}) = \sum_{\alpha=1}^{\mathrm{M}} \langle\alpha_\mathrm{B}|\rho|\alpha_B\rangle \tag{11}$$

1.5 Entanglement in bipartite pure states

The idea of entanglement forms a cornerstone in many applications of quantum information and computation. In fact, systems with states of large entanglement are valuable as resources for most quantum communication and computational protocols as well in quantum cryptography.

The Idea of entanglement is simple - if a pure state $|\psi\rangle$ (of system AB) is inexpressible as a direct product of some two states belonging to subsystems A and B, then we say that the state $|\psi\rangle$ is entangled.

We are now ready to explore the concept of entanglement of bipartite pure states and thus set the foundation on which the rest of this work relies upon.

Consider, once again, the composite system AB whose individual Hilbert spaces are $\mathcal{H}_\mathcal{A}^{(\mathrm{N})}$ and $\mathcal{H}_\mathcal{B}^{(\mathrm{M})}$respectively, then a normalised pure state $|\psi\rangle$ of the system AB resides in the tensor product space $\mathcal{H}_\mathcal{A}^{(\mathrm{N})} \otimes \mathcal{H}_\mathcal{B}^{(\mathrm{M})}$.

Without loss of generality, assume that the dimensions N, M satisfy $N \leq M$ and let $|i_A\rangle$ and $|\alpha_B\rangle$ represent orthonormal bases of A and B respectively. Then,

$$|\psi\rangle = \sum_{i,\alpha} x_{i\alpha} |i_A\rangle \otimes |\alpha_B\rangle \tag{12}$$

Therefore, we have,

$$\rho = |\psi\rangle \langle\psi| = \sum_{i,\alpha}\sum_{j,\beta} x_{i\alpha} x^*_{j\beta} |i_A\rangle\langle j_A| \otimes |\alpha_B\rangle\langle\beta_B| \tag{13}$$

where the roman indices run from 1 to N and the greek indices run from 1 to M. Now, using the definition of partial trace from equation 11, we have,

$$\rho^A = \mathrm{Tr_B}(\rho) = \sum_{\alpha=1}^{M} \langle\alpha_B|\rho|\alpha_B\rangle = \sum_{i,j=1}^{N}\sum_{\alpha=1}^{M} x_{i\alpha} x^*_{j\alpha} |i_A\rangle\langle j_A| = \sum_{i,j=1}^{N} w_{ij} |i_A\rangle\langle j_A| \tag{14}$$

where w_{ij} forms the entries of the $N \times N$ martix $W = XX^\dagger$. Similarly, we can obtain the expression for the reduced density matrix ρ^B from the $M \times M$ matrix $\tilde{W} = X^\dagger X$.

That is, the reduced density matrix for system A residing in the pure state $|\psi\rangle$ is given by $XX^\dagger$ and that of system B is $X^\dagger X$.

Furthermore, using the schmidt decomposition formula, we have that,

$$\rho^A = \sum_{i=1}^{N} \lambda_i |\lambda_i^A\rangle\langle\lambda_i^A| \tag{15}$$

Thus, from this diagonal representation of ρ^A, we can deduce that the eigenvalues of the matrix W are none other than the λ_i with eigenvectors $|\lambda_i^A\rangle$, $i = 1, 2, \ldots, N$. Similarly, the eigenvalues of ρ^B are the same additionally, the rest of the $N - M$ eigenvalues are zero.

The trace constraint on ρ^A ensures that $\sum_i \lambda_i = 1$.

We now define the Von Neumann Entropy $S(\rho)$ for as follows:

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i \tag{16}$$

Note that this entropy of entanglement is same regardless of over which reduced density matrix this operation is performed since the state of the system AB is a pure state. This urges the viewpoint that the entropy of entanglement be considered as a property of the composite system AB rather than being a property of the components of the bipartite system.

With this measure, let us look at some specific cases: Suppose one of the eigenvalues say λ_1 is 1 and the rest are all zero, then, the entropy of entanglement is 0 (with the convention that $0 \log 0 = 0$) and this means that the state

$|\psi\rangle$ is completely seperable (i.e., $|\psi\rangle = |\lambda_1^A\rangle|\lambda_1^B\rangle$ and thus unentangled. In other cases, the state is entangled. Maximal entanglement occurs when $\lambda_i = \frac{1}{N}\,\forall\, i$, this means that, $S(\rho) = \log N$. Thus, the measure of the entropy of bipartite entanglement ranges from 0 to $\log N$ reflecting the spectrum of completely unentangled to maximally entangled.

2 Random States

2.1 Generating random pure state rdm

2.1.1 Introduction

We begin this section by defining a sequence of random matrix ensembles each consequently related to the next, with the intent of answering the question of how one generates reduced density matrices of random pure states.

Firstly, consider a matrix X (of dimension $N \times M$, $N \leq M$) whose elements $x_{i\alpha}$ each are drawn from the normal distribution $\mathcal{N}(0,1)$. X is then said to belong to the gaussian random matrix ensemble $G_\beta(N,M)$. The parameter β is $1,2$ respectively as the entries are real or complex numbers (where in the case of complex numbers, the real part and the imaginary part are each drawn in random from $\mathcal{N}(0,1)$).

The Wishart Ensemble of random matrices are those matrices W of the form $W = XX^\dagger$where X is drawn from $G_\beta(N,M)$.

In the next section, we do not confine the nature of the probability distribution from which the elements of matrix X is drawn from but rather provide a general framework for the generation of reduced density matrices from random pure states whose coefficients are the entries of X.

2.1.2 Deriving the reduced density matrix from random pure states

Let X be a $N \times M$ random matrix ($N \leq M$) whose elements $x_{i\alpha}$ are either real or complex numbers being drawn according to some probability distribution, say $\mathbb{P}$.

Now, consider a state $|\psi\rangle$ belonging to the composite system $\mathcal{H}_{\mathcal{AB}}^{(\mathrm{NM})}$given by:

$$|\psi\rangle = \sum_i \sum_\alpha x_{i\alpha} |i\rangle \otimes |\alpha\rangle \tag{17}$$

where, $|i\rangle$ and $|\alpha\rangle$ form an orthonormal bases respectively. To enforce the normalisation condition, we divide the state $|\psi\rangle$ by $\sqrt{\langle\psi|\psi\rangle} = \sqrt{\sum_i \sum_\alpha |x_{i\alpha}|^2}$

Then, the density matrix of this pure state is given by the following

$$\rho = |\psi\rangle\langle\psi| \tag{18}$$

and by taking the partial trace over system B, we get the reduced density matrix corresponding to system A:

$$\rho_A = \mathrm{Tr_B}(|\psi\rangle\langle\psi|) = \frac{XX^\dagger}{\sum_i \sum_\alpha |x_{i\alpha}|^2} = \frac{XX^\dagger}{\mathrm{Tr}XX^\dagger} \tag{19}$$

i.e given that the distribution of the elements $x_{i\alpha}$ are drawn independently (and identically) from $\mathbb{P}$ to form the state $|\psi\rangle$, it can be mapped directly onto the space of reduced density matrix of system A as follows:

$$\rho_A = \frac{\mathrm{Tr_B}(|\psi\rangle\langle\psi|)}{\langle\psi|\psi\rangle} = \frac{XX^\dagger}{\mathrm{Tr}XX^\dagger} \tag{20}$$

2.2 Hilbert-Schmidt Ensemble and Other Ensembles

Let X be drawn from the Ginibre Ensemble $G_\beta(N, M)$. Then, the matrix $W = XX^\dagger$ is distributed according to the Wishart Ensemble Distribution.

By requiring that the trace of ρ_A be 1 (i.e. dividing W by $\mathrm{Tr}(W)$), we generate reduced density matrices belonging to what is known as the Hilbert-Schmidt Ensemble.

Due to the elements each having been sampled from $\mathcal{N}(0,1)$ (which possesses rotational symmetry), we have that state $|\psi\rangle$ generated from X upon normalisation is equivalent to having been chosen from a uniform probability distribution over the Sphere S^{NM-1} in the case that $\beta = 1$ and S^{2NM-1} in the case that $\beta = 2$.

The joint probability distribution of the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ of the Hilbert-Schmidt Ensemble is given by [1]:

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = B_{M,N}\delta(\sum_{i=1}^{N} \lambda_i - 1) \prod_{i=1}^{N} \lambda_i^{\frac{\beta}{2}(M-N+1)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \tag{21}$$

where $\beta = 1, 2$ corresponding to whether the entries of X are real or complex respectively. The term $\delta(\sum_{i=1}^{N} \lambda_i - 1)$ in the above expression ensures the trace constraint requirement of density matrices is satisfied. Furthermore, considering the case of $M = N$, we arrive at the following:

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = C_N\delta(\sum_{i=1}^{N} \lambda_i - 1) \prod_{j<k} |\lambda_j - \lambda_k|^\beta \tag{22}$$

Apart from the HS Ensemble, we also consider reduced density matrices generated by sampling the elements of X according to the following distributions:

- Laplace distribution: $P(x) = \frac{\exp(-|x|)}{2}$, $x \in \mathbb{R}$
- Uniform distribution: $P(x) = \frac{1}{2}$, $x \in [-1, 1]$
- Cauchy distribution: $P(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$

We note here that the pure states generated under each of these cases do not satisfy the uniform distribution on the sphere of pure states. Exact expression for the eigenvalue distributions in these cases is yet to be discovered.

Another point of note is that the matrices obtained over $\mathbb{R}$, i.e, the real valued ensembles exhibit time reversal symmetry whereas complex valued ensemble is characteristic of time reversal symmetry breaking system.

2.3 Entropy of Entanglement for HS ensemble

The Exact formula for average entropy of entanglement in the case of the Hilbert-Schmidt ensemble was found by Don Page[2] and is as follows:

$$\langle S_A \rangle = \sum_{k=N+1}^{NM} \frac{1}{k} - \frac{N-1}{2M} \tag{23}$$

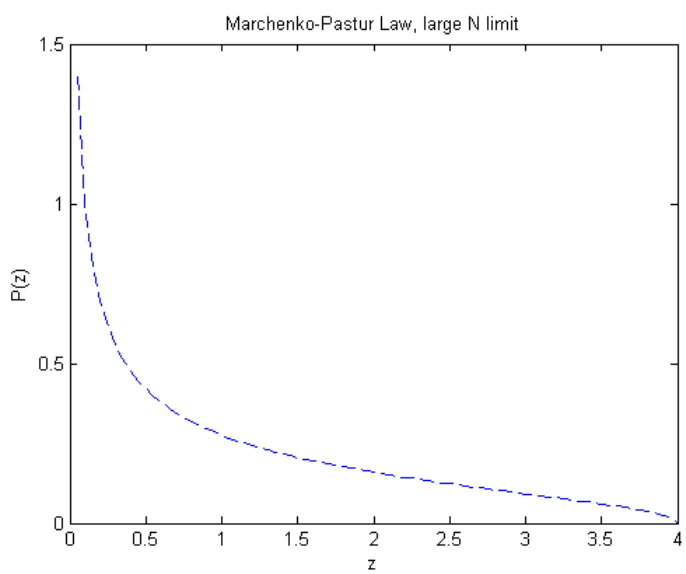
Suppose $N = M$ takes asymptotically large values, we see that,

$$\langle S_A \rangle = \sum_{k=N+1}^{NM} \frac{1}{k} - \frac{N-1}{2M} \approx \ln N - \frac{1}{2} \tag{24}$$

2.4 Large N limit on X - The Marchenko-Pastur Law

There is a well known theorem due to Vladimir Marchenko and Leonid Pastur regarding the eigenvalue distribution of large dimensional random matrices $XX^\dagger$ whose entries are drawn from a finite variance, zero mean probability distribution. A simplified version of the same is presented here. We assume that X is a square matrix of dimension N whose entries are drawn from some zero mean probability distribution whose variance, $\sigma^2 = 1$. Let $z = N\lambda$, where λ is an eigenvalue of $XX^\dagger$, then, the probability density function $P(z)$ [4] is given by :

$$P(z) = \frac{\sqrt{4-z}}{2\pi\sqrt{z}} \tag{25}$$



2.5 Physical significance of entropy of entanglement

2.5.1 Qubits, Qutrits and Qudits

Classically, Information is stored in bits, usually encoded in binary (base 2 representation). Analogous to the classical 0-1 bit is the *qubit*, a unit of quantum information. A qubit is any two-state quantum system. Formally, let $|\psi\rangle \in \mathcal{H}^2$ be a normalised state, we can represent it as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where, $\{|0\rangle, |1\rangle\}$ is the standard orthonormal basis of $\mathcal{H}^2$ and $|\alpha|^2 + |\beta|^2 = 1$. The state $|\psi\rangle$ upon measurement using the standard basis results in the state $|0\rangle$ with probability $|\alpha|^2$ and in state $|1\rangle$ with probability $|\beta|^2$ unlike a classical bit which either encodes 0 or 1.

Similarly, a *qutrit* is defined as that unit of quantum information encoding information via normalised states belonging to a three-state quantum system, i.e, $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$. In general, an *n-qudit* encodes information via states in a quantum system of dimension n, i.e, as a normalised state $|\psi\rangle \in \mathcal{H}^n$.

2.5.2 Entanglement between two qudits

Consider two *n-qudits* A and B, their composite system is described by vectors belonging to the bipartite system $\mathcal{H}^n \otimes \mathcal{H}^n$. Suppose ρ represents the density matrix of some pure state $|\psi\rangle \in \mathcal{H}^n \otimes \mathcal{H}^n$, then, computationally, we know that the entanglement between the two qudits is given by $S(\rho^A)\,[= S(\rho^B)]$.

Roughly, one may understand the entropy of entanglement to be a measure of the correlation between the outcome of individual measurements on the two qudits A and B, that is, suppose one were to measure the state of the first qudit alone without measuring the the second qudit, then, the amount of entanglement between the two qudits gives a measure of the influence that measuring the first qudit's state has on the measurement of the state of the second qudit.

Let us consider the example case of two qubits whose joint state is prepared as follows: $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \in \mathcal{H}^2 \otimes \mathcal{H}^2$.

This state is maximally entangled, i.e, $S(\rho^A) = log2$. In otherwords, if we are to measure the state of the first qubit in the standard basis, it is easily seen that, it also forces the second qubit to collapse into the same state as the first qubit.

In contrast, if the joint state is seperable, i.e, $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle$, then, $S(\rho^A) = 0$, i.e, any measurement on the subsystem A does not affect the outcome of the measurement of subsystem B and vice-versa. The two systems are completely uncorrelated ,i.e, unentangled.

In the following section, we determine the distribution of the entropy of entanglement as well as the average entropy for the case of random pure states of two qubits ($N = 2$), two qutrits ($N = 3$) and two 4-qudits ($N = 4$).

3 Results

For the four distributions expressed in the previous section, the marginal probability distribution of the eigenvalue of the reduced density matrix is found for $N = 2, 3, 4$ for x being drawn from the set $\mathbb{C}$ and $\mathbb{R}$. In the case of HS ensemble, the distribution of eigenvalues is verified against the marginal probability distribution derived from equation (21) for the complex case.[2] The distribution of the Entropy of Entanglement is also found for each of these cases. The average entropy of entanglement is computed and compared with Don Page's Formula.

3.1 Comparison with analytical expressions for HS ensemble

3.1.1 N=2, $\beta = 2$

The Joint probability density of the eigenvalues is given by:

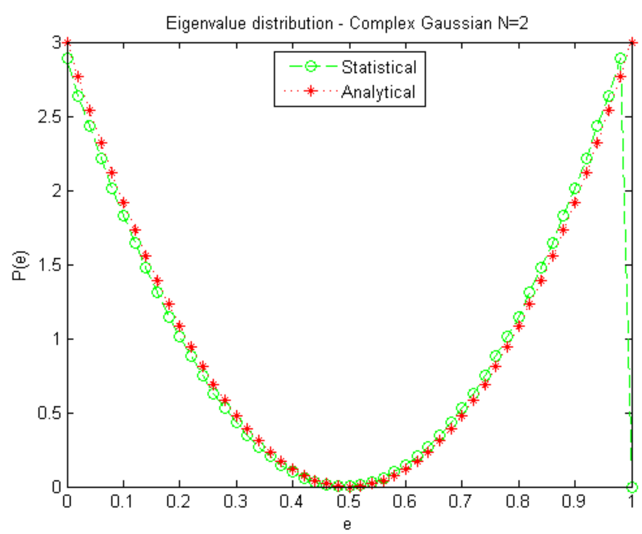
$$P(\lambda_1, \lambda_2) = C_2 \delta(\lambda_1 + \lambda_2 - 1)(\lambda_1 - \lambda_2)^2 \tag{26}$$

$$\Rightarrow P(\lambda) = \int P(\lambda_1, \lambda_2)\, d\lambda_2 = C_2(2\lambda - 1)^2, 0 \leq \lambda \leq 1 \tag{27}$$

Normalisation constraint yields $C_2 = 3$ which implies,

$$P(\lambda) = 3(2\lambda - 1)^2,\ 0 \leq \lambda \leq 1. \tag{28}$$

The eigenvalue distribution found statistically is compared with the analytical expression in the plot below:



[2]It is noted here that the analytical expressions of the marginal probability distribution in the case of Complex HS Ensemble for $N = 2, 3, 4$ were not found elsewhere in literature.

The statistical average of the entropy of entanglement between two qubits was found to be 0.3333 in agreement with the predicted value of 0.3333 from the formula.

3.1.2 N=3, $\beta = 2$

The Joint probability density of eigenvalues is given by:

$$P(\lambda_1, \lambda_2, \lambda_3) = C_3 \delta(\lambda_1 + \lambda_2 + \lambda_3 - 1)(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 \tag{29}$$

$$\Rightarrow P(\lambda_1, \lambda_2) = \int P(\lambda_1, \lambda_2, \lambda_3)\, d\lambda_3 = C_3(\lambda_1 - \lambda_2)^2(2\lambda_1 + \lambda_2 - 1)^2(2\lambda_2 + \lambda_1 - 1)^2$$

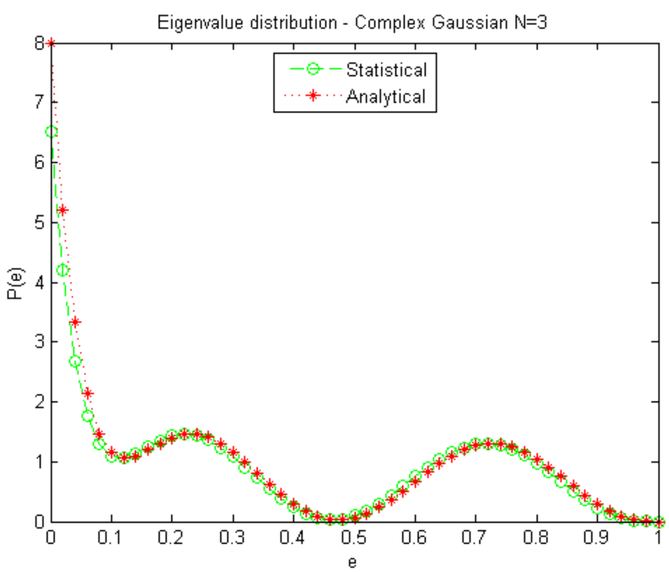
$$\Rightarrow P(\lambda_1) = C_3 \int_0^{1-\lambda_1} (\lambda_1 - \lambda_2)^2(2\lambda_1 + \lambda_2 - 1)^2(2\lambda_2 + \lambda_1 - 1)^2 d\lambda_2$$

$$\Rightarrow P(\lambda) = -\frac{C_3}{210}(-1+\lambda)^3(1 - 18\lambda + 132\lambda^2 - 354\lambda^3 + 309\lambda^4) \tag{30}$$

Normalisation yields $C_3 = \frac{1}{1680}$ which implies,

$$P(\lambda) = -8(-1+\lambda)^3(1 - 18\lambda + 132\lambda^2 - 354\lambda^3 + 309\lambda^4),\ 0 \leq \lambda \leq 1 \tag{31}$$

The eigenvalue distribution found statistically is compared with the analytical expression in the plot below:



The statistical average of the entropy of entanglement between two qutrits was found to be 0.6622 in close agreement with the predicted value of 0.6623 from the formula.

3.1.3 N=4, $\beta = 2$

The joint probability density is given by:

$$\begin{aligned} P(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= C_4 \delta(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - 1) \times \\ & (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_1 - \lambda_4)^2 (\lambda_2 - \lambda_3)^2 (\lambda_2 - \lambda_4)^2 (\lambda_3 - \lambda_4)^2 \end{aligned}$$

$$\Rightarrow \quad P(\lambda_1, \lambda_2, \lambda_3) = C_4 (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 \times$$
$$(2\lambda_1 + \lambda_2 + \lambda_3 - 1)^2 (\lambda_2 - \lambda_3)^2 (\lambda_1 + 2\lambda_2 + \lambda_3 - 1)^2 (\lambda_1 + \lambda_2 + 2\lambda_3 - 1)^2$$

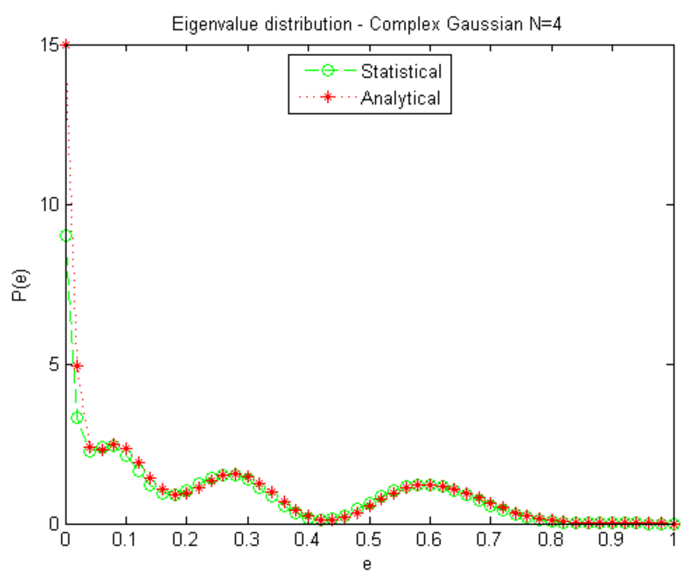
On Integrating out λ_2, λ_3, we get:

$$\Rightarrow P(\lambda) = C_4 \frac{(-1+\lambda)^8 (1 - 48\lambda + 1044\lambda^2 - 9904\lambda^3 + 44934\lambda^4 - 94128\lambda^5 + 73116\lambda^6)}{25225200}$$

Normalisation yields $C_3 = \frac{1}{378378000}$ which implies,

$$P(\lambda) = 15(-1+\lambda)^8 (1 - 48\lambda + 1044\lambda^2 - 9904\lambda^3 + 44934\lambda^4 - 94128\lambda^5 + 73116\lambda^6), \; 0 \leq \lambda \leq 1$$

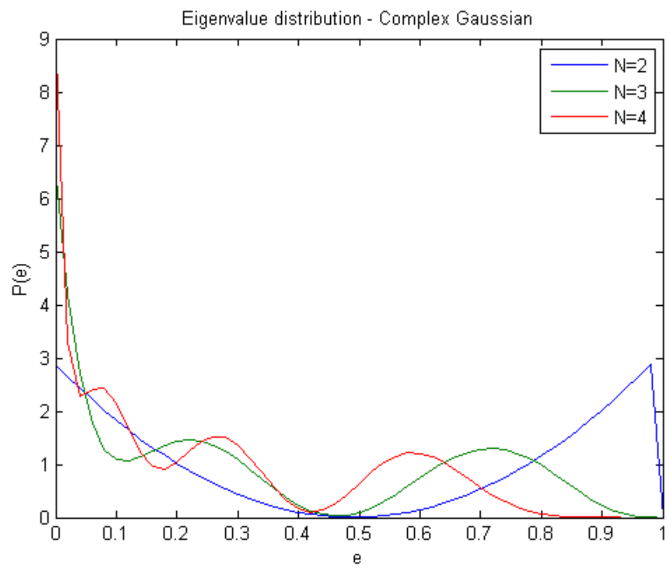
The eigenvalue distribution found statistically is compared with the analytical expression in the plot below:

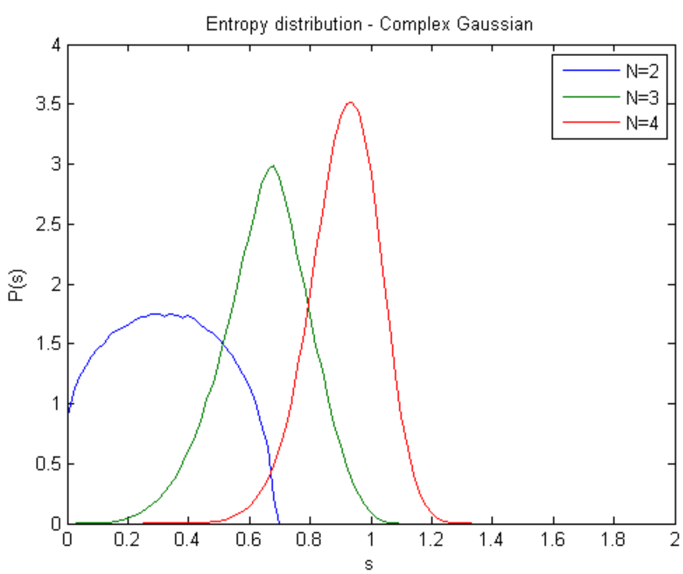


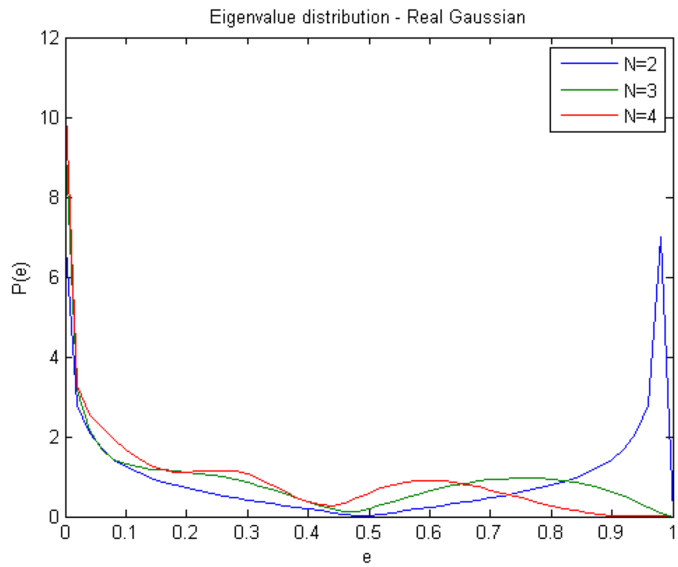
The statistical average of entanglement between two 4-qudits was found to be 0.9222 in close agreement with the predicted value of 0.9224 from the formula.

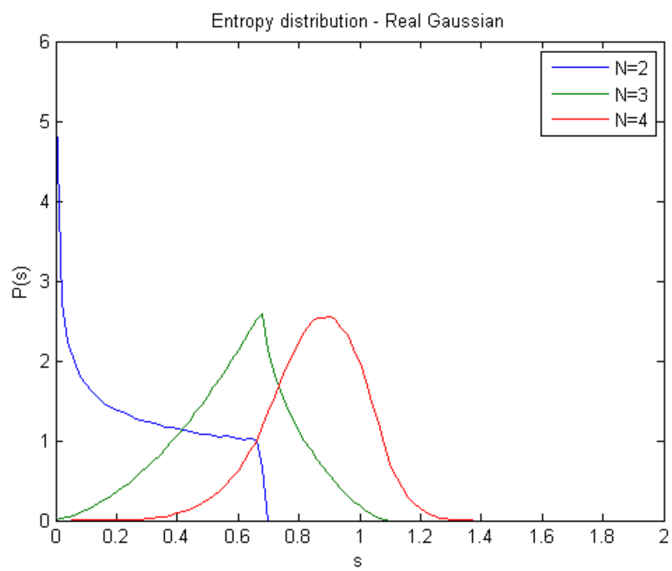
3.2 Marginal Distribution of eigenvalues and Distribution of Entropy of Entanglement

3.2.1 Gaussian - Complex and Real Cases

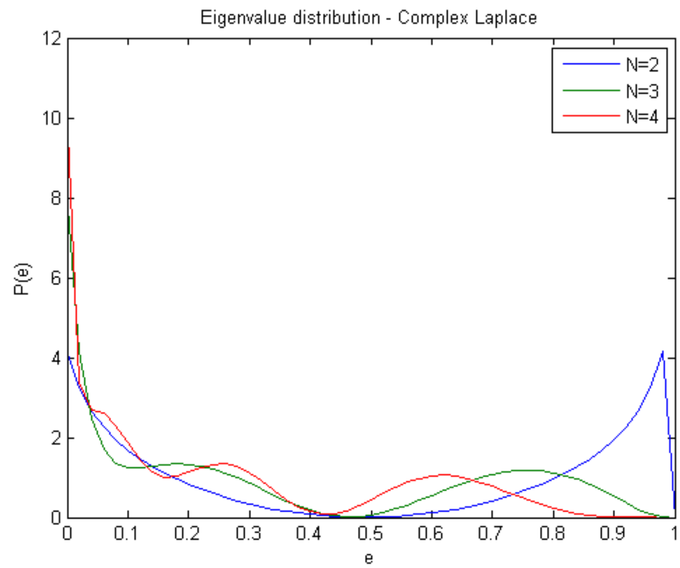


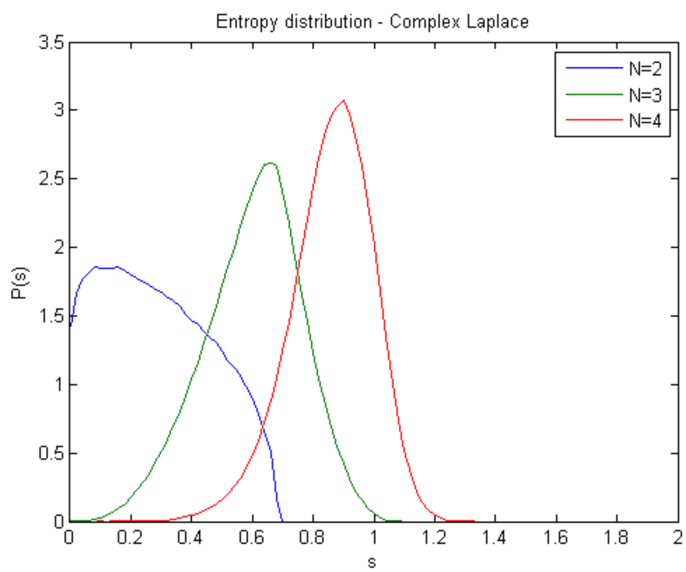


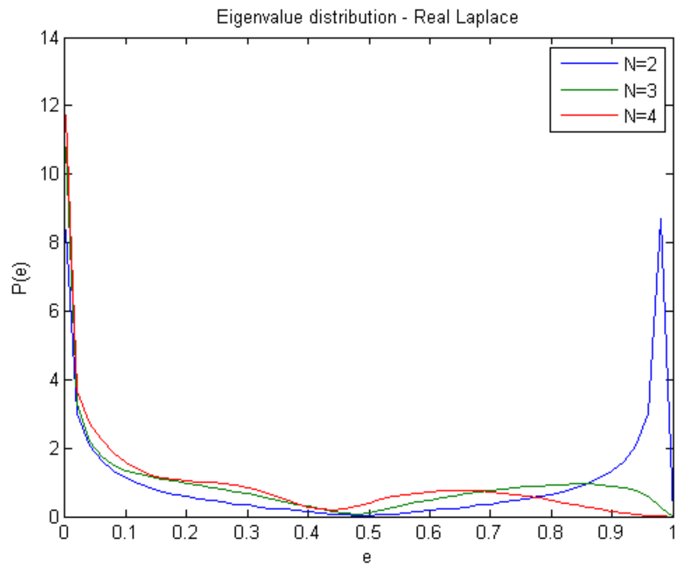

Eigenvalue distribution - Real Gaussian
N=2
N=3
N=4
P(e)
e
0
2
4
6
8
10
12
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Entropy distribution - Real Gaussian
N=2
N=3
N=4
P(s)
s
0
1
2
3
4
5
6
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

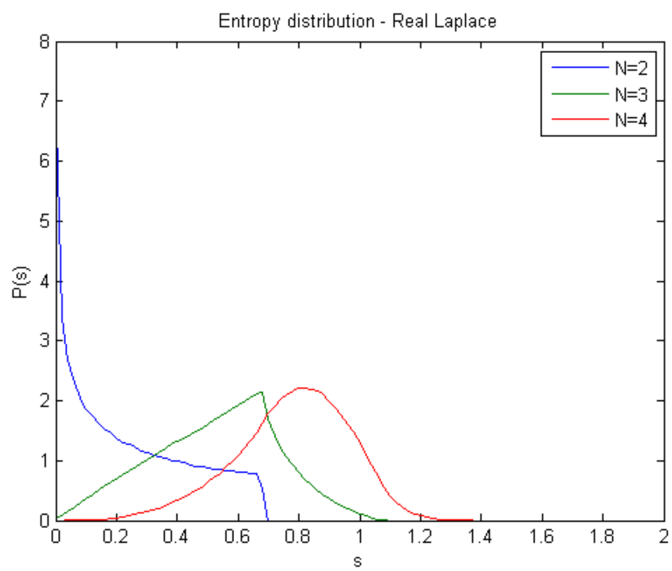
3.2.2 Laplace - Complex and Real cases





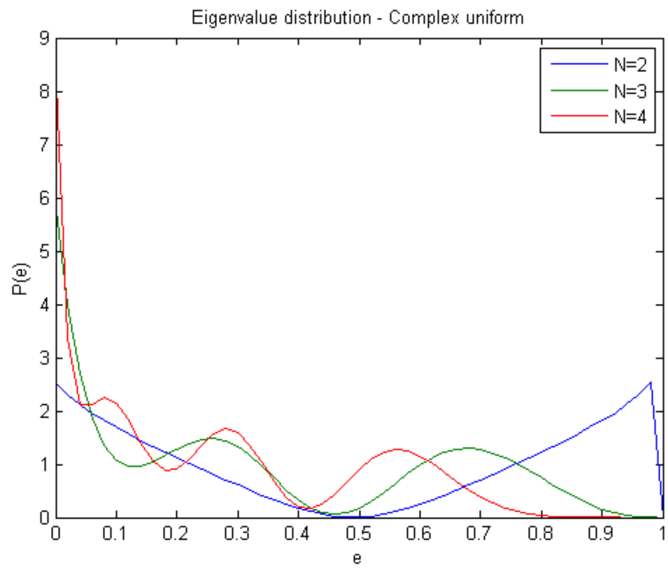


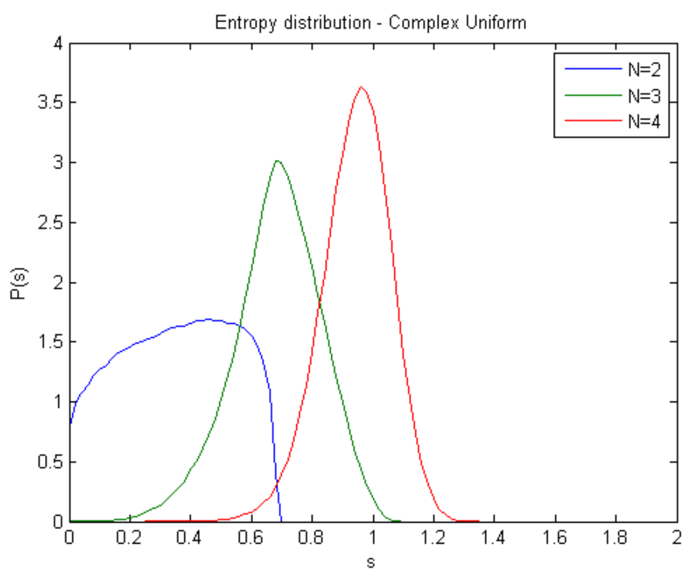
Eigenvalue distribution - Real Laplace
N=2
N=3
N=4
P(e)
e

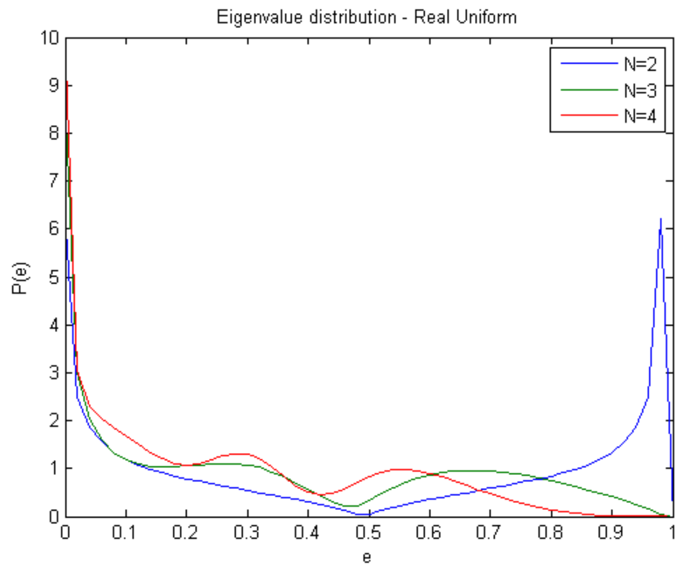


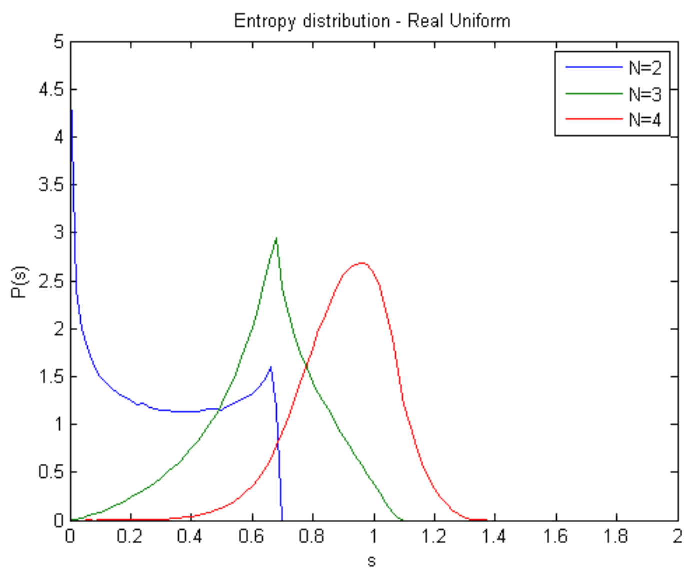
Entropy distribution - Real Laplace
N=2
N=3
N=4
P(s)
s

3.2.3 Uniform - Complex and Real cases

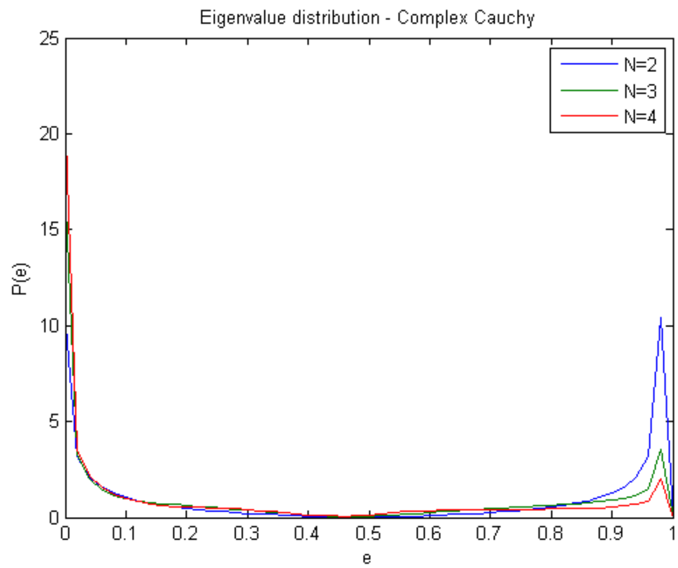


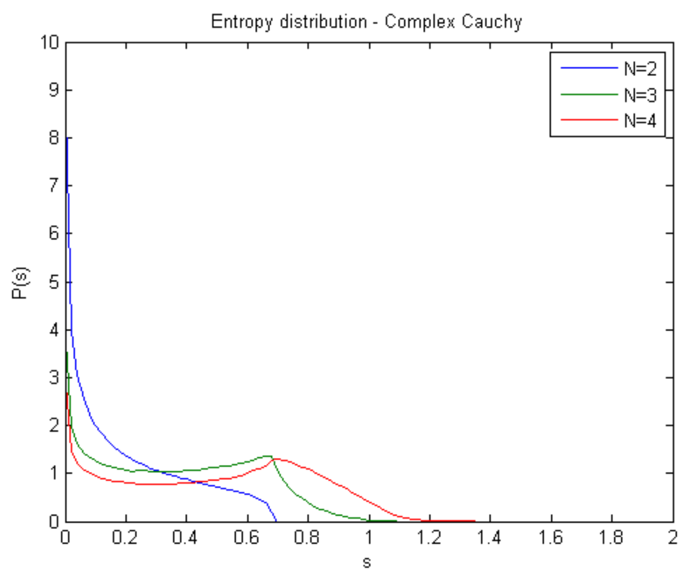


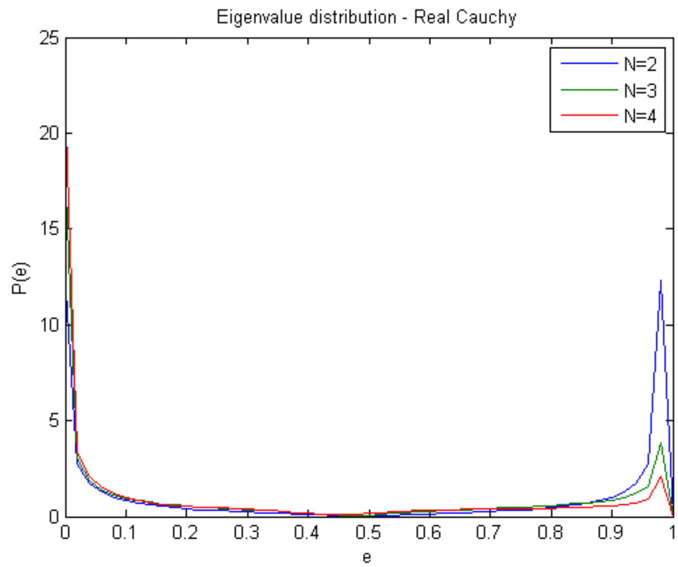

Eigenvalue distribution - Real Uniform
N=2
N=3
N=4
P(e)
e

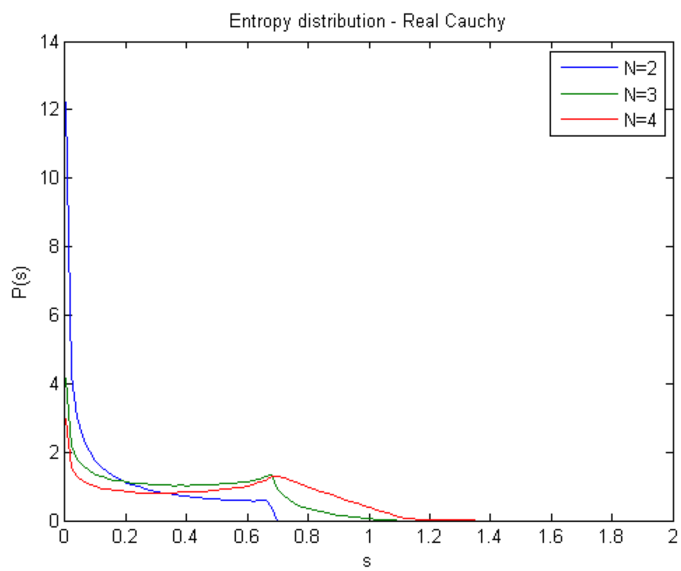

Entropy distribution - Real Uniform
N=2
N=3
N=4
P(s)
s

3.2.4 Cauchy - Complex and Real cases



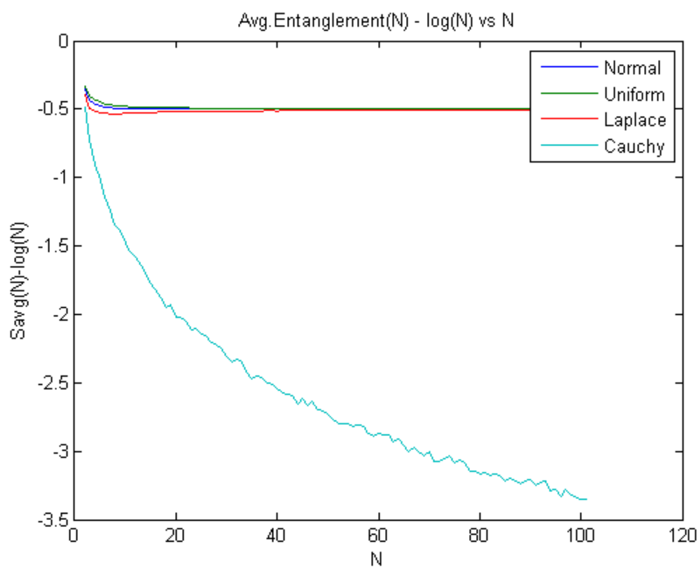



Eigenvalue distribution - Real Cauchy
N=2
N=3
N=4
P(e)
e
0
5
10
15
20
25
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Entropy distribution - Real Cauchy
N=2
N=3
N=4
P(s)
s
0
2
4
6
8
10
12
14
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

3.3 Comparison of Entropy of Entanglement

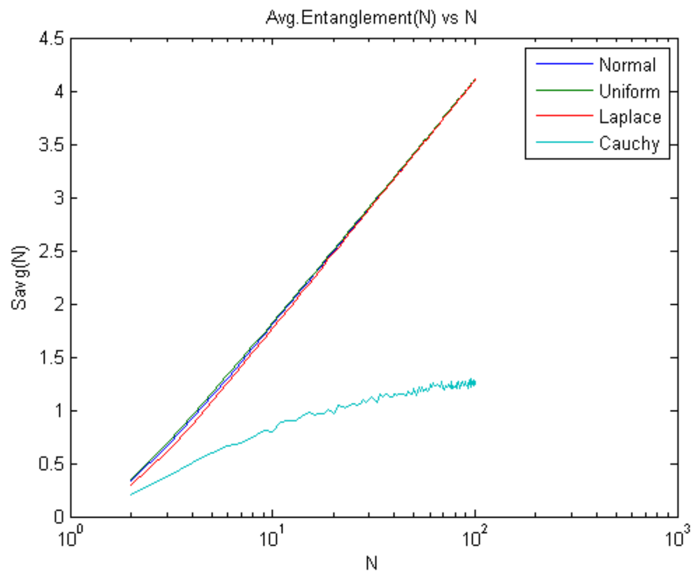
The graphs from the numerical analysis plotted in the previous section indicates a strong similarity in the mariginal distribution of the eigenvalues as well as the entropy of entanglement in the case of X drawn from the three finite variance distributions, namely Gaussian, Laplace and the Uniform distribution while the cauchy distributed states show drastically low levels of entanglement. Upon comparison of the average of the entropy of entanglement for $N = 2$ to 100, it was found that the average entropy of the three finite variance distributions behaves asymptotically as expected from equation (24).



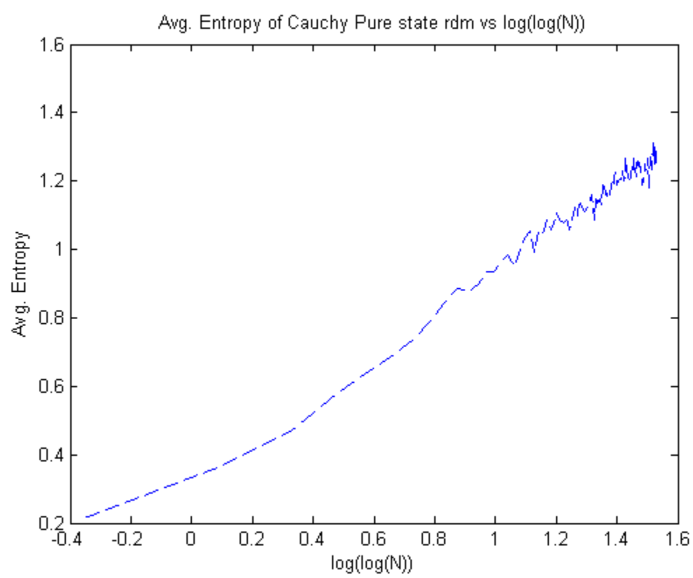
3.3.1 Table of average entanglement

$P(x)\backslash N$	2	3	4
Gaussian	0.3334	0.6622	0.9225
Laplace	0.2979	0.6116	0.8689
Uniform	0.3607	0.6949	0.9531
Cauchy	0.2026	0.3726	0.4925

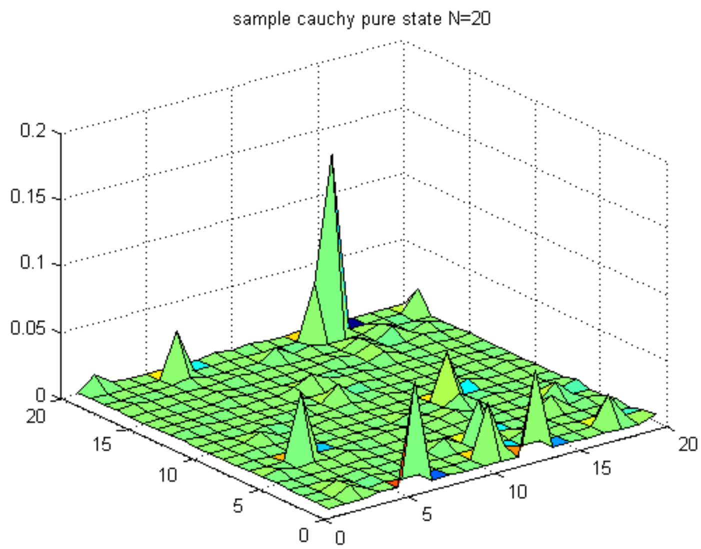
As expected, the three finite variance distributions lead to entropies that scale linearly with $\log N$:

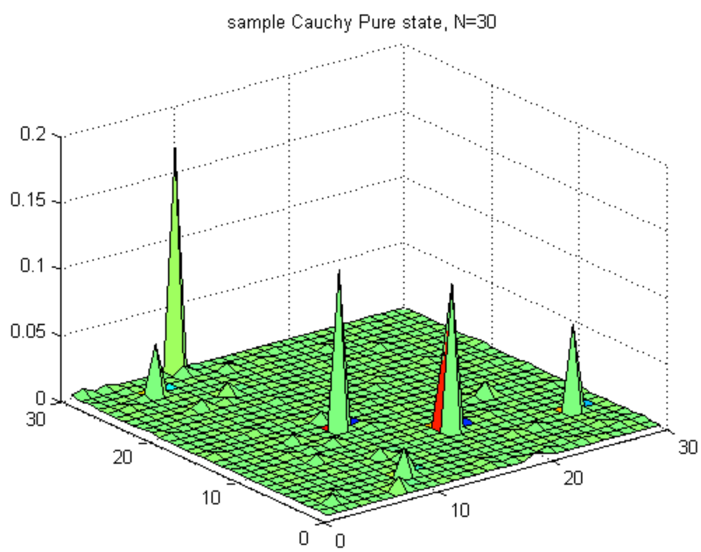


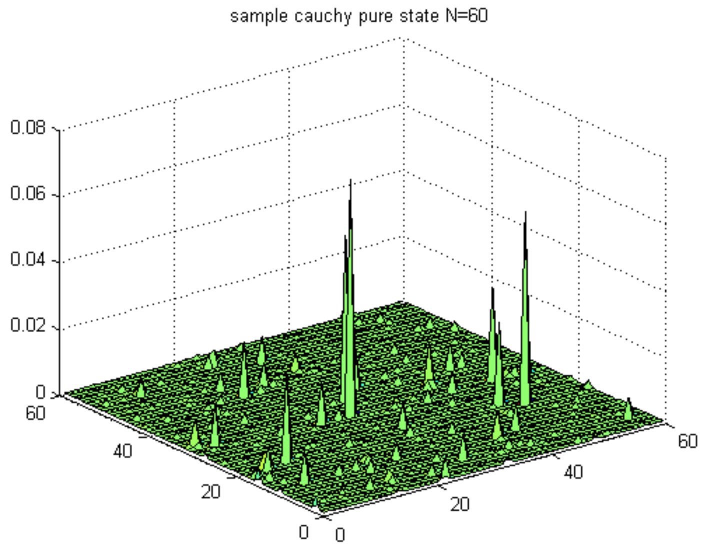
It is found that the Cauchy distributed pure states led to entanglement distribution that scales linearly with $\log \log N$:

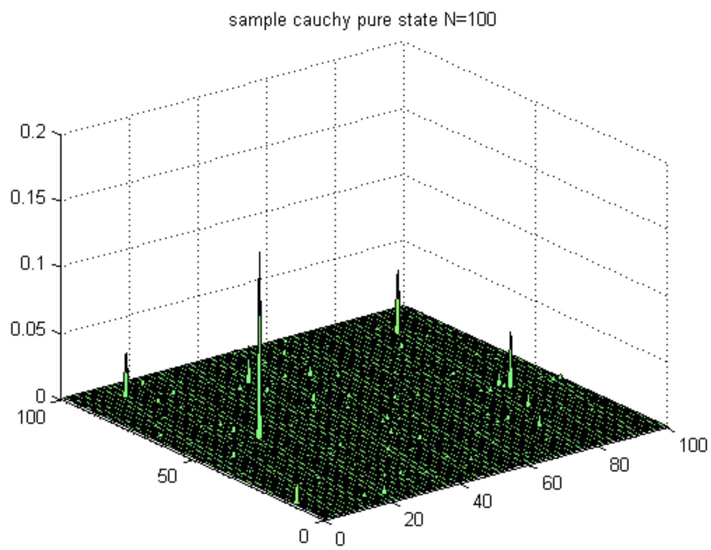


Upon inspection, the cauchy distributed pure states were found to be highly localised, some sample states are shown below:


sample cauchy pure state N=20
0.2
0.15
0.1
0.05
0
20
15
10
5
0
0
5
10
15
20


sample Cauchy Pure state, N=30
0.2
0.15
0.1
0.05
0
30
20
10
0
0
10
20
30


sample cauchy pure state N=60
0.08
0.06
0.04
0.02
0
60
40
20
0
0
20
40
60


sample cauchy pure state N=100
0.2
0.15
0.1
0.05
0
100
50
0
0
20
40
60
80
100

3.4 Conclusion

Apart from the pure states generated using the well known HS ensemble (which in the complex case is equivalent to picking pure states uniformly at random), it was found that pure states generated using other distributions such as the Laplace and Uniform distribution also possess similar behavior in terms of distribution of eigenvalues of reduced density matrices as well as distribution of entanglement in real and complex cases respectively.

The exact analytical expression for the marginal eigenvalue distribution was found in the case of the complex HS ensemble for $N = 2, 3, 4$ and confirmed to be correct by comparison with statistically generated values of the same.

When the pure states are populated using the Cauchy distribution, it is found that the resultant reduced density matrices are highly localised (i.e, have low entropies of entanglement).

Futher study is needed to understand the reason for this behavior as well as the $\log \log N$ behavior of the scaling law for the entanglement when the pure states are populated according to cauchy states.

4 Numerical Simulation Code

4.1 Gaussian

Marginal Eigenvalue Distribution Code (Complex):

```
l=zeros(1,2000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(2)+i*randn(2);
    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    j=j+2;
end
a=histc(l,(0:0.02:1));
a=a/(2000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,3000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(3)+i*randn(3);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    j=j+3;
end
a=histc(l,(0:0.02:1));
a=a/(3000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,4000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(4)+i*randn(4);
```

```
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    j=j+4;
end
a=histc(l,(0:0.02:1));
a=a/(4000000*0.02);
plot((0:0.02:1),a);
legend('N=2','N=3','N=4');
xlabel('e');
ylabel('P(e)');
title('Eigenvalue distribution - Complex Gaussian');
hold off
```

Entropy of Entanglement Distribution Code (Complex):

```
l=zeros(1,2000000);
m=0;
j=1;
S=zeros(1,1000000);
for m = 1:1000000
    a=randn(2)+i*randn(2);
    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2));
    j=j+2;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,3000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(3)+i*randn(3);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
```

```
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3));
    j=j+3;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,4000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(4)+i*randn(4);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3))-c(4)*log(c(4));
    j=j+4;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
legend('N=2','N=3','N=4');
xlabel('s');
ylabel('P(s)');
title('Entropy distribution - Complex Gaussian');
hold all
```

Marginal Eigenvalue Distribution Code (Real):

```
l=zeros(1,2000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(2);
    b=a'*a;
```

```
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    j=j+2;
end
a=histc(l,(0:0.02:1));
a=a/(2000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,3000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(3);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    j=j+3;
end
a=histc(l,(0:0.02:1));
a=a/(3000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,4000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(4);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    j=j+4;
end
a=histc(l,(0:0.02:1));
a=a/(4000000*0.02);
plot((0:0.02:1),a);
legend('N=2','N=3','N=4');
xlabel('e');
```

```
ylabel('P(e)');
title('Eigenvalue distribution - Real Gaussian');
hold off
```

Entropy of Entanglement Distribution Code (Real):

```
l=zeros(1,2000000);
m=0;
j=1;
S=zeros(1,1000000);
for m = 1:1000000
    a=randn(2);
    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2));
    j=j+2;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,3000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=randn(3);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3));
    j=j+3;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,4000000);
S=zeros(1,1000000);
```

```
m=0;
j=1;
for m = 1:1000000
    a=randn(4);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3))-c(4)*log(c(4));
    j=j+4;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
legend('N=2','N=3','N=4');
xlabel('s');
ylabel('P(s)');
title('Entropy distribution - Real Gaussian');
hold all
```

4.2 Laplace

Marginal Eigenvalue Distribution Code (Complex):

```
l=zeros(1,2000000);
m=0;
j=1;
for m = 1:1000000
    a=exprnd(1,2,2);
    k=unidrnd(2,2,2);
    k=k-[1,1;1,1];
    k=k*2-[1,1;1,1];
    a=a.*k;

    d=exprnd(1,2,2);
    k=unidrnd(2,2,2);
    k=k-[1,1;1,1];
    k=k*2-[1,1;1,1];
    d=d.*k;

a=a+i*d;
```

```
b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    j=j+2;
end
a=histc(l,(0:0.02:1));
a=a/(2000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,3000000);
m=0;
j=1;
for m = 1:1000000
    a=exprnd(1,3,3);
    k=unidrnd(2,3,3);
    k=k-[1,1,1;1,1,1;1,1,1];
    k=k*2-[1,1,1;1,1,1;1,1,1];
    a=a.*k;

    d=exprnd(1,3,3);
    k=unidrnd(2,3,3);
    k=k-[1,1,1;1,1,1];
    k=k*2-[1,1,1;1,1,1];
    d=d.*k;

a=a+i*d;

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    j=j+3;
end
a=histc(l,(0:0.02:1));
a=a/(3000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,4000000);
m=0;
j=1;
for m = 1:1000000
```

```
    a=exprnd(1,4,4);
    k=unidrnd(2,4,4);
    k=k-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    k=k*2-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    a=a.*k;

    d=exprnd(1,4,4);
    k=unidrnd(2,4,4);
    k=k-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    k=k*2-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    d=d.*k;

a=a+i*d;
b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    j=j+4;
end
a=histc(l,(0:0.02:1));
a=a/(4000000*0.02);
plot((0:0.02:1),a);
legend('N=2','N=3','N=4');
xlabel('e');
ylabel('P(e)');
title('Eigenvalue distribution - Complex Laplace');
hold off
```

Entropy of Entanglement Distribution Code (Complex):

```
l=zeros(1,2000000);
m=0;
j=1;
S=zeros(1,1000000);
for m = 1:1000000
    a=exprnd(1,2,2);
    k=unidrnd(2,2,2);
    k=k-[1,1;1,1];
    k=k*2-[1,1;1,1];
    a=a.*k;

    d=exprnd(1,2,2);
    k=unidrnd(2,2,2);
    k=k-[1,1;1,1];
```

```
    k=k*2-[1,1;1,1];
    d=d.*k;

a=a+i*d;

    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2));
    j=j+2;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,3000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=exprnd(1,3,3);
    k=unidrnd(2,3,3);
    k=k-[1,1,1;1,1,1;1,1,1];
    k=k*2-[1,1,1;1,1,1;1,1,1];
    a=a.*k;

    d=exprnd(1,3,3);
    k=unidrnd(2,3,3);
    k=k-[1,1,1;1,1,1];
    k=k*2-[1,1,1;1,1,1];
    d=d.*k;

a=a+i*d;

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3));
    j=j+3;
end
```

```
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,4000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
a=exprnd(1,4,4);
    k=unidrnd(2,4,4);
    k=k-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    k=k*2-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    a=a.*k;

    d=exprnd(1,4,4);
    k=unidrnd(2,4,4);
    k=k-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    k=k*2-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    d=d.*k;

a=a+i*d;
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3))-c(4)*log(c(4));
    j=j+4;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
legend('N=2','N=3','N=4');
xlabel('s');
ylabel('P(s)');
title('Entropy distribution - Complex Laplace');
hold all
```

Marginal Eigenvalue Distribution Code (real):

```
l=zeros(1,2000000);
```

```
m=0;
j=1;
for m = 1:1000000
    a=exprnd(1,2,2);
    k=unidrnd(2,2,2);
    k=k-[1,1;1,1];
    k=k*2-[1,1;1,1];
    a=a.*k;

b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    j=j+2;
end
a=histc(l,(0:0.02:1));
a=a/(2000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,3000000);
m=0;
j=1;
for m = 1:1000000
    a=exprnd(1,3,3);
    k=unidrnd(2,3,3);
    k=k-[1,1,1;1,1,1;1,1,1];
    k=k*2-[1,1,1;1,1,1;1,1,1];
    a=a.*k;

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    j=j+3;
end
a=histc(l,(0:0.02:1));
a=a/(3000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,4000000);
m=0;
j=1;
for m = 1:1000000
```

```
    a=exprnd(1,4,4);
    k=unidrnd(2,4,4);
    k=k-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    k=k*2-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    a=a.*k;

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    j=j+4;
end
a=histc(l,(0:0.02:1));
a=a/(4000000*0.02);
plot((0:0.02:1),a);
legend('N=2','N=3','N=4');
xlabel('e');
ylabel('P(e)');
title('Eigenvalue distribution - Real Laplace');
hold off
```

Entropy of Entanglement Distribution Code (real):

```
l=zeros(1,2000000);
m=0;
j=1;
S=zeros(1,1000000);
for m = 1:1000000
    a=exprnd(1,2,2);
    k=unidrnd(2,2,2);
    k=k-[1,1;1,1];
    k=k*2-[1,1;1,1];
    a=a.*k;
    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2));
    j=j+2;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
```

```
plot((0:0.02:2),a);
hold all
l=zeros(1,3000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=exprnd(1,3,3);
    k=unidrnd(2,3,3);
    k=k-[1,1,1;1,1,1;1,1,1];
    k=k*2-[1,1,1;1,1,1;1,1,1];
    a=a.*k;

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3));
    j=j+3;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,4000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
a=exprnd(1,4,4);
    k=unidrnd(2,4,4);
    k=k-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    k=k*2-[1,1,1,1;1,1,1,1;1,1,1,1;1,1,1,1];
    a=a.*k;

    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3))-c(4)*log(c(4));
```

```
    j=j+4;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
legend('N=2','N=3','N=4');
xlabel('s');
ylabel('P(s)');
title('Entropy distribution - Real Laplace');
hold all
```

4.3 Uniform

Marginal Eigenvalue Distribution Code (Complex):

```
l=zeros(1,2000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(2)*2-[1,1;1,1];
d=rand(2)*2-[1,1;1,1];

a=a+i*d;

b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    j=j+2;
end
a=histc(l,(0:0.02:1));
a=a/(2000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,3000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(3)*2-ones(3,3);
d=rand(3)*2-ones(3,3);

a=a+i*d;

b=a'*a;
```

```
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    j=j+3;
end
a=histc(l,(0:0.02:1));
a=a/(3000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,4000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(4)*2-ones(4,4);
d=rand(4)*2-ones(4,4);

a=a+i*d;

    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    j=j+4;
end
a=histc(l,(0:0.02:1));
a=a/(4000000*0.02);
plot((0:0.02:1),a);
legend('N=2','N=3','N=4');
xlabel('e');
ylabel('P(e)');
title('Eigenvalue distribution - Complex Uniform');
hold off
```

Entropy of Entanglement Distribution Code (Complex):

```
l=zeros(1,2000000);
m=0;
j=1;
S=zeros(1,1000000);
for m = 1:1000000
    a=rand(2)*2-[1,1;1,1];
d=rand(2)*2-[1,1;1,1];
```

```
a=a+i*d;

b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2));
    j=j+2;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,3000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(2)*2-ones(3,3);
d=rand(2)*2-ones(3,3);

a=a+i*d;

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3));
    j=j+3;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,4000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(4)*2-ones(4,4);
```

```
d=rand(4)*2-ones(4,4);

a=a+i*d;

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3))-c(4)*log(c(4));
    j=j+4;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
legend('N=2','N=3','N=4');
xlabel('s');
ylabel('P(s)');
title('Entropy distribution - Complex Uniform');
hold all
```

Marginal Eigenvalue Distribution Code (Real):

```
l=zeros(1,2000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(2)*2-[1,1;1,1];

b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    j=j+2;
end
a=histc(l,(0:0.02:1));
a=a/(2000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,3000000);
m=0;
j=1;
```

```
for m = 1:1000000
    a=rand(3)*2-ones(3,3);

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    j=j+3;
end
a=histc(l,(0:0.02:1));
a=a/(3000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,4000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(4)*2-ones(4,4);

    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    j=j+4;
end
a=histc(l,(0:0.02:1));
a=a/(4000000*0.02);
plot((0:0.02:1),a);
legend('N=2','N=3','N=4');
xlabel('e');
ylabel('P(e)');
title('Eigenvalue distribution - Real Uniform');
hold off
```

Entropy of Entanglement Distribution Code(real):

```
l=zeros(1,2000000);
m=0;
j=1;
S=zeros(1,1000000);
for m = 1:1000000
    a=rand(2)*2-[1,1;1,1];
```

```
b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2));
    j=j+2;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,3000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(2)*2-ones(3,3);

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3));
    j=j+3;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,4000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=rand(4)*2-ones(4,4);

b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
```

```
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3))-c(4)*log(c(4));
    j=j+4;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
legend('N=2','N=3','N=4');
xlabel('s');
ylabel('P(s)');
title('Entropy distribution - Real Uniform');
hold all
```

4.4 Cauchy

Marginal Eigenvalue Distribution Code (complex):

```
l=zeros(1,2000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,2,2)+i*trnd(1,2,2);
    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    j=j+2;
end
a=histc(l,(0:0.02:1));
a=a/(2000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,3000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,3,3)+i*trnd(1,3,3);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
```

```
    l(j+2)=c(3);
    j=j+3;
end
a=histc(l,(0:0.02:1));
a=a/(3000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,4000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,4,4)+i*trnd(1,4,4);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    j=j+4;
end
a=histc(l,(0:0.02:1));
a=a/(4000000*0.02);
plot((0:0.02:1),a);
legend('N=2','N=3','N=4');
xlabel('e');
ylabel('P(e)');
title('Eigenvalue distribution - Complex cauchy');
hold off
```

Entropy of Entanglement Distribution Code (complex):

```
l=zeros(1,2000000);
m=0;
j=1;
S=zeros(1,1000000);
for m = 1:1000000
    a=trnd(1,2,2)+i*trnd(1,2,2);
    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2));
    j=j+2;
end
sum(S)/1000000
```

```
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,3000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,3,3)+i*trnd(1,3,3);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3));
    j=j+3;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,4000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,4,4)+i*trnd(1,4,4);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3))-c(4)*log(c(4));
    j=j+4;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
legend('N=2','N=3','N=4');
xlabel('s');
```

```
ylabel('P(s)');
title('Entropy distribution - Complex Cauchy);
hold all
```

Marginal Eigenvalue Distribution Code (real):

```
l=zeros(1,2000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,2,2);
    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    j=j+2;
end
a=histc(l,(0:0.02:1));
a=a/(2000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,3000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,3,3);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    j=j+3;
end
a=histc(l,(0:0.02:1));
a=a/(3000000*0.02);
plot((0:0.02:1),a);
hold all
l=zeros(1,4000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,4,4);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
```

```
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    j=j+4;
end
a=histc(l,(0:0.02:1));
a=a/(4000000*0.02);
plot((0:0.02:1),a);
legend('N=2','N=3','N=4');
xlabel('e');
ylabel('P(e)');
title('Eigenvalue distribution - Real cauchy');
hold off
```

Entropy of Entanglement Distribution Code (real):

```
l=zeros(1,2000000);
m=0;
j=1;
S=zeros(1,1000000);
for m = 1:1000000
    a=trnd(1,2,2);
    b=a'*a;
    b=b/(b(1,1)+b(2,2));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2));
    j=j+2;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,3000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,3,3);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3));
    c=eig(b);
    l(j)=c(1);
```

```
    l(j+1)=c(2);
    l(j+2)=c(3);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3));
    j=j+3;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
hold all
l=zeros(1,4000000);
S=zeros(1,1000000);
m=0;
j=1;
for m = 1:1000000
    a=trnd(1,4,4);
    b=a'*a;
    b=b/(b(1,1)+b(2,2)+b(3,3)+b(4,4));
    c=eig(b);
    l(j)=c(1);
    l(j+1)=c(2);
    l(j+2)=c(3);
    l(j+3)=c(4);
    S(m)=-c(1)*log(c(1))-c(2)*log(c(2))-c(3)*log(c(3))-c(4)*log(c(4));
    j=j+4;
end
sum(S)/1000000
a=histc(S,(0:0.02:2));
a=a/(1000000*0.02);
plot((0:0.02:2),a);
legend('N=2','N=3','N=4');
xlabel('s');
ylabel('P(s)');
title('Entropy distribution - Real Cauchy);
hold all
```

4.5 Comparison of Average Entropy:

```
N=2;
S=zeros(4,100);
for N=2:101
    for l=1:1000
        a=randn(N)+i*randn(N);
        b=a'*a;
        q=0;
```

```
        for m=1:N
            q=q+b(m,m);
        end
        b=b/q;
        c=eig(b);
        S(1,N-1)=S(1,N-1)-c'*(log(c));
    end
    S(1,N-1)=S(1,N-1)/1000 - log(N);
end
plot(2:101,S(1,:));
hold all;

N=2;
for N=2:101
    for l=1:1000
                a=rand(N)*2-ones(N,N);
        d=rand(N)*2-ones(N,N);
        a=a+i*d;
                b=a'*a;
        q=0;
        for m=1:N
            q=q+b(m,m);
        end
        b=b/q;
        c=eig(b);
        S(2,N-1)=S(2,N-1)-c'*(log(c));
    end
    S(2,N-1)=S(2,N-1)/1000 - log(N);
end
plot(2:101,S(2,:));
hold all;

N=2;
for N=2:101
    for l=1:1000
        a=exprnd(1,N,N);
        k=unidrnd(2,N,N);
        k=k-ones(N,N);
        k=k*2-ones(N,N);
        a=a.*k;
             d=exprnd(1,N,N);
        k=unidrnd(2,N,N);
        k=k-ones(N,N);
        k=k*2-ones(N,N);
        d=d.*k;
```

```
a=a+i*d;

        b=a'*a;
        q=0;
        for m=1:N
            q=q+b(m,m);
        end
        b=b/q;
        c=eig(b);
        S(3,N-1)=S(3,N-1)-c'*(log(c));
    end
    S(3,N-1)=S(3,N-1)/1000 - log(N);
end
plot(2:101,S(3,:));
hold all;
N=2;
for N=2:101
    for l=1:1000
        a=trnd(1,N,N)+i*trnd(1,N,N);
        b=a'*a;
        q=0;
        for m=1:N
            q=q+b(m,m);
        end
        b=b/q;
        c=eig(b);
        S(4,N-1)=S(4,N-1)-c'*(log(c));
    end
    S(4,N-1)=S(4,N-1)/1000 - log(N);
end
plot(2:101,S(4,:));
legend('Normal','Uniform','Laplace','Cauchy');
xlabel('N');
ylabel('Savg(N)-log(N)');
title('Avg.Entanglement(N) - log(N) vs N');
```

4.6 Generation of Cauchy Pure states

```
a=trnd(1,N,N)+i*trnd(1,N,N);
    a=a/sum(sum(abs(a)));
    figure;
    surf(1:N,1:N,abs(a),gradient(abs(a)));
```

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